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EXCITATION OF ION-ACOUSTIC WAVES AND ELECTRON HEATING IN A PLASMA IN AN EXTERNAL ELECTRIC FIELD

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We consider a plasma with cold ions in a strong external electric field that is periodic in time. The collisionless mechanism responsible for electron heating under these conditions is analyzed. This heating mechanism operates as follows: because of the external electric field the electrons acquire a velocity with respect to the ions; ion-acoustic waves are excited and these tend to smooth the electron distribution function. The smoothing process is repeated in the periodic electric field, leading to flattening of the electron distribution function (consequently an increase in electron temperature) in the velocity range corresponding to twice the velocity acquired by the electrons in the external field.

VARIOUS kinds of waves can be excited in a plasma when the static particle distribution function departs from equilibrium. If, for example, a strong current flows through the plasma, i.e., the ions and electrons (with Maxwellian velocity distributions) move with respect to each other with a constant velocity greater than some critical value, one finds that ion-acoustic waves are excited. [1] The quasilinear theory [2,3] takes account of the feedback effect of the growing waves on the particle distribution function for the weakly unstable case, where the growth rate is small compared with the frequency and the energy density of the wave is small compared with the thermal energy density of the particles. Under these conditions only the fine details of the particle distribution function are affected (for example, when ion-acoustic waves are excited the electron distribution function is flattened in a narrow velocity region about the velocity of the ion-acoustic wave). The gross plasma parameters, such as the moments of the distribution function, are not affected significantly. Below we describe a case in which the instability is weak but in which one of the gross parameters, i.e., the electron temperature, is changed significantly.

We shall treat a spatially uniform problem: a plasma in a strong magnetic field in which the ions and electrons can be described by a one-dimensional distribution function $f(v)$, where v is the velocity in the direction of the magnetic field.

Let us assume that there is an external electric field $E = E_0 \cos\omega t$ in the same direction as the magnetic field. Under the effect of this field the electrons move with respect to the ions and if $T_i \ll T$ (T_i, T are respectively the ion and electron temperatures) ion-acoustic waves are excited in the plasma; these waves are characterized by a phase velocity u ranging from a velocity of the order of the ion thermal velocity $u_{Ti} = \sqrt{T_i/M}$ to the ion acoustic velocity $u_0 = \sqrt{T/M}$ (cf. [1]). In general, the feedback effect of the ion-acoustic waves leads to the formation of a plateau on the particle distribution function. [3] In what follows we neglect the diffusion of the heavy ions under the effect of the waves and assume, for simplicity, that the ions are cold, $T_i = 0$ (so that $u_{Ti} = 0$). We assume that the plateau on the electron distribution function is formed so rapidly that it is in existence at any instant of time (cf. Appendix); our problem then is to analyze the time variation of the electron distribution function.

In the coordinate system in which the mean electron velocity $\bar{v}$ vanishes (Fig. 1) the ion distribution function and the associated plateau on the electron distribution function execute periodic motion with amplitude $V_0 = eE_0/m\omega$ (the field amplitude E_0 is limited by the condition $V_0 < v_T$; if this condition is not satisfied, the growth rate is comparable with the frequency and the quasi-linear theory no longer applies). Assume that at time t the ions as a whole have a velocity $V = V_0 \sin\omega t$

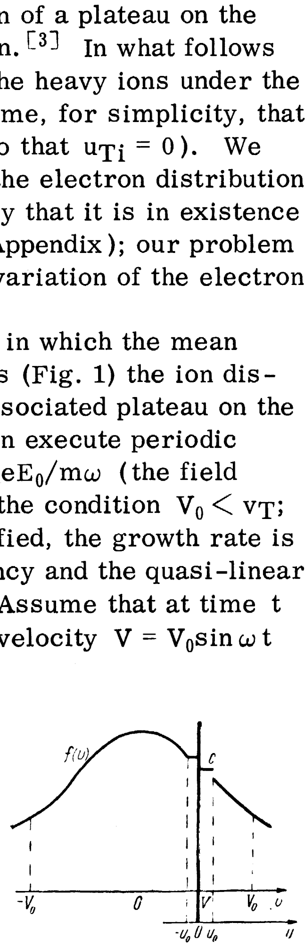

FIG. 1. Electronic (f) and ionic (solid vertical line) distribution functions. The plateaus of width u_0 execute periodic motion in the region $(-V_0, V_0)$. The case shown in the figure corresponds to plateau motion towards increasing v.

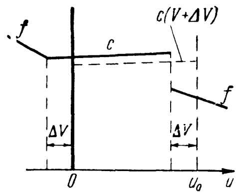

FIG. 2. Smoothing of the distribution function in the region $(0, u_0)$.

and that the height of the plateau on the electron distribution function $f(v) = c$ in the range $(V, V + u_0)$; in the ion coordinate system $(v = V + u)$ we have $f(u) = c$ in the range $0 < u < u_0$ (u is the statistical velocity of the electron in the ion coordinate system) (Fig. 2). In a time Δt the external electric field shifts the plateau by an amount $\Delta V = em^{-1}E_0 \cos \omega t \cdot \Delta t$ and the electron distribution function in the plateau region acquires the step shape shown in Fig. 2 by the solid line. The number of electrons in the plateau region is then reduced by $|\Delta V| [c - f(u_0)]$. On the other hand, this reduction is also equal to $u_0 [c(V) - c(V + \Delta V)]$, where $c(V + \Delta V)$ is the height of the plateau established after electron diffusion due to the ion-acoustic waves has occurred (the value $c(V + \Delta V)$ is shown in Fig. 2 by the dashed curve). Equating these expressions we find

$$\pm u_0 dc/dV = f(u_0 + V) - c(V), \tag{1}$$

where the plus sign is written when the plateau moves in the direction of increasing V (Figs. 1 and 2 illustrate this case) and the minus sign is written for motion in the opposite direction. It is evident from Fig. 2 that as the plateau moves there is a discontinuity at the fixed end and that beyond this region a continuous trail $f(V) = c$ remains.

We now consider the case of a strong electric field $u_0 \ll V_0$. If the width of the plateau u_0 is small it is evident from (1) that the distribution function $f(v)$ does not change appreciably in one period. Hence, from (1) we obtain an equation that describes the variation of $f(v)$ in a time much greater than the period of the external electric field $2\pi/\omega$.

We consider the time interval $\Delta\tau \sim 2\pi n/\omega$, $n \gg 1$, and find the quantity

$$\frac{\partial f}{\partial \tau} = \frac{\Delta f}{\Delta \tau} = \frac{(\Delta f)_n}{2\pi n / \omega},$$

where $(\Delta f)_n$ is the change in the distribution function in n periods as found from Eq. (1). The solution of Eq. (1) is

$$c = f \mp u_0 \partial f / \partial V + u_0^2 \partial^2 f / \partial V^2 \mp \ldots = L_\pm f,$$

$$L_\pm = 1 \mp u_0 \frac{\partial}{\partial V} + \left(u_0 \frac{\partial}{\partial V}\right)^2 \mp \ldots = \frac{1}{1 \pm u_0 \partial / \partial V}. \tag{2}$$

Thus, after a single traversal of the smoothing region the electron distribution function f is transformed into $L_\pm f$. In a full period the plateau moves in the forward and reverse directions so that $f \to L_-(L_+ f) = (L_- L_+) f$ and in n periods $f \to (L_+ L_-)^n f$; the change $(\Delta f)_n$ of the distribution function f in n periods under the effect of the smoothing region of width u_0 is

$$(\Delta f)_n = (L_+ L_-)^n f - f = [(L_+ L_-)^n - 1] f.$$

Substituting this expression in the relation $\partial f/\partial\tau = (\Delta f)_n \omega / 2\pi n$, to accuracy of order $(u_0/V_0)^2$ we obtain the equation for the distribution function:

$$\partial f / \partial \tau = D \partial^2 f / \partial v^2, \qquad D = u_0^2 \omega / 2\pi. \tag{3a}$$

Since the number of particles in the interval $(-V_0, V_0)$ is conserved the boundary condition for the diffusion equation (3a) is

$$(\partial f / \partial v)_{v = \pm V_0} = 0. \tag{3b}$$

Up to this point we have considered the diffusion of electrons due to waves whose phase velocities (in the ion coordinate system) lie in the range $(0, u_0)$. However, ion-acoustic waves are also excited in the range $(-u_0, 0)$. The plateaus on the electron distribution function f are different in these regions since the waves are damped on ions located at zero (we recall that the ions are cold $u_{Ti} = 0$; when $u_{Ti} \neq 0$ the plateau regions are separated by a space $|u| < u_{Ti}$ in which there are no waves because of strong damping on the ions). Hence, there are two independent smoothing regions, each of width u_0, rather than one region of width $2u_0$; thus, the diffusion coefficient D is twice as large as that given in Eq. (3a):

$$D = u_0^2 \omega / \pi = \omega T / \pi M. \tag{3c}$$

The electron diffusion mechanism in the range $(-V_0, V_0)$ can be understood as follows. Let us consider an insulated rod of length $2V_0$ in which the thermal conductivity is infinitely large in some region u_0 ($u_0 < 2V_0$) and zero elsewhere. The temperature f is independent of distance v (measured from the center of the rod) in the region u_0 and is equal to c; the temperature distribution f is maintained in time $f = f(v)$ in the remaining parts of the rod. If the region u_0 with infinite thermal conductivity is displaced by a distance ΔV (Fig. 2), the region beyond remains continuous $f(v) = c$ while the temperature c is changed in such a way that the amount of heat Q in the region u_0 is conserved ($Q = \int_0^{u_0} f dV = \text{const}$; the specific heat of the rod is taken as unity). Under these conditions the temperature c in the region u_0 is described by Eq. (1).

If $u_0 \ll V_0$ the temperature f does not change appreciably in a single traversal of the region u_0 along the rod; in this case the slow equalization of the temperature f in the rod can be described by the "heat conductivity equation" (3a) and (3b) in which $\omega/2\pi = \varphi$ is the transit time of the rod through the region u_0 in the forward and reverse directions (the motion of the region in time can be arbitrary and can vary from one period φ to another; we consider the case $V(t) = V_0 \sin \omega t$). If the rod contains two regions of infinite thermal conductivity separated by some range in which the thermal conductivity is zero (Fig. 1) the "thermal-conductivity coefficient" V is given by Eq. (3c). Finally, the width of the smooth region u_0 can depend on a function $\{f\}$, for example

$$\{f\} = \int_{-V_0}^{V_0} \frac{mV^2}{2} f \, dV,$$

where m is a constant.

An exact solution of Eqs. (3a)—(3c) that takes account of the time variation of the temperature t and the diffusion coefficient D is given in the Appendix. We present the basic results (for the case in which the initial electron distribution function is Maxwellian).

In accordance with Eqs. (3a)—(3c) the distribution function f(v) is equalized in the range $(-V_0, V_0)$ (Fig. 3) as $\tau \to \infty$, where

$$f_\infty = \frac{1}{2V_0} \int_{-V_0}^{V_0} f_0 \, dv. \tag{4a}$$

The characteristic equilibration time t_D is

$$t_D \sim V_0^2/D \sim V_0^2/\omega u_0^2. \tag{4b}$$

The electron temperature is increased by the following amount by virtue of the work done by the external electric field:

$$\Delta T = \int_{-V_0}^{V_0} \frac{mv^2}{2} [f_\infty - f_0] \, dv \tag{4c}$$

(the function f is normalized to unity).

The electron temperature increment ΔT grows as the amplitude of the electron oscillations V_0 increases and, for a fixed frequency ω, is proportional to the amplitude of the external field E_0. If the amplitude of the electric field E_0 is such that V_0 is comparable with the electron thermal velocity $V_0 \sim V_T$ the change ΔT is comparable with T, that is to say, the electron temperature T increases by approximately a factor of two by virtue of the work done by the external electric field.

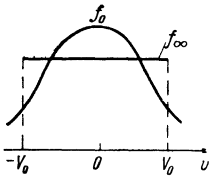

FIG. 3. Smoothing of distribution function in the interval $(-V_0, V_0)$ as $\tau \to \infty$.

It is of interest to compare the increase in energy density $(n\Delta T)$ with the Joule heat generated in the same time t_D. If the frequency of the external electric field ω is much larger than the collision frequency ν, the Joule heat generated per unit time is

$$Q \sim E_0^2 \omega \varepsilon'' \sim E_0^2 \nu \omega_0^2 / \omega^2,$$

since $\epsilon'' = \mathrm{Im}\, \epsilon = \mathrm{Im}\, [1 - \omega_0^2/\omega(\omega + i\nu)]$. When $V_0 \sim v_T$ the temperature change $\Delta T \sim T$ so that $n\Delta T/t_D Q \sim 1/t_D \nu$.

These estimates show that the Joule heating can be much smaller than the collisionless heating if the equilibration time t_D is much smaller than the collision time $1/\nu$, that is to say, if the following inequality is satisfied:

$$1 \ll \frac{m}{M} \frac{\omega}{\omega_0} N_D \frac{1}{\lambda},$$

where N_D is the number of particles in a Debye sphere and λ is the Coulomb logarithm ($\lambda \sim 10$).

We have assumed up to this point that the electrons acquire a periodic velocity with respect to the ions in the external electric field. However, the motion of the electrons with respect to the ions can also result from electric fields in a wave of arbitrary amplitude which might be formed in the plasma. Such a wave must be rapidly damped because of the excitation of ion-acoustic waves and because of electron heating. We use energy conservation to estimate the damping time t_E for such a wave:

$$E_0^2(\tau)/8\pi + nT(\tau) + \mathscr{E}(\tau) = \text{const.} \tag{5}$$

Here, E_0 is the amplitude of the electric field of the wave in which the electrons acquire a velocity with respect to the ions, T is the electron temperature, $\mathscr{E}$ is the energy of the ion-acoustic waves. Neglecting the change in the latter (an appropriate estimate is given in the Appendix) we have from Eq. (5)

$$t_E \sim -\frac{E_0^2}{8\pi} \Big/ \frac{d}{d\tau}\left(\frac{E_0^2}{8\pi}\right) \sim \frac{E_0^2/8\pi}{nT} \frac{T}{dT/d\tau} \sim \frac{E_0^2/8\pi}{nT} t_D.$$

We have shown above that the amplitude E_0 is limited by the inequality $V_0 = eE_0/m\omega < v_T$; when $V_0 \sim v_T$ the damping time is given by

$$t_E \sim \frac{\omega^2}{\omega_0^2} t_D \sim \frac{1}{\omega}\left(\frac{\omega}{\omega_{0i}}\right)^2. \qquad (6)$$

When $\omega \sim \omega_{0i}$ the wave is damped in a time of the order of one period $1/\omega$. This damping mechanism was invoked [3] in the interpretation of experiments carried out by Zavoĭskiĭ et al [4] (in which connection the present problem was studied). However, in these experiments the amplitude of the field E_0 was so large that the maximum electron velocity with respect to the ions V_0 was greater than the electron thermal velocity v_T. Under these conditions the waves excited in the plasma have growth rates comparable with the frequency, so that the damping time and increase in electron temperature given above can differ from the corresponding experimental quantities.

The author is indebted to A. A. Vedenov for suggesting this problem and for guidance and to M. A. Leontovich for valuable discussions.

APPENDIX

The derivation of Eq. (1) is based on the assumption that the plateau in the distribution function exists at any time in each diffusion region $(-u_0, 0)$ and $(0, u_0)$. This assumption is justified if there are always ion-acoustic waves with phase velocities u in these regions and if the wave energy is large enough so that the characteristic electron diffusion time $\langle t \rangle$ (for the establishment of the plateau) in velocity space is much smaller than the period of the external electric field.

We first investigate the way in which the energy of the ion acoustic-waves changes in motions of the plateau region. The equations of the quasi-linear theory, which describe the diffusion of electrons due to wave effects in the region $0 < u < u_0$, shown in Fig. 2, are of the form: [3]

$$\partial\varepsilon/\partial t = A\varepsilon\partial f/\partial u, \qquad \varepsilon = E^2/8\pi,$$
$$\partial f/\partial t = \partial(\mathcal{D}\partial f/\partial u)/\partial u, \qquad \mathcal{D} = B\varepsilon. \qquad (A.1)$$

The functions A and B in (A.1) depend only on the velocity u. If the ion-acoustic waves are characterized by the dispersion relation

$$\omega^2 = \omega_{0i}^2 k^2/(k^2 + D_e^{-2})$$

the functions A and B are given by

$$A = \pi\omega_{0e} u\,|u|\left[1 - \frac{u^2}{u_0^2}\right]^{1/2}, \qquad B = \frac{4\pi\,(e/m)^2}{|u|\,(1 - u^2/u_0^2)},$$
$$AB^{-1} = \left(\frac{m}{2e}\right)^2 \omega_{0e} u^3 \left(1 - \frac{u^2}{u_0^2}\right)^{3/2}. \qquad (A.2)$$

Here, ω_{0i} and ω_{0e} are the ion and electron plasma frequencies, D_e is the Debye radius, $u_0 = \sqrt{T/M}$ is the ion-acoustic velocity.

The equations in (A.1) have the integral

$$\frac{\partial}{\partial u} A^{-1} B \frac{\partial\varepsilon}{\partial t} - \frac{\partial f}{\partial t} = 0. \qquad (A.3)$$

We introduce below in place of the time t the variable $V = V_0 \sin\omega t$. Inasmuch as there is a plateau in the region $(0, u_0)$, $f(u, V) = c(V)$ (Fig. 2), after integration of (A.3) over velocity from 0 to u we have (taking account of the fact that $\epsilon_{u=0} = 0$ because of ion damping)

$$A^{-1}B\partial\varepsilon/\partial V = u\,dc/dV. \qquad (A.4)$$

Hence

$$\varepsilon - \varepsilon_0 = uAB^{-1}(c - c_0). \qquad (A.5)$$

The relation in (A.4) can also be obtained by integrating (A.3) over velocity between the limits $(u, u_0 + 0)$. Since $\epsilon_{u=u_0+0} = 0$, we have

$$A^{-1}B\frac{\partial\varepsilon}{\partial V} = -\int_u^{u_0+0} \frac{\partial f}{\partial V}\,du. \qquad (A.6)$$

The function f has a discontinuity at the point u_0:

$$f(u, V) = c + (f - c)\,\theta(u - u_0 + V(t) - v), \qquad (A.7)$$

where $\varphi(x) = 1$ when $x > 0$, $\varphi(x) = 0$ when $x < 0$; v is the ion coordinate in the coordinate system in which the electrons are at rest. Taking account of (1), from (A.6) and (A.7) we have

$$A^{-1}B\frac{\partial\varepsilon}{\partial V} = -\int_u^{u_0-0}\frac{\partial f}{\partial V}\,du - \int_{u_0-0}^{u_0+0}\frac{\partial f}{\partial V}\,du$$
$$= -(u_0 - u)\frac{dc}{dV} - (f - c) = u\frac{dc}{dV}.$$

For motion of the region $(0, u_0)$ in the direction of lower V in place of (A.5) we have

$$\varepsilon - \varepsilon_0 = (u - u_0)AB^{-1}(c - c_0). \qquad (A.8)$$

We introduce the notation

$$y = u_0|AB^{-1}|, \qquad z = |uAB^{-1}|. \qquad (A.9)$$

Then, (A.5) and (A.6) can be written in the form

$$\varepsilon - \varepsilon_0 = (c - c_0) \times \begin{cases} z \\ -(y - z), \end{cases} \qquad (A.10)$$

where the upper factor is written for motion of the region in the direction of increasing V while the lower is written for motion in the direction of decreasing V. For the region $(-u_0, 0)$ the factors in (A.10) must be taken locally. Now, using (A.10) we must find an equation describing the change in wave energy after long intervals of time in the same way as (3) is obtained from (1). For this purpose, using (A.10) we compute the change of energy $\Delta\epsilon$ in a time $\Delta\tau$ which extends over many periods of the external electric field (Fig. 4).

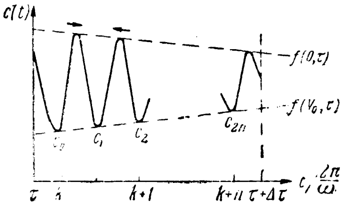

FIG. 4. Time dependence of the plateau level for the case of an even distribution function $f(v) = -f(v)$. The arrows indicate the direction of motion of the plateaus toward increasing v (the plateau level varies from c_0 to c_1) and in the opposite direction.

According to (A.10) and Fig. 4, in which we have shown the time dependence of the level of the plateau in the region $(0, u_0)$, the change of ϵ in one period is

$$(\Delta\varepsilon)_1 = z(c_1 - c_0) - (y - z)(c_2 - c_1) = (z - y/2)(c_2 - c_0),$$

while the change in n periods is

$$(\Delta\varepsilon)_n = (z - y/2)(c_{2n} - c_0).$$

Since the difference $\epsilon_\sim$ between the highest and lowest values of the energy of any wave is finite [from (A.10)] it is evident that $\epsilon_\sim = y[f(0, \tau) - f(V_0, \tau)]$, if n is high enough the changes of ϵ in the interval from τ to $2\pi k/\omega$ and from $2\pi(k + n)/\omega$ to $\tau + \Delta\tau$ can be neglected:

$$\frac{(\Delta\varepsilon)_n}{2\pi n/\omega} \to \frac{\partial\varepsilon}{\partial\tau}.$$

Substituting the expression for $(\Delta\epsilon)_n$ and taking account of the fact that

$$c_{2n} - c_0 = \Delta f(V_0, \tau)\left[1 + O\left(\frac{1}{n}\right)\right],$$

$$2\pi n/\omega = \Delta\tau\left[1 + O\left(\frac{1}{n}\right)\right],$$

we have

$$\frac{\partial\varepsilon}{\partial\tau} = \left(z - \frac{y}{2}\right)\frac{\partial f(V_0, \tau)}{\partial\tau}. \qquad \text{(A.11)}$$

According to (A.2) and (A.9) the quantities y and z are weak functions of the time τ since $u_0 = u_0(\tau) = \sqrt{T(\tau)/M}$. In carrying out an estimate we neglect this dependence. From (A.11) we have

$$\varepsilon(u, \tau) - \varepsilon(u, 0) = (z - y/2)[f(V_0, \tau) - f(V_0, 0)]. \qquad \text{(A.12)}$$

Using the last relation we can determine when the wave existence criterion $\epsilon(u, \tau) > 0$ is satisfied in the regions $0 < |u| < u_0$. Since the quantity $f(V_0, \tau) - f(V_0, 0)$ increases monotonically during the electron diffusion time, it follows from (A.9) and (A.12) that waves with phase velocity u in the range $0 < |u| < u_0/2$ will be damped. This condition imposes a limitation on the initial spectrum of excited waves $\epsilon(u, 0)$:

$$\varepsilon(u, \infty) = \varepsilon(u, 0) + (z - y/2)[f_\infty - f(V_0, 0)] > 0, \quad 0 < |u| < u_0/2. \qquad \text{(A.13)}$$

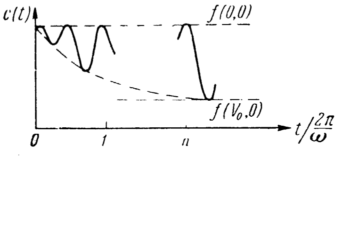

FIG. 5. Variation of the plateau level in the case of a gradual increase in the field amplitude to the stationary value.

The last condition on $\epsilon(u, 0)$ is satisfied, for example, when the external electric field grows to an amplitude E_0 in several periods as shown in Fig. 5 (the electron diffusion during the growth time is neglected).

Since this process is analogous to that shown in Fig. 4, we can apply (A.12) taking account of the notation used in Fig. 5 [at the beginning of the process, $f = f(0, 0)$, $\epsilon = 0$ and at the end $f = f(V_0, 0)$, $\epsilon = \epsilon(u, 0)$]. As a result we have

$$\varepsilon(u, 0) = \left(z - \frac{y}{2}\right) \times \begin{cases} f(V_0, 0) - f(0, 0), & 0 < |u| < u_0/2 \\ 0, & u_0/2 < |u| < u_0 \end{cases}, \qquad \text{(A.14)}$$

so that in the electron diffusion process, in accordance with (A.12),

$$\varepsilon(u, \tau) = \left(z - \frac{y}{2}\right) \times \begin{cases} f(V_0, \tau) - f(0, 0), & 0 < |u| < u_0/2 \\ f(V_0, \tau) - f(V_0, 0), & u_0/2 < |u| < u_0 \end{cases}. \qquad \text{(A.15)}$$

Using (A.1) and (A.2) we can write the second criterion for the existence of a plateau in the form

$$1/\omega \gg \langle t \rangle \sim u_0^2/\langle B \rangle \langle \varepsilon \rangle. \qquad \text{(A.16)}$$

For the initial distribution $\epsilon(u, 0)$ given in (A.14) the characteristic energy of the ion-acoustic waves $\langle\epsilon\rangle$ is taken to be

$$\langle\Delta\varepsilon\rangle \sim \langle y \rangle [f(0, 0) - f(V_0, 0)] \sim \langle y \rangle / v_T, \qquad \text{(A.17)}$$

which, taking account of (A.9) yields

$$1/\omega \gg v_T/\omega_{0e}u_0 \sim 1/\omega_{0i}.$$

If, however, $\langle\epsilon(u, 0)\rangle \gg \langle\Delta\epsilon\rangle$, in the estimate in (A.16) we must take $\langle\epsilon\rangle = \langle\epsilon(u, 0)\rangle$.

We now estimate the total change of wave energy. This is obtained by integrating the right side of (A.12) over k between the limits $(-\infty, \infty)$. Since y and z are functions of the phase velocity u it is convenient to convert to the integration variable u, which is expressed in terms of k by the dispersion relation $u^2 = \omega_{0i}^2/(k^2 + D_e^{-2})$. We have

$$\Delta\mathscr{E}(\tau) = -2\int_0^{u_0} \frac{\omega_{0i}\,du}{u^2\sqrt{1 - u^2/u_0^2}}\left(z - \frac{y}{2}\right)[f(V_0, \tau) - f(V_0, 0)]$$

$$\sim \frac{\omega_{0i}}{u_0}\langle\Delta\varepsilon\rangle \sim nT\left(\frac{m}{M}\right)^2.$$

We can now compare the energy changes in (5):

$$\Delta\left(\frac{E_0^2}{8\pi}\right):(n\Delta T):\Delta\mathscr{E}\sim\frac{E_0^2}{8\pi}:nT\,\frac{t_E}{t_D}:\Delta\mathscr{E}\sim 1:1:\left(\frac{m}{M}\,\frac{\omega_{0e}}{\omega}\right)^2. \tag{A.18}$$

In conclusion, we write the solution of (3) taking account of the time variation of the diffusion coefficient. Let us start with the equation

$$\partial f/\partial\tau = D\partial^2 f/\partial v^2 \tag{A.19}$$

with initial and boundary conditions in which D depends on the functional $\{f\}$ of a function to be determined so that it is an unknown function of time. The form of the functional $\{f\}$ and the function $D = D(\{f\})$ are known.

Replacing the time τ by the variable x

$$dx = D d\tau, \qquad x_{\tau=0} = 0, \tag{A.20}$$

we obtain from (A.19) an equation with exactly the same conditions, but for which $D = 1$. If it is possible to write the solution of the equation $f(x, v)$ in explicit form then $D(x) = D(\{f(x, v)\})$ is also known, and after substitution in (A.2) we have

$$\tau = \int_0^x \frac{dx'}{D(\{f(x, v)\})}. \tag{A.21}$$

The function to be determined $f(\tau, v)$ is found by eliminating the parameter x from (A.21) and the relation $f = f(x, v)$.

For the equations (3a)—(3c) we have

$$D = \frac{\omega}{\pi}\frac{T(\tau)}{M} = \frac{\omega}{\pi M}(T_0 + \{f\} - \{f_0\}),$$

$$\{f\} = \int_{-V_0}^{V_0}\frac{mv^2}{2} f\,dv, \qquad T_0 = T(0), \qquad f_0 = f(0, v),$$

$$f(x, V) = \sum_{n=0}^{\infty} f^{(n)} \exp\left\{-\frac{\pi^2}{V_0^2} n^2 x\right\}\cos n\pi\frac{v}{V_0},$$

$$f_0(v) = \sum_{n=0}^{\infty} f^{(n)} \cos n\pi \frac{v}{V_0},$$

$$\tau = \frac{\pi M}{\omega}\int_0^x\left[T_0 + \sum_{n=1}^{\infty} f^{(n)}\left(\exp\left\{-\left(\frac{\pi n}{V_0}\right)^2 x'\right\} - 1\right) m V_0^3 \frac{(-1)^n}{(n\pi)^2}\right]^{-1} dx'.$$

[1] Vedenov, Velikhov and Sagdeev, UFN **73**, 701 (1960), Soviet Phys. Uspekhi.

[2] Vedenov, Velikhov and Sagdeev, Nuclear Fusion **1**, 82, (1961).

[3] A. A. Vedenov, Atomnaya énergiya (Atomic energy) **13**, 5 (1962).

[4] Babykin, Gavrin, Zavoiskiĭ, Rudakov and Skoryupin JETP **43**, 411, 1962, Soviet Phys. JETP **16**, 295 (1963).

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