THEORY OF CYCLOTRON INSTABILITY IN A NON-UNIFORM PLASMA

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The stability of a non-uniform plasma in a uniform external magnetic field is investigated at frequencies that are multiples of the ion cyclotron frequency. It is shown that perturbations with wavelengths shorter than the ion Larmor radius may be unstable.

1. In the present work longitudinal oscillations ($\nabla \times \mathbf{E} \approx 0$) of a non-uniform plasma at low pressure ($\beta = 8\pi p/H_0^2 \ll 1$) are studied at frequencies that are multiples of the ion cyclotron frequency ($\omega \approx n\omega_{ci}$). Such oscillations may be unstable. The instability mechanism considered by us is similar to that which has been thoroughly investigated by a number of other authors [1-3] for the case of low frequencies ($\omega \ll \omega_{ci}$) and which they refer to as "drift" instabilities.

It is shown by us that oscillations at frequencies close to $\omega \approx n\omega_{ci}$ ($n \neq 0$) can also be unstable.

2. The dispersion equation for longitudinal oscillations of a low pressure non-uniform plasma has the form

$$1+\sum_{i,e}\frac{4\pi e^2 n_0}{k^2}\sum_{n=-\infty}^{\infty}\Big\{I_n(z)e^{-z}\Big[\frac{n\omega_c}{T}\frac{i\sqrt{\pi}W_n}{k_z v_T}+\frac{1}{T}(1+i\sqrt{\pi}x_n W_n)\Big]+\frac{1}{n_0}\frac{\partial}{\partial y}I_n(z)e^{-z}\frac{k_x n_0}{m\omega_c}\Big(1-\frac{n\omega}{z\omega_c}\Big)\frac{i\sqrt{\pi}W_n}{k_z v_T}\Big\}=0. \qquad (1)$$

Here $n_0 = n(y)$ is the plasma frequency, $T = T(y)$ is the temperature of each component of the plasma, $v_T = \sqrt{2T/m}$, $z = k_x^2\rho^2$, $\rho = \sqrt{T/m\omega_c^2}$ is the mean Larmor radius, $I_n(z)$ are the Bessel functions of imaginary argument, $\mathbf{k} = (k_x, 0, k_z)$ is the wave vector, m, e are mass and charge, $\omega_c = eH_0/mc$ is the cyclotron frequency of each kind of particle, $\mathbf{H}_0 = (0, 0, H_0)$ is the external magnetic field, and $W_n = W(x_n)$ is the probability integral with complex argument:

$$W(x)=e^{-x^2}\Big(1+\frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}dt\Big);\quad x_n=(\omega-n\omega_c)/k_z v_T.$$

The summation in (1) is carried out over the ions and electrons and over all integer n. It is assumed that the dependence of the perturbation on the time and space coordinates is chosen in the form

$$\exp(-i\omega t+ik_x x+ik_z z).$$

We shall henceforth assume that the temperature distribution of the electrons and ions is uniform so that $\partial T/\partial y = 0$. Generalization of the following results to the case $\partial T/\partial y \neq 0$ does not present any difficulty. The contributions of the second differential coefficients of density and temperature in (1) are not important in the subsequent discussions and have therefore been omitted.

Equation (1) is essentially a generalization of the dispersion equation in the paper by Rosenbluth, Krall, and Rostoker [6], for the case $k_z \neq 0$ (and also $\nabla T \neq 0$). If one sets $k_z = 0$ and $\nabla T = 0$ in (1), then the resultant equation is not identical with that of [6] and differs from it by terms which are not important for the aims of the work previously referred to [6] as well as for our further considerations. Evidently the terms indicated above were omitted in [6], although this fact was not discussed.

3. We consider first the growth of the cyclotron harmonics produced by the ions. We assume for simplicity that $T_e = 0$ ($T_i \neq 0$) and that the wave is propagated transverse to the magnetic field. For $\omega \approx n\omega_{ci}$ and $z_i \gg 1$, we obtain from (1)

$$1+k^2\Big(d_i^2+\frac{m_e}{m_i}\rho_i^2\Big)-\frac{kv_{dr}^i}{\omega}=\frac{1}{\sqrt{2\pi z_i}}\frac{\omega-kv_{dr}^i}{\omega-n\omega_{ci}};$$
$$d_i^2=T_i/4\pi e^2 n_0,\qquad v_{dr}^i=-\varkappa T_i/m_i\omega_{ci},\qquad \varkappa=n_0^{-1}dn_0/dy. \qquad (2)$$

If ω is not too close to $n\omega_{ci}$ the right-hand side of (2) can be neglected. We then obtain for the drift wave:

$$\omega^+=kv_{dr}^i/[1+k^2(d_i^2+m_e\rho_i^2/m_i)]. \qquad (3)$$

If the frequency ω^+ is small compared to the cyclotron frequency, then the term containing v_{dr}^i can be neglected in (2) and then from the dispersion equation we obtain the cyclotron branch of a uniform plasma [5]

$$\omega\approx n\omega_{ci}. \qquad (4)$$

Thus Eq. (2) describes the "intersection" of two branches, a drift branch and a cyclotron branch with number n. The condition for such an "intersection" has the form

$$\varkappa\rho_i \geqslant 2n\frac{m_e}{m_i}\left(1+\frac{\omega_{ce}^2}{\omega_{0e}^2}\right)^{1/2}, \qquad \omega_0^2=\frac{4\pi e^2 n_0}{m}. \tag{5}$$

The wave numbers corresponding to the "intersection" are roughly equal to:

$$k \approx k_n = \tfrac{1}{2}(d_i^2 + m_e\rho_i^2/m_i)^{-1}\{v_{\mathrm{dr}}^i/n\omega_{ci} \pm [(v_{\mathrm{dr}}^i/n\omega_{ci})^2 - 4(d_i^2+m_e\rho_i^2/m_i)]^{1/2}\}. \tag{6}$$

At the intersection point of the branches, the oscillation frequency is complex if

$$\frac{\varkappa\rho_i}{\sqrt{\pi}\,n}\left[2-\frac{k_n v_{\mathrm{dr}}^i}{n\omega_{ci}}+\left(\frac{k_n v_{\mathrm{dr}}^i}{n\omega_{ci}}\right)^2\right] > \left[\frac{k_n v_{\mathrm{dr}}^i}{n\omega_{ci}}-1\right]\left[2-\frac{k_n v_{\mathrm{dr}}^i}{n\omega_{ci}}\right]^2. \tag{7}$$

A sufficient condition for instability is determined by Eq. (5), in which n = 1. For denser plasmas ($\omega_{0e}^2 \gg \omega_{ce}^2$) it has the form:

$$\varkappa\rho_i \geqslant 2(m_e/m_i)^{1/2}. \tag{8}$$

A rarefied plasma ($\omega_{0e}^2 \ll \omega_{ce}^2$) can be unstable only if the inhomogeneity is even greater:

$$\varkappa\rho_i \geqslant 2v_A/c, \qquad v_A^2 = H_0^2/4\pi n_0 m_i. \tag{9}$$

Even when the gradients exceed the critical values in (8) or (9) the increments of the growing waves have an order of magnitude

$$\gamma = \mathrm{Im}\,\omega \sim (m_e/m_i)^{1/4}\omega_{ci}, \tag{10}$$

and the frequency intervals $\Delta\mathrm{Re}\,\omega$ where $\gamma \neq 0$ are

$$\Delta\mathrm{Re}\,\omega \sim (m_e/m_i)^{1/2}\omega_{ci}. \tag{11}$$

4. We now consider the case of waves propagated at an angle to the magnetic field ($k_z = 0$). If it is assumed that $|\omega - n\omega_{ci}| \gg k_z v_{Ti}$ and that $|\omega - n\omega_{ci}| \ll \omega_{ci}$, then the dispersion equation (1) can be rewritten in the following form:

$$1+\frac{4\pi e^2 n_0}{k^2}\left\{\frac{1}{T_e}+\frac{1}{T_i}+\frac{i\sqrt{\pi}W(\omega/k_z v_{Te})}{T_e k_z v_{Te}}(\omega - k_x v_{\mathrm{dr}}^e)\times I_0(z_e)e^{-z_e} - \frac{I_n(z_i)e^{-z_i}}{T_i(\omega-n\omega_{ci})}(\omega - k_x v_{\mathrm{dr}}^i)\right\}=0. \tag{12}$$

In order to derive (12) we also assume that $\omega_{ce}/k_z v_{Te} \ll 1$.

Solving (12) for $\Delta = \omega - \omega_{ci}$ we obtain

$$\Delta = \frac{n\omega_{ci}-k_x v_{\mathrm{dr}}^i}{\sqrt{2\pi z_i}}\left[1+\frac{T_i}{T_e}+k^2 d_i^2+\frac{i\sqrt{\pi}}{k_z v_{Te}}\times W\left(\frac{\omega}{k_z v_{Te}}\right)I_0(z_e)e^{-z_e}(n\omega_{ci}-k_x v_{\mathrm{dr}}^e)\right]^{-1}. \tag{13}$$

The condition $\Delta \ll \omega_{ci}$ is satisfied only if $z_i \gg 1$, and therefore we have replaced $I_n(z_i)$ in (13) by its asymptotic approximation.

From (13) it is evident that if the wave proceeds in the direction of the electron drift, i.e., $k_x v_{\mathrm{dr}}^e/\omega > 0$, then for its growth it is necessary that $k_x v_{\mathrm{dr}}^e/\omega > 1$; a wave traveling in the direction of the ion drift grows if $k_x v_{\mathrm{dr}}^i/\omega > 1$. These conditions may be conveniently written out in the following form:

$$\begin{aligned}\varkappa\rho_i &> n(m_e T_i/m_i T_e)^{1/2}/\sqrt{z_e}, & k_x v_{\mathrm{dr}}^e/\omega &> 0,\\ \varkappa\rho_i &> n(m_e T_e/m_i T_i)^{1/2}/\sqrt{z_e}, & k_x v_{\mathrm{dr}}^i/\omega &> 0.\end{aligned} \tag{14}$$

Thus, in contrast to the instability considered in [2], where it is assumed that $\omega \ll \omega_{ci}$ and where it has been shown that only waves propagating in the direction of the electron drifts are unstable, in our case (for $|\omega - n\omega_{ci}| \ll \omega_{ci}$) waves are unstable which propagate in the direction of both the electron and ion drifts.

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