THEORY OF PHOTON SCATTERING IN THE COULOMB FIELD OF A NUCLEUS AT HIGH FREQUENCIES

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The elastic scattering of photons in the Coulomb field of a nucleus at high frequencies $\omega \gg m$ (m = mass of the electron, ω = frequency of the incident photon) and large scattering angles $\theta \gg m/\omega$ is investigated by the dispersion relations method. The principal contribution to the cross section in this angle range is from the real part of the scattering amplitude (whereas for small-angle scattering the main contribution is from the imaginary part of the amplitude). An expression is found for the differential cross section of scattering into angles $\theta \gg m'/\omega$.

1. Elastic scattering of a photon in the Coulomb field of a nucleus at high frequencies has been considered only at zero angles [1] and at small angles.[2,3] 1) This process was considered in the research of Akhiezer and Pomeranchuk [3] for scattering angles $m/\omega \ll \theta \ll 1$ (m = mass of the electron, ω = frequency of the incident photon). In the work of Bethe and Rohrlich,[2] this process was studied in the angle range $\theta < m/\omega$. As has been made clear, the principal contribution in this angle range is made by the imaginary part of the scattering amplitude. The assumption has then been made by Zernik [5] that the imaginary part of the amplitude also remains larger than the real part for large angle scattering $\theta > m/\omega$. In that work, on the basis of the assumption introduced by Kessler, [6] the imaginary part of the scattering amplitude was calculated by numerical methods for particular values of the frequencies and angles of the scattering.

In the present work, the elastic scattering of a photon at large frequencies $\omega \gg m$ is studied by means of the method of dispersion relations in their invariant form.[7] This method makes it possible to find an analytic expression for the cross section of the scattering process under consideration for angles $\theta \gg m/\omega$.

First of all, an expression is found from the unitarity condition for the imaginary part of the scattering amplitude. Furthermore, the real part of the amplitude is found from the dispersion relations by means of the imaginary part already at hand (only the principal terms are kept here). We have reduced the problem of finding the asymptotes of the amplitudes at large momentum transfer $t \gg 4m^2$ (t = square of the momentum transferred to the nucleus) to the somewhat equivalent problem of the determination of the asymptote of the amplitude as $m \to 0$. Here, only those terms are taken into account which have the strongest singularity as $m \to 0$. In the angle range under consideration, the principal contribution is made by the real part of the scattering amplitude.

An expression is found for the differential cross section of elastic scattering of a photon, and the contribution of the region of large angles to the total cross section is estimated.

2. In an analysis of scattering processes by means of the dispersion relations method, it is convenient to use the scattering amplitude A which in the case of the scattering of a photon in the nuclear Coulomb field is connected with the matrix element M by the relation

$$M\delta(\omega_1-\omega_3) = \frac{Z^2e^6}{16\pi^2\omega}\int \frac{d^3q_1d^3q_2}{q_1^2q_2^2} A\delta^4(k_1+\mathbf{q}_1-k_3-\mathbf{q}_2), \tag{1}$$

where Ze is the charge of the nucleus, ω_1 and ω_3 are the frequencies of the photon before and after the scattering (here $\omega_1 = \omega_3 = \omega$), k_i is the 4-momentum of the photon, and $\mathbf{q}_1$ and $\mathbf{q}_2$ are the nucleus recoil momenta.

The total amplitude A can be represented in the form

$$A = A_1 + A_2 + A_3 + A_{1e} + A_{2e} + A_{3e}, \tag{2}$$

where the partial amplitudes A_1, A_2, A_3 corre-

1) Elastic scattering of a photon at an arbitrary angle for frequencies $\omega \sim m$ has been studied in the work of Eftimiu and Vrejoiu[4].

spond to the following scattering channels:

$$(k_1, e_1) + \mathbf{q}_1 \to (k_3, e_3) + \mathbf{q}_2, \qquad (3.\mathrm{I})$$
$$(-k_3, e_3) + \mathbf{q}_1 \to (-k_1, e_1) + \mathbf{q}_2, \qquad (3.\mathrm{II})$$
$$\mathbf{q}_1 \to (k_3, e_3) + (-k_1, e_1) + \mathbf{q}_2 \qquad (3.\mathrm{III})$$

(e_1 are the polarization vectors of the photon). The partial amplitudes A_{1e}, A_{2e}, A_{3e} take into account the exchange channels in the variables (k_3, e_3) and $\mathbf{q}_2$.

The scattering amplitude in the first channel A_1 is associated with the matrix element of the operator T entering into the definition of the scattering matrix $S = 1 + iT$, by the relation

$$\langle k_3, e_3 \mid T \mid k_1, e_1 \rangle = \frac{Z^2e^6}{16\pi^2\omega} \int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2} A_1 \delta^4(k_1 + \mathbf{q}_1 - k_3 - \mathbf{q}_2). \qquad (4)$$

To obtain the imaginary part of the amplitude A_1 we use the unitarity condition for the operator T:

$$T - T^+ = iT^+T, \qquad (5)$$

where T^+ is the Hermitian conjugate operator relative to the operator T. In order to obtain an expression for Im A_1 in the first non-vanishing approximation of perturbation theory (or order Z^2e^6), one must keep as intermediate states only the states of the free electron-positron pair. As a result we get

$$-\frac{i}{2}\langle k_3, e_3 | T - T^+ | k_1, e_1 \rangle$$
$$= \frac{Z^2e^6}{16\pi^2\omega} \int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2} \operatorname{Im} A_1 \delta^4 (k_1 + \mathbf{q}_1 - k_3 - \mathbf{q}_2)$$
$$= \pi^2 \frac{Z^2e^6}{\omega} \int \frac{d^3q_1 d^3q_2}{q_1^2 q_2^2} \sum_{\substack{p_1,\ -p_2,\ s_1,\ s_2 \\ (p_1+p_2=k_1+\mathbf{q}_1=k_3+\mathbf{q}_2)}} \frac{m^2}{p_1^0 p_2^0}$$
$$\times (A^{(1)+}_{e^+e^-\to\gamma} A^{(1)}_{\gamma\to e^+e^-} + A^{(2)+}_{e^+e^-\to\gamma} A^{(2)}_{\gamma\to e^+e^-}), \qquad (6)$$

where p_1, p_2 and $\mathbf{s}_1$, $\mathbf{s}_2$ are the momenta and the polarizations of the electron and the positron, while $A^{(i)}_{\gamma\to e^+e^-}$ and $A^{(i)+}_{e^+e^-\to\gamma}$ are the amplitudes of pair production by a photon in the nuclear Coulomb field and single photon annihilation of a pair, which are functions of p_1, p_2 and s_1, s_2.

The expressions for these amplitudes are well known:

$$A^{(1)}_{\gamma\to e^+e^-} = i\bar{u}(p_1)\hat{e}_1 \frac{i(\hat{p}_1 - \hat{k}_1) - m}{-2p_1k_1} \gamma_4 v(-p_2),$$
$$A^{(2)}_{\gamma\to e^+e^-} = i\bar{u}(p_1)\gamma_4 \frac{i(\hat{p}_1 - \hat{k}_1) - m}{-2p_1k_1} \hat{e}_1 v(-p_2),$$

$$A^{(1)+}_{e^+e^-\to\gamma} = -i\bar{v}(-p_2)\gamma_4 \frac{i(\hat{p}_1 - \hat{k}_3) - m}{-2p_1k_3} \hat{e}_3 u(p_1),$$
$$A^{(2)+}_{e^+e^-\to\gamma} = -i\bar{v}(-p_2)\hat{e}_3 \frac{i(\hat{p}_1 - \hat{k}_3) - m}{-2p_1k_3} \gamma_4 u(p_1), \qquad (7)$$

where $\hat{p} = p_\mu\gamma_\mu$, γ_μ are the Dirac matrices (the invariant normalization has been chosen for the spinors u and v).

In this approximation the imaginary part of the scattering amplitude in the first channel is represented by the Feynmann diagram shown in Fig. 1 (the dashed electron lines indicate free particles). It is convenient to characterize these diagrams by the scalar invariants

$$s = -(k_1 + \mathbf{q}_1)^2, \qquad t = (k_1 - k_3)^2,$$
$$u = (k_3 - \mathbf{q}_1)^2,$$

which are connected among themselves by virtue of the conservation law $k_1 + \mathbf{q}_1 = k_3 + \mathbf{q}_2$ by the relation $s + t + u = q_1^2 + q_2^2$.

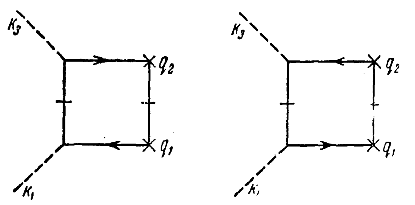

FIG. 1

The property of crossing symmetry (3) makes it possible to obtain an expression for the scattering amplitude in the second channel from the expression for the amplitude in the first channel by means of the following substitutions:

$$A_1 \to A_2 \text{ for } (k_1, e_1) \leftrightarrow (-k_3, e_3),\ s \leftrightarrow -u,\ t \to t. \qquad (8)$$

Along with channel I, one must also consider the channels I_e and III separately. One can then obtain expressions for the amplitudes A_{3e} and A_{2e} from the expressions for the amplitudes in these channels by means of the substitutions

$$A_{1e} \to A_{3e}, \qquad A_3 \to A_{2e}$$
$$(k_1, e_1) \leftrightarrow (-k_3, e_3), \qquad (8')$$
$$s \leftrightarrow -u,\ t \to t.$$

We note that in channels I, II and I_e, III_e, t has the meaning of the square of the momentum $(\mathbf{q}_1 - \mathbf{q}_2)^2$ transferred to the nucleus.

The kinematics of the elastic scattering of the photon in the Coulomb field of the nucleus is shown in Fig. 2.

The imaginary parts of the amplitude A_1 and A_{1e}, as will be seen below, are different from zero for $s \geq 4m^2$, while Im A_2, Im A_{3e} and Im A_3, Im A_{2e} for $u \leq -4m^2$ and $t \leq -4m^2$. Under the condition that the momenta of the external photon lines lie on definite energy surfaces, i.e., for $k_i^2 = 0$ and definite values of q_1^2 and $q_2^2 = -s + t + u - q_1^2$ (considered as the squares of certain external

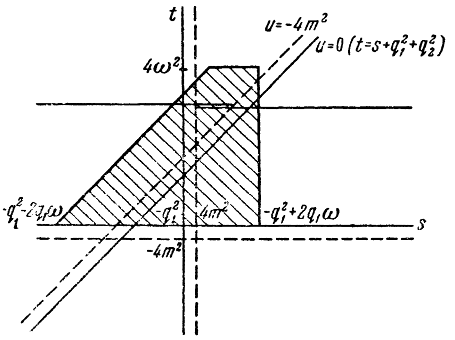

FIG. 2

imaginary masses), the physical regions in channels I, II, I_e, III_e and III, II_e are the half-planes $t \geq 0$ and $t \leq 0$. For a given frequency of the incident phonon ω and transfer momentum q_1^2 the region of change of the variables s, t, and u is the region shaded in Fig. 2.

We now write the explicit expression for Im A_1:

$$\operatorname{Im} A_1 = \operatorname{Im} T^{(1)}_{\mu\sigma} e_{1\mu} e_{3\sigma}$$

$$= \frac{1}{4\pi^2} \int d^4v\delta\,(kv)\,\delta\,(v^2 - s + 4m^2)\,\frac{S}{(1)\,(2)}\,, \tag{9}$$

$$S = \mathrm{Sp}\left(\frac{i}{2}\,(\hat{k} + \hat{v}) - m\right) \gamma_\mu \left(\frac{i}{2}\,(\hat{v} - \hat{p}) - m\right)$$

$$\times \gamma_4 \left(\frac{i}{2}\,(\hat{k} - \hat{v}) + m\right) \gamma_4 \left(\frac{i}{2}\,(\hat{v} - \hat{p}') - m\right) \gamma_\sigma e_{1\mu} e_{3\sigma}, \tag{9'}$$

where

$$v = p_1 - p_2, \quad k = k_1 + \mathbf{q}_1, \quad p = k_1 - \mathbf{q}_1,$$

$$p' = k_3 - \mathbf{q}_2 = 2k_3 - k_1 - \mathbf{q}_1 \quad (k^2 = -s, \quad p^2 = s + 2q_1^2,$$

$$p'^2 = s + 2q_2^2,\ pp' = s - 2t + q_1^2 + q_2^2,$$

$$pk = -q_1^2, \qquad p'k = -q_2^2),$$

$$(1) = vp - s - q_1^2, \quad (2) = vp' - s - q_2^2,$$

$$(3) = q_1^2 + q_2^2 - 2\omega^2 - 4i\omega v_4 + 2v_4v_4.$$

3. From the condition of gauge invariance of the matrix element, it follows that the dependence of M(1) on the scalar products of the polarization vectors e_i and the wave vectors k_i should be of the form

$$M = \frac{Z^2e^6}{16\pi^2\omega} f\left(\delta_{\mu\sigma} + \frac{2}{t}\,k_{3\mu}k_{1\sigma}\right) e_{1\mu}e_{3\sigma}, \tag{10}$$

where f is a scalar function which depends on the invariant t and perhaps on the frequency ω. Therefore, in order to determine the entire matrix element it suffices to know only the coefficient for $\delta_{\mu\sigma}$ in the matrix element. Our further problem consists of finding the expression for this coefficient.

First of all, the function f, and also the amplitude A, can be represented in the form $f = f_1 + f_2 + f_3 + f_{1e} + f_{2e} + f_{3e}$ where, for example, f_2 denotes the contribution of channel II. We now compute the contribution of the first channel, i.e., f_1. For $t \gg 4m^2$, the contribution to f_1 from the trace of (9′) can give only the terms written down below:

$$S \to \delta_{\mu\sigma}\Big[-\tfrac{1}{2}\,(1)\,(2) - \tfrac{1}{2}\,(1)\,(s + q_2^2 - 2\omega^2)$$

$$-\tfrac{1}{2}\,(2)\,(s + q_1^2 - 2\omega^2) - t\,(s - \omega^2)$$

$$+ i\omega v_4\,((1) + (2)) + tv_4v_4\Big] - q_{1\sigma}q_{1\mu}\,((1) + (2) + (3))$$

$$- v_\mu q_{1\sigma}\,((2) + (3)) - v_\sigma q_{1\mu}\,((1) + (3)) - v_\mu v_\sigma(3). \tag{11}$$

Now, substituting these expressions in (9) and integrating over v (see Appendix), we get for that part Im $T^{(1)}_{\mu\sigma}$ which gives us the contribution to the coefficient f_1

$$\operatorname{Im} T^{(1)}_{\mu\sigma} \to \tfrac{1}{4}\,\pi^{-2}\,(s \operatorname{Im} a_1\delta_{\mu\sigma} + \operatorname{Im} b_1 q_{1\mu}q_{1\sigma}); \tag{12}$$

$$\operatorname{Im} a_1 \approx -(J_1 + tJ_2), \qquad \operatorname{Im} b_1 \approx 0. \tag{12'}$$

[We retain here only those terms which give the maximum contribution to the real part of the scattering amplitude under the assumption $m \to 0$ and the same principal contribution to the matrix element (10).]

Thus, Im A_1 has in the approximation under consideration the form

$$\operatorname{Im} A_1 \approx -\tfrac{1}{4}\,\pi^{-2} s\,(J_1 + tJ_2)\,(e_1e_3 + 2t^{-1}(k_1e_3)\,(k_3e_1)). \tag{13}$$

From the dispersion relations it is now possible to find an expression for the real part of a_1. As $s \to \infty$, the value of Im a_1 falls off as $s^{-1} \ln s$; therefore, one can use the dispersion relations without subtraction in the determination of Re a_1. In the first scattering channel the dispersion relations have the form (for given values of q_1^2 and q_2^2)

$$\operatorname{Re} a_1 = \frac{\mathrm{P}}{\pi} \int_{4m^2}^{\infty} \frac{\operatorname{Im} a_1(s', t)}{s' - s}\, ds'. \tag{14}$$

Substituting in (14) the expression for Im a_1 (12′), we get for Re a_1 (for $t \gg 4m^2$)

$$\operatorname{Re} a_1 \approx \tfrac{1}{2}\, s^{-1} \ln^2 (t/m^2). \tag{15}$$

(Actually, only those terms are kept which have as $m \to 0$ a singularity, in the given case a doubly logarithmic one.) Thus, we have the following expression for Re A_1:

$$\operatorname{Re} A_1 \approx \tfrac{1}{8}\,\pi^{-2} \ln^2 (t/m^2)\,(e_1e_3 + 2t^{-1}\,(k_1e_3)\,(k_3e_1)). \tag{16}$$

The expression for the real part of the scattering amplitude in the second channel can be obtained from (16) by means of the substitution (8).

Furthermore, by means of direct calculations, we can establish that for $t \gg 4m^2$ the channels I_e and III_e give a smaller contribution (singly logarithmic) in comparison with channels I and II; therefore, we shall not consider these channels nor take their contribution into account.

It is also evident from the calculations that the principal terms of the imaginary part of the amplitude as $m \to 0$ are the singly logarithmic terms. We carry out the following integration of $\operatorname{Im} A_1$ over q_1 in the limits established by $\theta(s-4m^2)$, that is, when

$$-1 \leqslant \cos(\mathbf{q}_1, \mathbf{k}_1) \leqslant -(q_1^2+4m^2)/2q_1\omega \quad \text{and}$$

$$2m^2/\omega \leqslant q_1 \leqslant 2(\omega^2-m^2)/\omega,$$

leads only to singly logarithmic terms of the type $\ln(t/m^2)$. Therefore, in the approximation $t \gg 4m^2$, we also neglect the contribution of the imaginary part of the scattering amplitude. We then get for the total amplitude A

$$A \approx \tfrac{1}{4}\pi^{-2}(e_1e_3 + 2t^{-1}(k_1e_3)(k_3e_1))\ln^2(t/m^2). \tag{17}$$

It is now easy to find the expression for the coefficient f in (10). By a comparison of (10) with (1), it follows that

$$f = \frac{1}{4\pi^2}\int\frac{d^3q_1}{q_1^2(t-s+u-q_1^2)}\ln^2\frac{t}{m^2} \tag{18}$$

or

$$f = \frac{I}{2\pi\sqrt{t}}\ln^2\frac{t}{m^2}; \qquad I = \int_0^\infty\frac{dx}{x}\ln\left|\frac{1+x}{1-x}\right| = \frac{\pi^2}{2}. \tag{18$'$}$$

Finally, we have the expression for the matrix element M(10):

$$M \approx \frac{Z^2e^6}{2^6\pi\omega}\frac{1}{\sqrt{t}}\Big(e_1e_3+\frac{2}{t}(k_1e_3)(k_3e_1)\Big)\ln^2\frac{t}{m^2}. \tag{19}$$

4. The differential cross section of elastic scattering of the photon in the nuclear Coulomb field is related to the matrix element M by the expression

$$d\sigma = (2\pi)^{-4}\omega^2|M|^2\,do. \tag{20}$$

Substituting in (20) the expression (19) for M, we get the following for the differential cross section:

$$d\sigma \approx \frac{Z^4\alpha^6}{2^5\omega^2}\frac{do}{1-\cos\theta}\left(\mathbf{e}_1\mathbf{e}_3+\frac{(\mathbf{n}_1\mathbf{e}_3)(\mathbf{n}_3\mathbf{e}_1)}{1-\cos\theta}\right)^2\ln^4\frac{2\omega^2(1-\cos\theta)}{m^2}, \tag{21}$$

where θ is the scattering angle and $\mathbf{n}_1$, $\mathbf{n}_3$ are the unit vectors along the directions of propagation of photons before and after scattering. The cross section, averaged and summed over the polarizations of the photons in the initial and final states, has the form

$$d\sigma \approx \frac{Z^4\alpha^6}{2^5\omega^2}\frac{do}{1-\cos\theta}\ln^4\frac{2\omega^2(1-\cos\theta)}{m^2}. \tag{22}$$

Equations (21), (22) are applicable for $\omega \gg m$, $\theta \gg m/\omega$. We emphasize that in the region of angles under consideration ($\theta \gg m/\omega$) the principal contribution to the cross section is made by the real part of the scattering amplitude. The contribution of the imaginary part can be neglected in this case. (For small angle scattering $\theta < m/\omega$, the principal contribution to the cross section is made by the imaginary part of the scattering amplitude, as was shown in [2].)

For $\theta \ll 1$, Eq. (22) takes the form

$$d\sigma \approx \frac{Z^4\alpha^6}{\omega^2}\frac{do}{\theta^2}\ln^4\frac{\omega\theta}{m}. \tag{23}$$

If this formula is extended to the angles $\theta \sim m/\omega$, then we get the expression found in the work of Akhiezer and Pomeranchuk: [3]

$$d\sigma = a\frac{Z^2\alpha^6}{\omega^2}\frac{do}{\theta^2},$$

where a is some numerical constant.

Integrating (22) over θ from m/ω to π, we get the contribution of the region of the large angles to the total scattering cross section:

$$\sigma \approx \frac{2}{5}\pi\frac{Z^4\alpha^6}{\omega^2}\ln^5\frac{\omega}{m}. \tag{24}$$

This expression takes on a maximum value for a photon frequency $\omega \sim me^{5/2}$, i.e., on the boundary of the region of applicability of Eq. (24). For these values of frequencies, the contribution of the region of large angles is comparable with the contribution of the region of smaller angles and for $Z \sim 40$ it is of the order of magnitude $\sigma \sim 10^{-28}\ \text{cm}^2$.

In conclusion, I express my deep gratitude to Professor A. I. Akhiezer for his advice and discussions.

APPENDIX

In carrying out integration over v in (9), the following integrals are encountered:

$$J_1^{(1)} = \int d^4v\,\frac{\delta(kv)\,\delta(v^2-s+4m^2)}{(1)} = \theta(s-4m^2)\frac{s}{s+q_1^2}J_1,$$

$$J_1^{(2)} = \int d^4v\,\frac{\delta(kv)\,\delta(v^2-s+4m^2)}{(2)} = \theta(s-4m^2)\frac{s}{s+q_2^2}J_1,$$

$$J_2 = \int d^4v\,\frac{\delta(kv)\,\delta(v^2-s+4m^2)}{(1)(2)}$$

$$= \theta(s-4m^2)\frac{\pi}{2str_2}\ln\frac{2r_1(r_1+r_2)-\gamma\alpha}{2r_1(r_1-r_2)-\gamma\alpha}\,\frac{2r_1(r_1+r_2)-\gamma\alpha'}{2r_1(r_1-r_2)-\gamma\alpha'}, \tag{A.1}$$

where

$$J_1 = -\frac{\pi}{s} \ln \frac{1+r_1}{|1-r_1|},$$

$$r_1 = \sqrt{1-\frac{4m^2}{s}}, \quad r_2 = \sqrt{r_1^2 + \gamma\eta}, \quad \eta = \frac{(s+q_1^2)(s+q_2^2)}{s^2},$$

$$\gamma = \frac{4m^2}{t}, \quad \alpha = \frac{(q_1^2-q_2^2)(s+q_1^2)}{s^2}, \quad \alpha' = \frac{(q_2^2-q_1^2)(s+q_2^2)}{s^2}.$$

The largest contribution to the real part of the coefficient f_1 is made by those terms of Im a_1 which are proportional to $1/s$ as $s \to 0$ (these terms lead to doubly logarithmic expressions). In the computation of the integrals given below, only those terms are kept. We have

$$\int \frac{v_\mu \delta(kv)\,\delta(v^2 - s + 4m^2)}{(1)(2)} d^4v \approx -J_2 k_\mu,$$

$$\int \frac{v_\mu v_\sigma \delta(kv)\,\delta(v^2 - s + 4m^2)}{(1)(2)} d^4v \approx J_2 k_\mu k_\sigma,$$

$$\int \frac{v_\mu v_\sigma v_\nu \delta(kv)\,\delta(v^2 - s + 4m^2)}{(1)(2)} d^4v \approx -J_2 k_\mu k_\sigma k_\nu,$$

$$\int \frac{v_\mu v_\sigma v_\nu v_\rho \delta(kv)\,\delta(v^2 - s + 4m^2)}{(1)(2)} d^4v \approx J_2 k_\mu k_\sigma k_\nu k_\rho. \quad \text{(A.2)}$$

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