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REGGE POLES AND ASYMPTOTIC BEHAVIOR OF AMPLITUDES IN PERTURBATION THEORY

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For the case of scattering by a Yukawa potential a connection is established between the high momentum transfer behavior of the amplitude in low orders of perturbation theory and the position of the Regge poles. Use of higher order approximations makes it possible to show that perturbation theory is in agreement with the existence of only moving poles. The same approach is then applied to a study of relativistic field theory. It turns out that the simple picture, analogous to potential scattering, now does not agree with perturbation theory. As a most likely solution of this problem it is suggested that poles accumulate near negative integer points (and possibly near zero) because of the existence of many channels in field theory.

In a previous paper of Shekhter and the authors [1] the motion of Regge poles [2] was studied in the scattering of particles by a Yukawa potential in the case of a small coupling constant. In the present work we study the relation between these poles and the asymptotic behavior of the amplitude in perturbation theory. The clarification of this relation is of interest, in the first place, because the sum of the contributions of individual Regge poles represents a very substantial rearrangement of the perturbation theory series.

It is this very reason that makes the study of the scattering amplitude with the help of complex orbital angular momenta attractive. On the other hand, in the case of field theory, which is after all the case of principal interest, the Regge trajectories cannot be determined in the same way as in quantum mechanics.[1] This is related to the fact that in field theory one cannot write an equation for the pole trajectory in terms of the radial wave function for a definite orbital angular momentum. To obtain in this case information about the motion of the pole it is most convenient to make use of the asymptotic behavior in momentum transfer of the amplitude in perturbation theory.

In Sec. 1 we discuss the perturbation theory for the scattering of particles by a Yukawa potential. The asymptotic behavior of the amplitudes in low orders turns out to be related to the motion of the poles near their limit positions. Already the third order approximation makes it possible to verify the agreement of perturbation theory with the existence of only simple moving poles.

In Sec. 2 we attempt to study in the same way the relativistic field theory. It then turns out that the simple picture of Regge poles, analogous to the case of potential scattering, is not in agreement with perturbation theory. However aside from the obvious possibility that the expansion in the coupling constant is not valid or that there exist non-Regge-like singularities there is another way out related to the existence of many channels in field theory. An infinite number of channels could give rise to an accumulation of an infinite number of poles near negative integer points for small coupling constants and infinite energies. Such a supposition somewhat unexpectedly gets tangled with the recent result of Gribov and Pomeranchuk[3] on the accumulation of poles in a relativistic field theory near negative integers due to the existence of the crossed channel. In that case, for the poles connected with all thresholds it is reasonable to assume a motion similar to the motion of the poles for scattering by a Yukawa potential. In the presence of accumulating poles perturbation theory becomes ineffective for the study of individual Regge terms even for weak coupling.

1. SCATTERING BY A YUKAWA POTENTIAL

The expression for an individual Regge pole term in the partial wave amplitude has the form

$$f_l = \alpha r_i(\alpha, k^2)/[l - l_i(\alpha, k^2)], \tag{1}$$

which gives a contribution to the total amplitude equal to

$$A(k^2, z) = \alpha r_i(\alpha, k^2)(2l_i + 1) Q_{-l_i-1}(-z)/\cos \pi l_i. \qquad (2)$$

Here α, k^2 and z are the coupling constant, the square of the particle momentum and the cosine of the scattering angle. The function $l_i(\alpha, k^2)$ describes the trajectory of the i-th pole. The residue at that pole is defined in the form $\alpha r_i(\alpha, k^2)$, since for $\alpha = 0$ the amplitude should vanish. The contribution to the total amplitude is represented, according to Mandelstam,[4] in terms of the Legendre functions of the second kind. Such an expression is valid both to the right and to the left of the line $\mathrm{Re}\, l = -1/2$.

In going to perturbation theory one expands Eqs. (1) and (2) in a power series in α. For the partial wave amplitude in n-th order of perturbation theory one obtains, obviously, a sum of poles of different multiplicities—from simple poles up to n-tuple poles. All these poles lie at the point $l = l_i(0, k^2)$. The leading term resulting from them in the asymptotic behavior of the total amplitude consists of the product of $(-z)^{l_i(0,k^2)}$ and a polynomial of degree $n - 1$ in $\ln(-z)$. [A k-tuple pole in the partial wave amplitude gives rise to all the powers of $\ln(-z)$ from the $(k - 1)$-st to the zeroth.] The same result is, of course, also obtained by direct expansion of the right side of Eq. (2).

The n-th order of perturbation theory is determined by the values of the derivatives of $r_i(\alpha, k^2)$ and $l_i(\alpha, k^2)$ with respect to α evaluated at $\alpha = 0$, up to the $(n - 1)$-st derivative. Therefore in every successive order two new parameters appear—the highest derivatives of r_i and l_i. These parameters enter only into the coefficients of the simple and double poles in the expansion of f_l and, consequently, into the coefficients of the first and zeroth power of $\ln(-z)$ in the total amplitude.

The coefficients of the poles of higher multiplicity [higher powers of $\ln(-z)$] are determined by lower orders of perturbation theory. For example the coefficient of the highest power of the logarithm is proportional to $\alpha r_i(0, k^2)[l_i'(0, k^2)]^n/n!$. Thus a comparison of the expansion of Eqs. (1) and (2) in powers of α with the amplitude calculated from perturbation theory makes it possible to confirm already in third order (coefficient of the logarithm squared) the agreement of perturbation theory with the existence of only moving poles. It is interesting to note that the resultant structure has on the whole the character of a double expansion in α and $\alpha \ln(-z)$, similar to the one that occurs in field theory (see, for example, the article by L. D. Landau in [5]).

The amplitude for the scattering of a particle of mass m by a Yukawa potential $V(r) = \alpha e^{-\mu r}/r$ is given in first order of perturbation theory by

$$A^{(1)}(k^2, z) = -2\alpha m / [2k^2(1 - z) + \mu^2]. \qquad (3)$$

For $|z + \sqrt{z^2 - 1}| > |z_0 + \sqrt{z_0^2 - 1}|$ ($z_0 = 1 + \mu^2/2k^2$) the amplitude $A^{(1)}$ may be represented in the form

$$A^{(1)}(k^2, z) = \alpha m k^{-2} \sum_{i=0}^{\infty} P_i(z_0)\, Q_i(z)\,(2i + 1). \qquad (4)$$

Comparing Eqs. (4) and (2) we find in first order poles at negative integer points

$$l_i(0, k^2) = -i - 1 \qquad (i = 0, 1, \ldots) \qquad (5)$$

with residues

$$r_i(0, k^2) = -mk^{-2} P_i(1 + \mu^2/2k^2). \qquad (6)$$

The same result is of course obtained also by direct evaluation of the partial wave amplitude.

The second order amplitude may be expressed as a dispersion integral or evaluated exactly:

$$A^{(2)}(k^2, z) = \frac{\alpha^2 m^2}{2\pi} \int_0^{\infty} \frac{dk'^2}{k'^2 - k^2} \frac{1}{k'^3 \sqrt{K(z', z_0', z_0')}} \ln \frac{z_0'^2 - z' + \sqrt{K(z', z_0', z_0')}}{z_0'^2 - z' - \sqrt{K(z', z_0', z_0')}}$$

$$= \frac{-2\alpha^2 m^2}{\sqrt{(-q^2)(\mu^4 + 4\mu^2 k^2 + k^2 q^2)}} \ln \left[\frac{q^2 + 4\mu^2}{\mu^2} \times \frac{2k^2 q^2 + \mu^2(\mu^2 + 4k^2) + 2\sqrt{q^2 k^2 (\mu^4 + 4\mu^2 k^2 + k^2 q^2)}}{-q^2(\mu^2 - 4k^2) + 4\mu^2(\mu^2 + 4k^2) + 4\sqrt{-\mu^2 k^2 (\mu^4 + 4\mu^2 k^2 + k^2 q^2)}} \right]; \qquad (7)$$

$$q^2 = 2k^2(1 - z), \qquad z' = 1 - q^2/2k'^2, \qquad z_0' = 1 + \mu^2/2k'^2,$$

$$K(z_1, z_2, z) = z_1^2 + z_2^2 + z^2 - 2z_1 z_2 z - 1. \qquad (8)$$

For large z the expression (7) becomes, as could be expected, a sum of negative integer powers of z multiplied by expressions linear in the logarithm. A study of the terms not containing logarithms gives corrections to the residues with which we will not concern ourselves in what follows. To determine the corrections to the position of the poles one must express the coefficient of the logarithm in the form of a sum of Legendre functions of the second kind. This is easy to do by making use of the easily proved identity

$$\frac{1}{\sqrt{K(z_1, z_2, z)}} = \sum_{i=0}^{\infty} (2i + 1) P_i(z_1) P_i(z_2) Q_i(z),$$

$$z_2 > z z_1 + \sqrt{(1 - z^2)(1 - z_1^2)}. \qquad (9)$$

Then, by comparing the asymptotic behavior of Eq. (7) with the second term in the expansion of Eq. (2) in powers of α and making use of Eq. (6) for the residue, one can obtain the following trajectory $l_i(\alpha, k^2)$ accurate to terms linear in α:

$$l_i(\alpha, k^2) = -i - 1 - \frac{\alpha m}{\sqrt{-k^2}} P_i\left(1 + \frac{\mu^2}{2k^2}\right). \quad (10)$$

Thus even in this approximation the pole becomes a moving pole.

The expression (10) coincides with the result of the previous paper[1] [Eq. (7)] for the motion of the poles near negative integer points. This means that the asymptotic behavior of the amplitude in perturbation theory reflects the motion of the poles near the points to which these poles tend in the limit $\alpha \to 0$ or $k^2 \to -\infty$. The segments of the trajectory $l_i(\alpha, k^2)$ corresponding to small k^2 are at that excluded since they cannot be expanded in a power series in the coupling constant.[1]

In the oscillatory motion (10) ($k^2 < -\mu^2/4$) the pole passes several times through the negative integer point. As can be seen from Eq. (6) its residue vanishes at those instants. This is due to the behavior of the potential at small distances ($\sim 1/r$) (private communication from V. N. Gribov).

It is easy to show that for a superposition of Yukawa potentials

$$V(r) = \sum_s \alpha_s e^{-\mu_s r}/r$$

the residue of the i-th pole takes the form

$$-\frac{m}{k^2} \sum_s \alpha_s P_i\left(1 + \frac{\mu_s^2}{2k^2}\right), \quad (6a)$$

and the trajectory is determined by the equation

$$l_i(\alpha_s, k^2) = -i - 1 - \frac{m}{\sqrt{-k^2}} \sum_s \alpha_s P_i\left(1 + \frac{\mu_s^2}{2k^2}\right). \quad (10a)$$

A more detailed study of the motion of the poles in the whole energy interval has been given previously.[1]

As was explained above the agreement of perturbation theory with the existence of only moving poles can be ascertained by looking at the coefficient of the logarithm squared term in the third order amplitude. In Appendix I this amplitude is obtained in the form of a double integral:

$$A^{(3)}(k^2, z) = -4\alpha^3 m^3 \int_{9\mu^2}^{\infty} \frac{dt'}{t' + q^2} \int_{4\mu^2}^{\mu(\sqrt{t'}+\mu)} dt'' \frac{1}{\sqrt{\sigma(k^2, t', t'')\,\sigma(k^2, \mu^2, t'')}}; \quad (11)$$

$$\sigma(k^2, t', t'') = t't''\mu^2 + 2k^2(t't'' + t'\mu^2 + t''\mu^2) - k^2(t'^2 + t''^2 + \mu^4). \quad (12)$$

A detailed study of the asymptotic behavior of this amplitude turns out to be rather involved. However the main terms corresponding to each pole [$\sim \ln^2(-z)$] may be calculated rather simply. Such a calculation is given in Appendix II. It gives for the i-th pole the result

$$\tfrac{1}{2}(-1)^i \alpha^3 m^3 k^{-4} (2i + 1) P_i^3(z_0)\, Q_i(-z) \ln^2(-2z). \quad (13)$$

This expression agrees with the main term [$\sim \ln^2(-z)$] obtained in the expansion of Eq. (2) accurate to terms of order α^3:

$$\frac{\alpha^3}{2!} r_i(0, k^2)(2l_i^0 + 1) \frac{Q_{-l_i^0-1}(-z)}{\cos \pi l_i^0} [l_i'(0, k^2) \ln(-2z)]^2 \quad (14)$$

[$l_i^0 = l_i(0, k^2)$]. In place of $r_i(0, k^2)$, l_i^0 and $l_i'(0, k^2)$ in Eq. (14) one should substitute the corresponding expressions (6), (5) and (10).

It thus turns out to be possible to verify the "Regge-like" nature of the theory by studying the asymptotic behavior of the amplitude in perturbation theory.

2. REGGE POLES IN FIELD THEORY

Let us attempt to apply the method outlined in the previous Section to the study of the motion of the poles in field theory. For simplicity we consider scattering of pions by pions. Let the pion interaction be described by a $\lambda\varphi^4$ term in the Lagrangian. Below we shall take the $\pi\pi$ interaction into account in terms of nucleon loops.

The lowest order of perturbation theory the scattering amplitude is equal to λ. This means that in lowest order in the coupling constant there is only one pole lying at the point $l = 0$. It is clear that the contribution of that pole to the total amplitude would be infinite [Eq. (2)] if its residue did not vanish. The condition that this contribution be equal to λ leads to the requirement

$$-r_0(\lambda, k^2)/l_0(\lambda, k^2)|_{\lambda\to 0} = 1. \quad (15)$$

In the next approximation of perturbation theory three diagrams arise—the simplest loops, each depending on one of the three Mandelstam invariants s, t and u. In what follows t denotes the energy and s the momentum transfer, i.e. the energy in the crossed channel. In order to determine the correction to the pole trajectory one must extract the terms containing ln s. The contribution of the pole located at zero has the form $-(\lambda/\pi)\ln s$, which together with Eq. (15) gives the values $l_0(\lambda, t) = -\lambda/\pi$, $r_0(\lambda, t) = \lambda/\pi$,

The asymptotic expansion of the amplitude in this order contains not only a constant (and a constant multiplied by a logarithm) but also all negative integer powers of s. This means that at all negative integer points there appear poles, and because of the presence of $\ln s$ in all terms at each point there appear simultaneously simple and double poles. This behavior is substantially different from the simple picture that arose in the case of potential scattering.

As was already explained in the previous Section in third order one may verify the agreement between perturbation theory and the hypothesis of moving poles. Direct computation of the coefficient of the $\ln^2 s$ term in third order shows that the result does not coincide with the expected value. Thus here too there is a substantial difference as compared with scattering by a potential.

The question arises whether the indicated disagreements could not be eliminated by taking into account explicitly the interaction of pions via nucleon pairs. Since this interaction contains a new parameter—the pion-nucleon coupling constant—one can by an appropriate choice of the relation between the two coupling constants move the difficulty with the pole at zero over into fourth order of perturbation theory. It appears completely improbable that there should thereafter exist in fourth and all higher orders of perturbation theory the necessary relation between the coefficients of all higher powers of the logarithm. The second difficulty, connected with the poles at negative integer points, only becomes deeper with the introduction of nucleon loops. Indeed, in lowest order of perturbation theory for the πN interaction (nucleon box) there also appear simultaneously simple and double poles, but with coefficients that depend on t. Therefore they certainly cannot compensate the double poles due to the interaction $\lambda\varphi^4$.

Analogous difficulties arise for other forms of interactions, for example the scattering of nucleons by nucleons. The scattering of pions by nucleons and quantum electrodynamics we did not study.

Aside from the obvious ways out of this situation, consisting in the inapplicability of the expansion in the coupling constant or the presence of non-Regge-like singularities, there exists another possibility. It could be that in the limit as the coupling constant tends to zero more than one pole tends to each negative integer point.[1)] If, for example, two poles tend to each point then the number of parameters that determine the asymptotic behavior of the amplitude is doubled. Therefore, as is easy to see, the agreement between perturbation theory and the existence of only moving poles can now be verified only starting with the fifth order. For a larger number of coincident poles the agreement can be established only starting with yet higher orders.

Precisely this situation occurs in the nonrelativistic theory in the presence of several two-particle channels. In that case the scattering matrix can be expressed in the form of the ratio of two matrices in such a way that the poles in angular momentum are determined by the zeros of the determinant of the denominator matrix.[6] The number of rows and columns of this determinant is equal to the number of channels n. The vanishing of the determinant is possible due to the vanishing of any one of the eigenvalues of the corresponding matrix. If the interaction in every channel has the form of a Yukawa potential then the poles corresponding to the vanishing of each of the eigenvalues tend to the negative integer points in the limit of weak coupling or $k^2 \to -\infty$, analogously to the case of single channel scattering. Consequently n poles tend to each negative integer point.

In field theory the number of channels is infinite, with a majority of them being multi-particle channels. It has been shown by Gribov and Pomeranchuk[7] that to each threshold in field theory, including multi-particle thresholds, there corresponds an accumulation of poles at a certain point in the l-plane at the threshold energy. It follows from [1] that for scattering by a Yukawa potential the poles of Gribov and Pomeranchuk are the same as the poles that for infinite negative energies and infinitely weak coupling go to the negative integer points. In field theory the poles accumulating at any threshold also should, apparently, move to the negative integer points when the interaction tends to zero. This is supported by the fact that the asymptotic expansion of the Feynman diagrams contains only negative integer powers of the invariants. Then to each negative integer point there moves in field theory an infinite number of poles corresponding to the infinite number of channels.

The above outlined proposal is in an amazing correspondence to the recent result of Gribov and Pomeranchuk,[3] according to which there is at any energy an infinite number of poles in the neighborhood of each negative integer, with these poles tending to the negative integers precisely for infinite negative energy. These poles exist only in field theory and are related to the existence of the crossed reactions. From our point of view these accumulations consist of poles corresponding to different thresholds in the sense described above. At that the existence of an infinite number of poles in the neighborhood of the negative integer points at any finite energy appears completely natural,

[1)]The authors are grateful to I. T. Dyatlov who called their attention to this possibility.

since there always exists an infinite number of thresholds lying arbitrarily higher than the given energy. If the above indicated proposal on the nature of these poles is correct then we are dealing here, apparently, with the only known case of a phenomenon due to the crossed reactions being simultaneously closely related to the higher intermediate states of the direct reaction. Therefore a proof of the correctness of this proposal could have most fundamental consequences.

All of the above considerations apply to poles lying in the vicinity of the negative integer points. However, from the results obtained at the beginning of this section it follows that definite difficulties are also due to the presence of the poles in the neighborhood of $l = 0$. For that case we have no additional arguments that would indicate an accumulation of poles near zero. If nevertheless such an accumulation does take place then one should think that the residues of these poles tend to zero as the poles approach zero. It is obvious that the existence of these poles would lead to extremely important experimental consequences. Indeed, for scattering with sufficiently large momentum transfer the accumulation of the poles near zero would be the nearest singularity, which would determine the asymptotic behavior of the amplitude in energy (provided, of course, that the vacuum pole does not move to the right of zero).

In conclusion we would like to express our gratitude to V. N. Gribov, who awakened our interest in the questions here discussed, repeatedly discussed various aspects of this work and made a number of valuable comments. We are also grateful to I. T. Dyatlov, G. S. Danilov, I. Ya. Pomeranchuk, K. A. Ter-Martirosyan, V. M. Shekhter, and I. M. Shmushkevich for useful discussions.

APPENDIX I

The unitarity condition gives for the absorptive part in k^2 of the third order amplitude the following integral:

$$A_1^{(3)}(k^2, z) = \frac{k}{4\pi}\int d\Omega' \,[A^{(2)}(k^2, z_1')\, A^{(1)*}(k^2, z_2') + A^{(1)}(k^2, z_1')\, A^{(2)*}(k^2, z_2')]. \tag{A.1}$$

The first and second order amplitudes $A^{(1)}$ and $A^{(2)}$ are given by Eqs. (3) and (7) of the text. For $k^2 > 0$ the amplitude $A^{(2)}(k^2, z)$ satisfies a dispersion relation in the angle with a simple absorptive part:

$$A^{(2)}(k^2, z_1') = \frac{\alpha^2 m^2}{k^3}\int_{z_1}^{\infty}\frac{dz''}{z'' - z_1'}\frac{1}{\sqrt{-K(z'', z_0, z_0)}},$$
$$z_0 = 1 + \frac{\mu^2}{2k^2}, \qquad z_1 = 1 + \frac{2\mu^2}{k^2}. \tag{A.2}$$

After substitution of Eq. (A.2) into Eq. (A.1) the integration over $d\Omega'$ can be carried out. The result of that integration is most conveniently expressed in terms of a dispersion integral over z; thereafter the expression for $A_1^{(3)}$ is reduced to the form

$$A_1^{(3)}(k^2, z) = -\frac{\alpha^3 m^3}{k^6}\int_{z_1}\frac{dz''}{\sqrt{-K(z'', z_0, z_0)}}\int_{\gamma}\frac{dz'}{z' - z}\frac{1}{\sqrt{K(z', z_0, z_0)}}. \tag{A.3}$$

The integration is over a region where both of the expressions under the square root sign are positive:

$$\gamma = z_0 z'' + \sqrt{(z_0^2 - 1)(z''^2 - 1)}. \tag{A.4}$$

Let us introduce in place of the angle variables z, z' and z'' the momentum transfers t, t' and t'' [$t = -2k^2(1 - z)$ etc.] Then after interchanging the order of integration in t' and t'', $A_1^{(3)}$ may be written as a dispersion integral in t:

$$A_1^{(3)}(k^2, z) = -4\alpha^3 m^3\int_{t_0}^{\infty}\frac{dt'}{t' - t}\int_{4\mu^2}\frac{dt''}{\sqrt{\sigma(k^2, t'', \mu^2)}\sqrt{-\sigma(k^2, t'', t')}}. \tag{A.5}$$

The quantities σ are defined by Eq. (12) in the text. The integration over t'' is over the region where both of the expressions under the square root sign are positive:

$$t_0 = 9\mu^2 + 4\mu^2\,[\sqrt{(1 + \mu^2/k^2)(1 + \mu^2/4k^2)} - (1 - \mu^2/2k^2)]. \tag{A.6}$$

In order to obtain the amplitude $A^{(3)}$ one must substitute $A_1^{(3)}$ into the dispersion integral in energy. Afterwards the order of integration must be changed so that the integration over energy (k'^2) is carried out first. The limits of integration are determined by the zeros of the quadratic polynomial in k'^2 that stands under the square root sign. Therefore the integral is easily evaluated and afterwards the most complicated problem turns out to be the determination of the upper limit of the integration over t''. When that problem is solved one arrives at the expression (11) in the text ($t = -q^2$).

APPENDIX II

Expression (11) of the text may be rewritten in terms of the variables already used in Appendix I namely z, z' and z'' ($z = 1 + t/2k^2$ etc.). Then

$$A^{(3)}(k^2, z) = \frac{2\alpha^3 m^3}{k^4} \int_{z_2}^{\infty} \frac{dz'}{z'-z} \int_{z_1}^{\beta} \frac{dz''}{\sqrt{K(z'', z_0, z_0)}\sqrt{K(z'', z', z_0)}},$$

$$z_2 = 1 + 9\mu^2/2k^2, \qquad \beta = z_0 + \sqrt{(z_0-1)(z'-1)}\ . \quad \text{(A.7)}$$

If one makes use for both square roots of the identity (9) then $A^{(3)}(k^2, z)$ is brought to the form

$$A^{(3)}(k^2,z) = \frac{2\alpha^3 m^3}{k^4} \sum_{n,s=0}^{\infty} (2n+1)(2s+1)P_n^2(z_0)$$

$$\times P_s(z_0) \int_{z_2}^{\infty} \frac{dz'}{z'-z} Q_s(z') \int_{z_1}^{\beta} dz'' Q_n(z'') P_s(z''). \quad \text{(A.8)}$$

In order that the asymptotic form of $A^{(3)}(k^2, z)$ involve terms containing $\ln^2 z$ it is necessary that the integration over z'' give rise to the appearance of $\ln\beta$. Making use of the explicit expansion of Q_n and P_s in powers of z'' one can transform the coefficient of the $\ln\beta$ term into

$$\frac{1}{2\Gamma(k+1)} \sum_{m=0}^{k} (-1)^m C_k^m \frac{\Gamma(s-m+{}^1/_2)}{\Gamma(s-k-m+{}^3/_2)}. \quad \text{(A.9)}$$

Here C_k^m are the binomial coefficients, $k = (s-n)/2$ and the summation over s and n must be performed in such a way as to keep k a nonnegative integer. The ratio of the gamma functions is equal to a polynomial of degree $(k-1)$ in the quantity $s - k + {}^3/_2$. For $k \neq 0$ all the coefficients of this polynomial vanish as a consequence of the identity

$$\sum_{m=0}^{k} (-1)^m C_k^m m^l = 0,\ l = 0, 1, \ldots, k-1. \quad \text{(A.10)}$$

Consequently the coefficient of $\ln\beta$ is equal to $\delta_{sn}/(2s+1)$. Asymptotically $\ln\beta \approx ({}^1/_2)\ln z'$. After that there arise integrals of the form

$$\int_{z_2}^{\infty} \frac{dz'}{z'-z} \frac{\ln z'}{z'^p}.$$

Such an integral leads to the appearance of the term $-({}^1/_2) z^{-p} \ln^2 z$. Taking this into account we obtain expression (13) of the text for the coefficient of the logarithm squared term.

Note added in proof (January 19, 1963). 1. After the present paper work went to press there appeared a large number of papers dealing with the relation between perturbation theory and Regge poles. The first to come to our attention were the papers of the Dubna authors.[8,9] In[9] the trajectories of the first two Regge poles for a Yukawa potential are computed in perturbation theory. This is done using a method analogous to the method of the present paper. In[8] the Regge asymptotic behavior is obtained for a relativistic field theory by the method of the renormalization group (the $g\varphi^3$ and $h\varphi^4$ interactions). However the results of this article do not seem to us to be sufficiently well established since they are based on a study of the asymptotic behavior of the amplitude in a variable in which the renormalization group does not improve on perturbation theory.

2. Recently one of the authors[10] has proposed arguments, not connected with perturbation theory, in favor of accumulation of poles near zero. Since however these considerations are based on the interaction of pions with particles with spin it is not quite clear how these considerations might affect the properties of a pure $\lambda\varphi^4$ interaction.

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