SOVIET PHYSICS JETP VOLUME 17, NUMBER 2 AUGUST, 1963

TURBULENCE THEORY OF A WEAKLY NONEQUILIBRIUM LOW-DENSITY PLASMA AND STRUCTURE OF SHOCK WAVES

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Submitted to JETP editor July 24, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 592—602 (February, 1963)

A kinetic equation has been derived for wave interaction in a weakly nonequilibrium low-density plasma. The relation between the laminar and turbulence theories of collisionless shock waves in a strong magnetic field is discussed. The kinetic equation is used to estimate the turbulent width of shock waves.

1. INTRODUCTION

IT is well known that collective oscillations can be excited in a low-density plasma which is near equilibrium and that these oscillations can have an important effect on relaxation phenomena in the plasma The initial stage in the development of these oscillations can be treated as an instability growing from the stationary state, in which case the appropriate growth rates are determined from the linear theory of stability. [1]

Obviously the waves arising as a result of the instability cannot continue to grow indefinitely. The ultimate state is determined by nonlinear effects deriving from the interaction between oscillation modes. It is important, in this regard, to note that the dispersion properties of different modes can be determined completely by the linearized equations of motion while the nonlinear effects can be described in terms of a weak (for low amplitudes) interaction between modes. For this reason one can think of the system as a weakly nonideal gas of modes similar to the phonon gas in the quantum theory of solids. [2] As in the theory of phonons, it may be assumed that a weak interaction between different modes (for a large number of modes) leads to a rapid randomization of the phases of the individual modes and to a slow change in mode amplitude (cf. for example, [3]) so that a kinetic equation can be written for the modes.

In the present work we present a general method for the derivation of this kinetic equation from the hydrodynamic plasma equations under the assumption that the lowest-order elementary process is the interaction of three modes. In principle, this method can be applied with different assumptions as to the equations of motion of the plasma.

As a concrete example (in connection with certain applications to the theory of shock waves in a low-density plasma in a strong magnetic field) the basic equations of motion are taken to be the magnetohydrodynamic equations for a low-density gas in a strong magnetic field ($H^2/8\pi \gg p$) in which account is taken of dispersion effects that arise at frequencies comparable with or greater than the ion Larmor frequency $\omega_H = eH/m_ic$.

These equations are (cf. for example, [4,5])

$$\partial \mathbf{V}/\partial t + (\mathbf{V}\nabla)\,\mathbf{V} = (4\pi\rho)^{-1}\,[\operatorname{rot}\mathbf{H},\ \mathbf{H}], \tag{1.1}$$ *

$$\partial \mathbf{H}/\partial t = \operatorname{rot}\,[\mathbf{V}\mathbf{H}] - (m_i c/4\pi e)\operatorname{rot}\rho^{-1}\,[\operatorname{rot}\mathbf{H},\ \mathbf{H}], \tag{1.2}$$

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{V}) = 0, \qquad \operatorname{div}\mathbf{H} = 0. \tag{1.3}$$

The equations in (1.1)—(1.3) differ from the usual magnetodydrodynamic equations in the appearance in (1.2) of a term that becomes small when $\omega \ll \omega_H$. This term describes the characteristic dispersion effects at $\omega \gtrsim \omega_H$ so that the system (1.1)—(1.3) can be called the "magnetohydrodynamic equations with ion dispersion."

The kinetic equation obtained in Sec. 2 of the present work is found to have the same structure as the kinetic equation proposed on the basis of phenomenological considerations by Camac et al, [6] which, however, contains no explicit matrix elements or a method for obtaining them.

In Secs. 4 and 5 we consider certain applications of the kinetic equation for interacting modes in the structure of the shock front in a low-density plasma in a strong magnetic field. Sagdeev [7] has shown that within the framework of the laminar theory a shock front in a low-density plasma propagating perpendicularly or nearly perpendicularly to the

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*rot = curl, $[\mathbf{vH}] = \mathbf{v}\times\mathbf{H}$, $(\mathbf{V}\nabla) = (\mathbf{V}\cdot\nabla)$.

magnetic field exhibits an oscillatory structure. [1] This result has been extended in [9] to the case of an arbitrary angle between the shock front and the magnetic field. The question arises as to the stability of this oscillatory structure with respect to decay into two waves the sum of frequencies and wave vectors of which are equal respectively to the frequency and wave vector of the original wave (these "decay" instabilities of periodic waves have been described earlier).[10]

The matrix elements appearing in the kinetic equation are found to be closely connected with the growth rates for the decay instability mentioned above; thus it is possible to evaluate easily the instability of the oscillatory structure against decay into two waves (cf. also[5]) and to obtain the corresponding growth rates.

Using the kinetic equation for interacting modes we examine the nature of the turbulence arising from decay and make estimates of the width of the shock wave.

We note, finally, that the general properties of the matrix elements allow us to set important limitations on the classes of waves that can exhibit instability against decay. The appropriate results are given in Sec. 3.

2. KINETIC EQUATION FOR MODES

The system of equations in (1.1)—(1.3) can be written in the following form for small deviations from the equilibrium values:

$$i\partial\varphi/\partial t + \hat{H}_0\varphi + \hat{H}_1\{\varphi, \varphi\} = 0, \tag{2.1}$$

where φ is the state vector, represented in the form of a column consisting of the components of the velocity **v**, the perturbation of the magnetic field **h**, and the density ρ; $\hat{H}_0$ is a linear operator with real eigenvalues; $\hat{H}_1$ is a bilinear differential operator. The operator $\hat{H}_0$ can be written in the form of a matrix whose elements are differential operators while $\hat{H}_1\{\varphi, \varphi\}$ can be written in the form of a column vector.

For low-amplitude waves we neglect the nonlinear term and obtain the equation

$$i\partial\varphi/\partial t + \hat{H}_0\varphi = 0. \tag{2.2}$$

This equation has characteristic solutions of the form $\varphi_{\mathbf{k}}(\mathbf{r}, t) = \varphi_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{r})}$ where the frequency $\omega_{\mathbf{k}}$ is real by virtue of the properties of the operator $\hat{H}_0$. The vector φ can be expanded in eigenvectors of the operator $\hat{H}_0$:

$$\varphi = \sum_{\mathbf{k}} (C_{\mathbf{k}}^{(0)}\varphi_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{kr})} + C_{\mathbf{k}_-}^{(0)}\varphi_{\mathbf{k}_-}e^{i(\omega_{\mathbf{k}}t-\mathbf{kr})});$$
$$C_{\mathbf{k}_-}^{(0)} = C_{\mathbf{k}}^{(0)*}, \quad \varphi_{\mathbf{k}_-} = \varphi_{\mathbf{k}}^*, \tag{2.3}$$

where $C_{\mathbf{k}}^{(0)}$ and $C_{\mathbf{k}_-}^{(0)}$ are the complex amplitudes of the modes with wave vectors **k** and − **k** and frequencies $\omega_{\mathbf{k}}$ and $-\omega_{\mathbf{k}}$ when the interaction between modes is neglected.

The following postulate plays a basic role in the derivation of the kinetic equation: the phases of the various modes $\alpha_{\mathbf{k}} = \arg C_{\mathbf{k}}^{(0)}$ are distributed completely randomly. This assertion is to be understood in the following sense. It is assumed that the modes (2.3) are excited as a result of some instability (below we consider the concrete case of the so-called decay instability, which has been investigated in detail earlier by Oraevskiĭ and Sagdeev [10]). For some time interval after the instability develops the phases of the different $C_{\mathbf{k}}^{(0)}$ are obviously correlated. However, the nonlinear mode interaction described by the second term in (2.1) weakens the correlation. This weakening of the correlation occurs more rapidly the higher the mode that appears as a result of the development of the instability. [2]

The random phase approximation means that the correlation between the phases $C_{\mathbf{k}}$ vanishes completely in a time small compared with the time required for a change in $|C_{\mathbf{k}}|^2$ (i.e., the energy of individual modes) owing to the nonlinear interaction between modes. For this reason we can average over the phases in the derivation of the kinetic equation (in the sense of averaging over an ensemble of the aggregate of phases of the different $C_{\mathbf{k}}$):

$$\overline{C_{\mathbf{k}}^{(0)}C_{\mathbf{k}'}^{(0)}} = |C_{\mathbf{k}}^{(0)}|^2\delta_{\mathbf{k}',\mathbf{k}_-}. \tag{2.4}$$

We divide the plasma into a slowly varying background and a rapidly oscillating part; the latter represents the propagating waves in the plasma. The energy density of these waves is

$$\mathcal{E} = \sum_{\mathbf{k}} E_{\mathbf{k}}, \quad E_{\mathbf{k}} = |C_{\mathbf{k}}|^2\{\rho_0|\mathbf{v}_{\mathbf{k}}|^2/2 + |\mathbf{h}_{\mathbf{k}}|^2/8\pi\}. \tag{2.5}$$

Normalizing the state vector $\varphi_{\mathbf{k}}$ by means of the condition

$$\rho_0|\mathbf{v}_{\mathbf{k}}|^2/2 + |\mathbf{h}_{\mathbf{k}}|^2/8\pi = \omega_{\mathbf{k}}, \tag{2.6}$$

we can interpret the square of the modulus of the wave amplitude $n_{\mathbf{k}} = |C_{\mathbf{k}}|^2$ as the number of quasiparticles with energy $\omega_{\mathbf{k}}$.

It is clear that the total energy and momentum

[1]The laminar theory of shock waves in a collisionless plasma has also been treated by a number of other authors.[8]

[2]However, the amplitudes of these modes must obviously be small so that perturbation theory can be applied.

of the background and quasiparticles are conserved. Furthermore, it can be shown in the semiclassical approximation that if the background varies slowly the adiabatic invariant $n_k = E_k/\omega_k$ (the oscillator energy divided by frequency) is conserved for each quasiparticle [6] i.e., the total derivative

$$D\,(E_k/\omega_k)\,/Dt \equiv Dn_k/Dt = 0. \tag{2.7}$$

The change in the number of quasiparticles n_k in a given state due to collisions between quasiparticles is described by the nonlinear term in (2.1) $H_1\{\varphi, \varphi\}$. Perturbation theory can be used if this nonlinear term is small. In the expansion in (2.3) we now add to the state vector the orthogonal component φ'_k arising from the nonlinear interaction.

It will be shown later [cf. Eq. (2.9)] that φ'_k is a first-order quantity if the amplitude C_k is a first-order quantity. Thus, the solution that takes account of the interaction between modes is sought in the form

$$\varphi = \sum_{\mathbf{k}} \{C_{\mathbf{k}}(t)\,(\varphi_{\mathbf{k}} + \varphi'_{\mathbf{k}})\, e^{-i(\omega_{\mathbf{k}} t - \mathbf{kr})} + C_{\mathbf{k}_-}(t)\,(\varphi_{\mathbf{k}_-} + \varphi'_{\mathbf{k}_-})\, e^{i(\omega_{\mathbf{k}} t - \mathbf{kr})}\}. \tag{2.8}$$

Substituting (2.8) in (2.1) we have to second order

$$-\,[\omega_{\mathbf{k}} + \hat{H}_0(\mathbf{k})]\, C_{\mathbf{k}} \varphi'_{\mathbf{k}} = i\frac{\partial C_{\mathbf{k}}}{\partial t}\varphi_{\mathbf{k}} + \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} C_{\mathbf{k}'} C_{\mathbf{k}''} H_1\{\varphi_{\mathbf{k}'}, \varphi_{\mathbf{k}''}\}\, e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})t}. \tag{2.9}$$

The condition for solving (2.9) to determine φ'_k is that the right side must be orthogonal to the solution of the conjugate equation

$$\tilde{\psi}_{\mathbf{k}}\,(\omega_{\mathbf{k}} + H_0(\mathbf{k})) = 0, \tag{2.10}$$

where $\tilde{\psi}_k$ is a row vector. Multiplying (2.9) scalarly by $\tilde{\psi}_k$ from the left we have

$$\Big(\tilde{\psi}_{\mathbf{k}},\ \Big\{i\frac{\partial C_{\mathbf{k}}}{\partial t}\varphi_{\mathbf{k}} + \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} C_{\mathbf{k}'} C_{\mathbf{k}''} H_1\{\varphi_{\mathbf{k}'}\varphi_{\mathbf{k}''}\}\, e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})t}\Big\}\Big) = 0.$$

Rewriting this in the form

$$\frac{\partial C_{\mathbf{k}}}{\partial t} = -i \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} V_{\mathbf{kk}'\mathbf{k}''} C_{\mathbf{k}'} C_{\mathbf{k}''} e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})t}, \tag{2.11}$$

where

$$V_{\mathbf{kk}'\mathbf{k}''} = -\,(\tilde{\psi}_{\mathbf{k}},\ H_1\{\varphi_{\mathbf{k}'},\ \varphi_{\mathbf{k}''}\})/(\tilde{\psi}_{\mathbf{k}},\ \varphi_{\mathbf{k}}),$$

we obtain the dynamic equation (2.11) for the $C_k(t)$.

The eigenvectors φ_k of the operator $\hat{H}_0$, the explicit form of (2.9) for the actual magnetohydrodynamic equations with ion dispersion, and the solution of $\tilde{\psi}_k$ for the homogeneous complex equation are given in Appendix 1. Using the expressions in Appendix 1 we obtain an explicit expression for the matrix element from (2.11)

$$\begin{aligned} V_{\mathbf{kk}'\mathbf{k}''} = {} & \sqrt{\frac{\omega\omega'\omega''}{H_0^2/8\pi}}\Big[8\Big(\frac{B}{A}+1\Big)\Big(\frac{B'}{A'}+1\Big) \times \Big(\frac{B''}{B''}+1\Big)\Big]^{-1/2} \frac{c_A}{|\,\omega\,|}\Big\{-\frac{k_x c_A^2 k'_y k''_y}{\omega_H^2 A' A''} \\ & + k_x\Big[1 + B\frac{\omega'\omega''}{k_y c_A^2}\Big(\frac{k''_y}{k''^2} + \frac{k'_y}{k'^2}\Big)\Big] \\ & + k'_x\Big[(1-B)\,B'\frac{\omega''}{\omega'} - B'B''\frac{\omega}{\omega'} + \frac{\omega\omega'}{k'^2 c_A^2} + \frac{\omega\omega''}{k''^2 c_A^2} B'\frac{k''_y}{k'_y}\Big] + k''_x\Big[(1-B)\,B''\frac{\omega'}{\omega''} - B'B''\frac{\omega}{\omega''} \\ & + \frac{\omega\omega''}{k''^2 c_A^2} + \frac{\omega\omega'}{k'^2 c_A^2} B''\frac{k'_y}{k''_y}\Big] + \frac{Bk'_y}{A'}\Big[\frac{k_x}{k_y} + \frac{k'_x k'_y c_A^2}{\omega\omega'} - \frac{k''_x k'_y c_A^2}{\omega\omega'} \\ & - \Big(\frac{k'_y}{k_y}\frac{\omega''}{\omega'} + \frac{\omega''}{\omega}\Big)\frac{[\mathbf{k}'\mathbf{k}'']_z}{k''^2}\Big] + \frac{Bk''_y}{A''}\Big[\frac{k_x}{k_y} + \frac{k''_x k''_y c_A^2}{\omega\omega''} - \frac{k'_x k''_y c_A^2}{\omega\omega'} \\ & - \Big(\frac{k''_y}{k_y}\frac{\omega'}{\omega''} + \frac{\omega'}{\omega}\Big)\frac{[k''k']_z}{k'^2}\Big]\Big\}, \end{aligned} \tag{2.12}$$

where H_0 is the fixed magnetic field along the y axis, $A = 1 - k_y^2 c_A^2/\omega^2$, $B = \omega^2/k^2c_A^2 - 1$ and the corresponding quantities with one or two primes are obtained by replacing ω and $\mathbf{k}$ and ω' and $\mathbf{k}'$ or ω'' and $\mathbf{k}''$ respectively.

Perturbation theory is used to determine $C_k(t)$ at time t. From (2.11) we have

$$C_{\mathbf{k}}(t) = C_{\mathbf{k}}^{(0)} + C_{\mathbf{k}}^{(1)} + C_{\mathbf{k}}^{(2)} + \ldots,$$

$$C_{\mathbf{k}}^{(1)} = -\,i\sum_{\mathbf{k}'\mathbf{k}''} C_{\mathbf{k}'}^{(0)} C_{\mathbf{k}''}^{(0)} \int_0^t V_{\mathbf{kk}'\mathbf{k}''}(t')\,dt',$$

$$C_{\mathbf{k}}^{(2)} = -\sum_{\mathbf{k}'\mathbf{k}''}\sum_{\mathbf{q}'\mathbf{q}''} C_{\mathbf{k}'}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \int_0^t dt' \int_0^{t'} dt'' V_{\mathbf{kk}'\mathbf{k}''}(t')\, V_{\mathbf{k}''\mathbf{q}'\mathbf{q}''}(t'') - \sum_{\mathbf{k}'\mathbf{k}''}\sum_{\mathbf{q}'\mathbf{q}''} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \int_0^t dt' \int_0^{t'} dt'' V_{\mathbf{kk}'\mathbf{k}''}(t')\, V_{\mathbf{k}'\mathbf{q}'\mathbf{q}''}(t'');$$

$$V_{\mathbf{kk}'\mathbf{k}''}(t) = V_{\mathbf{kk}'\mathbf{k}''} \exp\{-\,i\,(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega_{\mathbf{k}})\,t\},$$

$$V_{\mathbf{kk}'\mathbf{k}''} \neq 0 \quad \text{for } \mathbf{k} = \mathbf{k}' + \mathbf{k}''. \tag{2.13}$$

The quantity $C_k^{(0)}$ is time independent and corresponds to the solution in the absence of interaction between modes. The change in the number of quasiparticles $|C_k(t)|^2 - |C_k^{(0)}|^2$ averaged over the phases of $C_k^{(0)}$ by means of (2.4) is (to second order)

$$\overline{|C_{\mathbf{k}}(t)|^2} - \overline{|C_{\mathbf{k}}^{(0)}|^2} = \overline{|C_{\mathbf{k}}^{(1)}|^2} + \overline{(C_{\mathbf{k}}^{(0)} C_{\mathbf{k}}^{(2)*} + C_{\mathbf{k}}^{(0)*} C_{\mathbf{k}}^{(2)})}. \tag{2.14}$$

Substituting (2.13) in (2.14), averaging over the phases of the $C_{\mathbf{k}}^{(0)}$, and using the relations

$$\left|\int_0^t V_{\mathbf{kk}'\mathbf{k}''}(t')\,dt'\right|^2 \to t2\pi\,|V_{\mathbf{kk}'\mathbf{k}''}|^2\delta(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}});$$

$$V_{\mathbf{kk}'\mathbf{k}''} = V_{\mathbf{kk}''\mathbf{k}'}, \qquad V_{\mathbf{kk}'\mathbf{k}''} = -V^*_{\mathbf{k}_-,\,\mathbf{k}'_-,\,\mathbf{k}''_-}, \tag{2.15a}$$

$$V_{\mathbf{kk}'\mathbf{k}''} = V^*_{\mathbf{k}'',\,\mathbf{k}'_-,\,\mathbf{k}}, \qquad V_{\mathbf{kk}'\mathbf{k}''_-} = -V^*_{\mathbf{k}''_-,\,\mathbf{k}'_-,\,\mathbf{k}} \tag{2.15b}$$

[(2.15a) and (2.15b) can be verified using (2.12); it is important that (2.15b) be satisfied for the condition $\omega_{\mathbf{k}} = \omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}$, $\omega_{\mathbf{k}'}$, $\omega_{\mathbf{k}''} > 0$] we obtain the change in the number of quasiparticles per unit time due to collisions:

$$\left(\frac{\partial n_{\mathbf{k}}}{\partial t}\right)_S = 4\pi\sum_{\mathbf{k}'\mathbf{k}''}|V_{\mathbf{kk}'\mathbf{k}''}|^2\{(n_{\mathbf{k}'}n_{\mathbf{k}''} - n_{\mathbf{k}}n_{\mathbf{k}'} - n_{\mathbf{k}}n_{\mathbf{k}''})$$
$$\times\,\delta(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})\,\delta_{\mathbf{k}'+\mathbf{k}'',\,\mathbf{k}} + 2(n_{\mathbf{k}'}n_{\mathbf{k}''}$$
$$+\,n_{\mathbf{k}}n_{\mathbf{k}''} - n_{\mathbf{k}}n_{\mathbf{k}'})\,\delta(\omega_{\mathbf{k}''}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}})\,\delta_{\mathbf{k}'',\,\mathbf{k}'+\mathbf{k}}\}. \tag{2.16}$$

In this equation the summation is taken only over modes with positive frequencies $\omega_{\mathbf{k}}$, $\omega_{\mathbf{k}'}$, and $\omega_{\mathbf{k}''}$. This relation yields the conservation of energy $\omega_{\mathbf{k}}$ and quasi-momentum k for "collisions" between modes. We note that (2.16) is of the same form as the kinetic equation proposed on the basis of phenomenological considerations in [6]; however, [6] contains neither the explicit matrix elements nor the method of obtaining them.

It is interesting to note the analogy between (2.16) and the kinetic equation for phonons in a solid.[2] If we write $n_{\mathbf{k}} \gg 1$ in the latter (neglecting the explicit form of the matrix elements) it becomes (2.16). We note also that $(\partial n_{\mathbf{k}}/\partial t)_S$ vanishes when

$$n_{\mathbf{k}} = T/\omega_{\mathbf{k}}, \tag{2.17}$$

where T is the effective temperature of the mode gas (in energy units). The expression in (2.17) is nothing more than the Rayleigh-Jeans distribution.

3. CRITERIA FOR DECAY INSTABILITY

Using the formalism developed in the preceding section we can formulate easily the criteria for the decay instability treated earlier by Oraevskiĭ and Sagdeev.[10] Specifically, suppose that a wave characterized by frequency $\omega_{\mathbf{k}}$ and wave vector k propagates in the plasma. Then perturbations in the form of two waves, with frequencies $\omega_{\mathbf{k}'}$, $\omega_{\mathbf{k}''} > 0$ and wave vectors k′, k″ that satisfy the conditions

$$\mathbf{k}' = \mathbf{k} - \mathbf{k}'', \qquad \omega_{\mathbf{k}'} = \omega_{\mathbf{k}} - \omega_{\mathbf{k}''}, \tag{3.1}$$

grow with time.

From (2.11) and (2.15) we have a system of two equations for the amplitudes $C_{\mathbf{k}'}$ and $C_{\mathbf{k}''}$:

$$\partial C_{\mathbf{k}'}/\partial t = -iV_{\mathbf{k}'\mathbf{k}\mathbf{k}''_-}C^{(0)}_{\mathbf{k}}C_{\mathbf{k}''},$$
$$\partial C_{\mathbf{k}''_-}/\partial t = -iV_{\mathbf{k}''_-,\,\mathbf{k}_-,\,\mathbf{k}'}C^{(0)}_{\mathbf{k}_-}C_{\mathbf{k}'} \equiv iV^*_{\mathbf{k}'\mathbf{k}\mathbf{k}''_-}C^{(0)}_{\mathbf{k}_-}C_{\mathbf{k}'}.$$

Since the perturbations $C_{\mathbf{k}'}$ and $C_{\mathbf{k}''}$ are small the amplitude $C^{(0)}_{\mathbf{k}}$ may be taken as constant ($\partial C^{(0)}_{\mathbf{k}}/\partial t \sim C_{\mathbf{k}'}C_{\mathbf{k}''} \sim 0$). We seek a solution of this system in the form $e^{\nu t}$ and obtain the growth rate

$$\nu^2 = |V_{\mathbf{k}'\mathbf{k}\mathbf{k}''_-}|^2\,|C^{(0)}_{\mathbf{k}}|^2. \tag{3.2}$$

If the perturbations are such that $\omega_{\mathbf{k}'} > \omega_{\mathbf{k}''} > 0$ and the decay conditions

$$\mathbf{k}' = \mathbf{k} + \mathbf{k}'', \qquad \omega_{\mathbf{k}'} = \omega_{\mathbf{k}} + \omega_{\mathbf{k}''}, \tag{3.3}$$

are satisfied, we have a system of equations for determination of the amplitudes of the small perturbations:

$$\partial C_{\mathbf{k}'}/\partial t = -iV_{\mathbf{k}'\mathbf{k}\mathbf{k}''}C^{(0)}_{\mathbf{k}}C_{\mathbf{k}''},$$
$$\partial C_{\mathbf{k}''}/\partial t = -iV_{\mathbf{k}''\mathbf{k}_-,\,\mathbf{k}'}C^{(0)}_{\mathbf{k}_-}C_{\mathbf{k}'} \equiv -iV^*_{\mathbf{k}'\mathbf{k}\mathbf{k}''}C^{(0)}_{\mathbf{k}_-}C_{\mathbf{k}''}.$$

Using the solution $e^{\nu t}$ we find

$$\nu^2 = -|V_{\mathbf{k}'\mathbf{k}\mathbf{k}''}|^2\,|C^{(0)}_{\mathbf{k}}|^2 < 0, \tag{3.4}$$

that is to say, the wave is stable.

Thus, the decay instability can only lead to the excitation of modes with frequencies lower than the original frequency.

4. DECAY INSTABILITY OF THE OSCILLATORY STRUCTURE OF SHOCK WAVES

We apply the results of the preceding section to the question of stability of the oscillatory structure of weak shock waves propagating in a strong magnetic field. If the shock wave propagates at right angles to the magnetic field its oscillatory structure can be regarded as a wave of frequency ω_0 and wave vector $\mathbf{k}_0$ [7] where

$$k_0 = \sqrt{{}^1\!/_2(1-M^{-2})}\sqrt{m_i/m_e}\,\omega_H/c_A, \qquad \omega_0 = k_0 c_A M, \tag{4.1}$$

and M is the Mach number. A curve of the laminar profile is shown in Fig. 1a. The arrow indicates the direction of motion of the wave.

If $(M-1) \gg m_e/m_i$ the wave frequency ω_0 is appreciably greater than the ion Larmor frequency ω_H. In this case, the conditions (3.1) for a perturbation in the form of two fast magnetoacoustic waves, one of which propagates almost perpendicularly to the magnetic field [cf. Eq. (A.3)],

$$k'(1 + k_y'^2 c_A^2/\omega_H^2) = k_0 c_A M - \omega_+(\mathbf{k}''),$$
$$k'_x = k_0 - k''_x, \qquad k'_y = -k''_y, \tag{4.2}$$

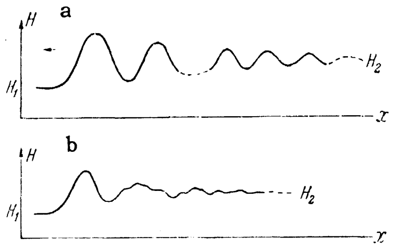

FIG. 1. Qualitative profile of the change in magnetic field in a shock wave propagating at right angles to H: a) laminar case, b) turbulent case.

can be satisfied only when $k'_y c_A \lesssim \sqrt{M-1}\,\omega_H \ll \omega_H$. In this case the matrix element $V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$ is

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \approx \sqrt{\frac{\omega\omega'\omega''}{H_0^2/8\pi}}\frac{1}{\sqrt{2}}\left(1+\frac{1}{2}\frac{k''_x}{k''}\right). \qquad (4.3)$$

The conditions (3.1) can also be satisfied for a perturbation in the form of a fast $[\omega''=\omega_+(k'')]$ wave and a slow $[\omega'=\omega_-(k')]$ wave. The expression for the matrix element $V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$ is easily obtained from (2.12) in two limiting cases:

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \approx \sqrt{\frac{\omega\omega'\omega''}{H_0^2/8\pi}}\frac{1}{\sqrt{8}}\frac{\omega}{\omega_H}\frac{k''_y c_A}{\omega_H} \quad \text{for } k''_y c_A \gg \omega_H, \qquad (4.4a)$$

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \approx -\sqrt{\frac{\omega\omega'\omega''}{H_0^2/8\pi}}\frac{1}{\sqrt{8}}\frac{k'_x c_A}{\omega_H} \quad \text{for } k''_y c_A \lessapprox \omega_H. \qquad (4.4b)$$

However, in (4.4a) the frequency of the slow wave is very close to the ion Larmor frequency $\omega' \approx \omega_H$ and this wave is strongly damped [11] if one takes account of even a very small thermal spread in the ion velocities. Hence these waves do not exist in practice and the decay indicated by (4.4a) does not occur.

The growth rates for the decay instabilities (4.2) and (4.4b) are of the same order for small Mach numbers

$$\nu_{1,2} \approx (M-1)\,\omega_0. \qquad (4.5)$$

As the Mach number increases $M-1 > \sqrt[3]{m_e/m_i}$ the increment in (4.5) becomes of the same order as the frequency of the Alfven wave $\omega' = k'_y c_A$ in the decay (4.4b) and, formally, can even be greater, as follows from (4.5). In this case the criteria for the application of perturbation theory $(\nu_2 \lesssim \omega', \omega'')$ are no longer satisfied. It is reasonable to assume, however, that the growth rates at these large Mach numbers cannot exceed the frequency of the perturbation, i.e., $\nu_2 \lesssim \omega'$. The interaction being considered is actually a resonance effect [cf. (3.1)] and the notion of a resonance is no longer meaningful if $\nu_2 = \operatorname{Im}\omega' > \operatorname{Re}\omega'$.

In what follows we treat Mach numbers that are not too small $(M-1) > \sqrt[3]{m_e/m_i}$ so that it can be assumed that $\nu_1 > \nu_2$; consequently the oscillatory structure of the shock wave decays in accordance with (4.2).

If, however, the shock wave propagates at an angle with respect to the magnetic field $(\theta \gg \sqrt{m_e/m_i})$ so that there is an ion dispersion effect, the oscillatory structure can be regarded as a wave characterized by frequency ω_0 and wave vector $\mathbf{k}_0$ [9] (a curve of the laminar profile is shown in Fig. 2a) given by

$$\omega_0 = k_0 c_A M, \qquad k_0 \approx \sqrt{(1-M^{-2})}\,\mathrm{tg}^{-1}\theta\,\omega_H/c_A. \qquad (4.6)^*$$

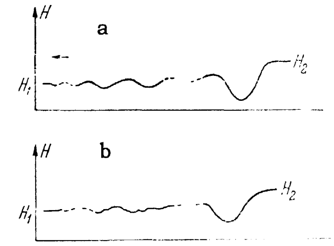

FIG. 2. Qualitative profile of the change in magnetic field in a shock wave propagating at an angle to H: a) laminar case, b) turbulent case.

In this case the condition in (3.1) is satisfied by the frequencies $\omega' = k'_y c_A$ and $\omega''_y c_A$ and wave vectors $\mathbf{k}'$ and $\mathbf{k}''$ of a perturbation in the form of two Alfvên waves. The matrix element describing the interaction of these waves is

$$V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} = \sqrt{\frac{\omega\omega'\omega''}{H_0^2/8\pi}}\frac{1}{\sqrt{8}}\frac{k_x}{k}, \qquad (4.7)$$

while the growth rate is of order

$$\nu \approx 8^{-1/2}(M-1)^{3/2}\omega_H\,\mathrm{tg}^{-1}\theta \quad \text{for } \mathrm{tg}\,\theta \sim 1. \qquad (4.8)$$

5. EFFECT OF TURBULENCE ON THE STRUCTURE OF A SHOCK FRONT

In this section we evaluate the effect of the turbulent oscillations arising from the decay instability on the laminar oscillatory structure of a shock wave. We first write the complete kinetic equation for the waves using (2.7), (2.16) and (3.2):

$$\frac{\partial n_{\mathbf{k}}}{\partial t}+\frac{\partial n_{\mathbf{k}}}{\partial x}\frac{dx}{dt}+\frac{\partial n_{\mathbf{k}}}{\partial k_x}\frac{dk_x}{dx}\frac{dx}{dt}=\left(\frac{\partial n_{\mathbf{k}}}{\partial t}\right)_S+2\nu n_{\mathbf{k}}. \qquad (5.1)$$

Here, the change in the number of modes in a given volume is equated to the change due to collisions and the increment due to the decay instability. The derivative is determined from the condition that the frequency is a constant in the coordinate system moving with the shock wave $\omega_k + k_x u$ while $dx/dt = \partial\omega_k/\partial k_x + u$ is the group velocity in this coordinate system (u is the plasma flow velocity with respect to the shock wave).

Equation (5.1) has been used without a wave source in [6] to construct the structure of the shock wave in the turbulent mode. The energy source for

*tg = tan.

the turbulent oscillations proposed by these authors was the change in frequency in the rest system of the plasma (which is consequently proportional to its energy) moving through a flow field of variable velocity. However, in considering quasiparticles moving together with a shock wave one must work with the energy in the coordinate system fixed in the shock wave. It can be shown from the equation

$$\frac{\partial n_k}{\partial t}+\frac{\partial n_k}{\partial x}\left(u+\frac{\partial \omega_k}{\partial k_x}\right)-k_x\frac{du}{dx}\frac{\partial n_k}{\partial k_x}=\left(\frac{\partial n_k}{\partial t}\right)_S, \quad (5.1a)$$

used in [6] that this energy is obviously conserved because the number of quasiparticles n_k is conserved in the slowly varying flow field as is their "energy" $\omega_k + k_x u$; the energy $\Sigma n_k(\omega_k + k_x u)$ must also be conserved in the collisions. Thus there is no energy source for the waves in this scheme and a question arises as to where the energy comes from. The decay instability of the oscillatory structure of the shock wave can be such a source.

As an example we analyze the propagation of a weak shock wave across a strong magnetic field ($H^2/8\pi \gg p$). As shown in the preceding section, the oscillatory structure of the shock wave is unstable and decays with the growth rate given by (4.5).

The low-frequency waves produced in the decay (frequencies $\sim k_{2y}c_A \sim \sqrt{M-1}\,\omega_H$) propagate at large angles with respect to the direction of propagation of the shock wave and are rapidly carried away by the flow. However, waves with frequencies $\omega \gg \omega_H$ are propagated in essentially the same direction as the shock wave (the angular spread of these waves is of order $k_{2y}/k_0 \sim \sqrt{m_e/m_i}$) and hence are practically not affected. When these waves collide with each other low-frequency waves are produced; the destruction of the latter results in the eventual dissipation of the energy of the high-frequency waves.

Using (4.3) we can estimate the change in the number of these waves due to collisions:

$$(\partial N_k/\partial t)_S = N_k/\tau, \qquad \tau = 2/\pi\bar{\beta}\omega'',$$
$$\bar{\beta} = \frac{1}{4}\sum_k N_k\omega_k/(H_0^2/8\pi), \quad (5.2)$$

where τ is the collision time for the waves, N_k is the number of high-frequency waves with frequency $\omega_k \sim \omega_0$, $\bar{\beta}$ is the ratio of wave energy to magnetic field energy.

Since the number of waves produced per unit time $\nu N \sim (M-1)\,\omega_0 N$ cannot be balanced by the loss of these waves due to collisions $N/\tau \sim \bar{\beta}\omega'' N$, it is reasonable to assume that all the energy of the laminar shock wave is transferred into energy of waves in the group, i.e.,

$$\bar{\beta} \cong (M-1)^2. \quad (5.3)$$

Equating the change in the number of waves N in the volume to the wave loss (5.2) due to collisions we obtain an estimate for the wave distribution in space (we use the coordinate system fixed in the packet)

$$(\partial\omega_k/\partial k_x + u)\,\partial N_k/\partial x \cong (\partial N_k/\partial t)_S$$

or

$$(M-1)\,c_A\,\partial N/\partial x = {}^1/_2\,\pi\bar{\beta}\omega'' N,$$

i.e., the change in density can be approximated by the exponential relation

$$N \sim e^{x/L}, \qquad L = 2c_A/\pi(M-1)^{5/2}\omega_H. \quad (5.4)$$

The corresponding turbulent profile is shown in Fig. 1b.

If the shock wave propagates at an angle to the magnetic field H_0 that is larger than $\sqrt{m_e/m_i}$ the ion inertia term becomes important. In this case the wave is unstable and decays into waves traveling at very different angles with respect to the direction of propagation of the shock wave [cf. (4.8)]. Hence, a large part of the energy converted into turbulent oscillations as a result of decay is carried away by the flow.

Because of the small energy in the turbulent oscillations $\bar{\beta} \approx (M-1)^2 \ll 1$ the collision time for these oscillations is very large so that the basic process that reduces the number of oscillations is their destruction. Equating the growth in oscillations caused by decay with the destruction, we obtain the distribution of oscillations in space:

$$2\nu n_k = c_A\,\partial n_k/\partial x.$$

Using (4.8) we write this relation in the form

$$(M-1)^{3/2}\omega_H/4\,\mathrm{tg}\,\theta = c_A\,\partial n_k/\partial x, \qquad n_k \sim e^{x/L}, \quad (5.5)$$

where

$$L = 4c_A\,\mathrm{tg}\,\theta/(M-1)^{3/2}\omega_H, \qquad \mathrm{tg}\,\theta \sim 1.$$

Thus, when the decay instability of the shock is introduced the length of the oscillatory structure cannot reach the value c_A/ν_{ei} (ν_{ei} is the frequency of electron-ion collisions) that obtains in a collisionless plasma since the appearance of the oscillatory structure leads to its own decay. The width of the shock wave is found to be relatively sensitive to the direction of propagation of the wave and is determined by the characteristic dimensions appearing in (5.4) and (5.5). However, the order-of-magnitude of the frequency of the turbulent oscillations coincides with the frequency of the oscillations of the laminar structure and is sensitive to the angle. The pattern of the turbulent profile in this case is shown in Fig. 2b.

The authors wish to thank R. Z. Sagdeev for stimulating discussions and valuable comments.

APPENDIX

When the magnetohydrodynamic equations with ion dispersion (1.1) and (1.3) are introduced in (2.1) the latter assumes [for $A = \exp\{-i(\omega_{k'} + \omega_{k''} - \omega_k)t\}$ the form

$$-\omega_k C_k v_k' - C_k \frac{[[kh_k'] H_0]}{4\pi\rho_0} = i\frac{\partial C_k}{\partial t} v_k$$

$$-\sum_{k'+k''=k} C_{k'}C_{k''}\Big\{(v_{k'}k'')\, v_{k''} - \frac{1}{4\pi\rho_0}\Big[[k'h_{k'}],\ \Big(h_{k''} - \frac{\rho_{k''}}{\rho_0}H_0\Big)\Big]\Big\} A,$$

$$-\omega_k C_k h_k' - C_k [k\,[v_k' H_0]] = i\frac{\partial C_k}{\partial t} h_k$$

$$+\sum_{k'+k''=k} C_{k'}C_{k''}\Big\{[k\,[v_{k'}h_{k''}]] - i\frac{cm_i}{4\pi\rho_0 e}\Big[k\Big[[k'h_{k'}]\Big(h_{k''} - \frac{\rho_{k''}}{\rho_0}H_0\Big)\Big]\Big]\Big\} A,$$

$$-\omega_k C_k \rho_k' + \rho_0 C_k (kv_k') = i\frac{\partial C_k}{\partial t}\rho_k - \sum_{k'+k''=k} C_{k'}C_{k''}\rho_{k'}(v_{k''}k)\, A. \qquad (A.1)$$

It is easily shown that we need write only the components along the x and z axes for (1.3). The The third equation is equivalent to div $\mathbf{h} = 0$, i.e., $h_x = -k_y h_y / k_x$. If the right side of (A.1) is set equal to zero we have

$$(\omega_k + H_0(k))\,\varphi_k = 0 \qquad (A.2)$$

for determining the eigenvector φ_k. Solving (A.2) we find the components of the eigenvector φ_k of the operator $\hat{H}_0$:

$$\varphi_2 = 0; \quad \varphi_3 = -\frac{i\omega}{\omega_H A}\frac{k_y^2 c_A^2}{\omega^2}\varphi_1; \quad \varphi_4 = -\frac{k_y \omega H_0}{k^2 c_A^2}\varphi_1;$$

$$\varphi_5 = \frac{ik_y H_0}{\omega_H A}\varphi_1; \quad \varphi_6 = \frac{k_x}{\omega} n_0 \varphi_1; \quad A \equiv 1 - \frac{k_y^2 c_A^2}{\omega^2}.$$

Writing (2.10) in explicit form by means of (A.2) we obtain the solution of the conjugate equation:

$$\psi_2 = 0; \quad \psi_3 = -\frac{\omega_H}{i\omega}B\psi_1; \quad \psi_4 = -\frac{\omega}{k_y H_0}\psi_1;$$

$$\psi_5 = \frac{\omega_H}{ik_y H_0}B\psi_1; \quad \psi_6 = 0, \quad B \equiv \frac{\omega^2}{k^2 c_A^2} - 1.$$

The natural frequencies are determined by the expression

$$\omega_\pm = \tfrac{1}{2} c_A \Big(\sqrt{(k + k_y)^2 + c_A^2 k^2 k_y^2/\omega_H^2} \pm \sqrt{(k - k_y)^2 + c_A^2 k_y^2 k^2/\omega_H^2}\Big). \qquad (A.3)$$

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Translated by H. Lashinsky
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