POSSIBILITY OF SUPERCONDUCTIVITY IN ANTIFERROMAGNETS

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Interaction between conduction electrons in an antiferromagnet, due to spin wave exchange, is investigated. It is shown that the interaction is repulsive if an electron pair is in the singlet state and attractive if in the triplet state with a zero total spin projection. Attraction is greatest in the p-state. Spin wave exchange in an antiferromagnet may produce superconductivity.

MODERN superconductivity theory [1-3] makes use of the notion of effective interaction between the conduction electrons, resulting from phonon exchange. This interaction has the sign of attraction and leads to the production of Cooper pairs in the S-state and the appearance of an energy gap in the excitation spectrum. One of the main accomplishments of this theory is the explanation of the isotopic effect, which is the consequence of the phonon mechanism of superconductivity.

There is, however, a known superconductor (ruthenium) which does not display the isotopic effect [4]. This suggests that superconductivity may be caused by some other (non-phonon) mechanism.

This note is devoted to an investigation of one of the possibilities, namely interaction between the conduction electrons in an antiferromagnet, caused by exchange of virtual spin waves. It turns out that this interaction is repulsive if the particle pair is in the singlet state and attractive for a pair in the triplet state with summary spin projection equal to zero. The attraction is the most intense in the p-state.

The possibility of formation of Cooper pairs in a state with nonzero momentum was predicted in [5-7]. An investigation of the stability of the states proposed in these papers, carried out by Vaks, Galitskiĭ, and Larkin [8], has shown that the ground state is a complicated superposition of states with different Cooper-pair momentum projections. We shall follow the simple scheme of Gor'kov and Galitskiĭ [7], which helps explain the very existence of superconductivity.

An analogous problem was investigated for the case of a ferromagnet in several papers [9,10]. The conclusion that an effective attraction exists in the triplet state was drawn, however, only by Akhiezer and Pomeranchuk [11]. It is stated in [9,10] that exchange of spin waves leads to a repulsion between the conduction electrons, but the procedure used in these investigations is applicable only for the analysis of the s-state of the electron pair.

1. We describe the interaction between the conduction electrons and the sublattice magnetization within the framework of the sd-exchange model [1,2]. The energy of the system of s-electrons in the field of d-electrons is written in the form

$$H_{int} = \sum_{\alpha} \int d^3\mathbf{r}\psi_\alpha^+(\mathbf{r})\, H(\mathbf{r})\, \psi_\alpha(\mathbf{r}), \qquad (1)$$

where ψ^+ and ψ are the operators for the creation and annihilation of the conduction electrons, while the energy density $H(\mathbf{r})$ is equal to

$$H(\mathbf{r}) = -2I\mathbf{s}\mathbf{M}/M_0. \qquad (2)$$

In the last formula I is the parameter of the sd-exchange interaction, which has the dimension of energy, $\mathbf{s}$ is the s-electron spin operator, $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ is the density operator of the summary magnetic moment of the sublattices, which are assumed to be identical, and M_0 is the maximum sublattice magnetization.

The operators $\mathbf{M}_1$ and $\mathbf{M}_2$ must be expressed in terms of the spin-wave operators. To this end we write the phenomenological Hamiltonian of the system of d-electrons, neglecting their spatial motion [13].

$$H_{sp} = \int d^3\mathbf{r}\left\{\frac{1}{2}\alpha\left(\frac{\partial\mathbf{M}_1}{\partial x_i}\frac{\partial\mathbf{M}_1}{\partial x_i} + \frac{\partial\mathbf{M}_2}{\partial x_i}\frac{\partial\mathbf{M}_2}{\partial x_i}\right) + \alpha_{12}\frac{\partial\mathbf{M}_1}{\partial x_i}\frac{\partial\mathbf{M}_2}{\partial x_i} + \delta\mathbf{M}_1\mathbf{M}_2 - \frac{1}{2}\beta\left[(\mathbf{n}\mathbf{M}_1)^2 + (\mathbf{n}\mathbf{M}_2)^2\right] + \frac{H^2}{8\pi}\right\}, \qquad (3)$$

where the field $\mathbf{H}$ produced by the spin waves is determined from the equations

$$\operatorname{curl}\mathbf{H} = 0, \qquad \operatorname{div}(\mathbf{H} + 4\pi\mathbf{M}) = 0, \qquad (4)$$

and the constants δ, α, α_{12}, and β are positive, with $\alpha > \alpha_{12}$; $\mathbf{n}$ is the unit vector in the direction of the easiest magnetization axis (the z axis).

The Hamiltonian H_{sp} is diagonalized by means of the substitution

$$M_x^{(n)} - iM_y^{(n)} = \left(\frac{2\mu M_0}{V}\right)^{1/2} \sum_{\mathbf{k},m} (u_m^{(n)} b_{\mathbf{k}}^{(m)} + v_m^{(n)*} b_{-\mathbf{k}}^{+(m)}) e^{i\mathbf{kr}}, \quad (5)$$

$$M_z^{(1)} = -M_z^{(2)} = M_0 - M_\perp^{(n)(2)}/2M_0, \quad (6)$$

where b and b^+ are spin-wave creation and annihilation operators satisfying the Bose commutation relations, the index n labels the sublattices and m the oscillation modes (n, m = 1, 2), μ is the spectroscopic splitting factor, and V is the volume of the crystal. Considering the temperature to be low, the occupation numbers of the spin waves are assumed equal to zero.

The momentum projection operators should satisfy the well known commutation relations. The commutation relations lead to several simple equations connecting the transformation parameters (5), which will not be written out here. If we neglect the last term in the right half of (3) (magnetic energy), then the system of equations for $u_m^{(n)}$ and $v_m^{(n)}$ does not have a unique solution, and the energies of the different oscillation modes coincide:

$$\omega_q^{(1,2)} \equiv \omega_q = [(2\delta + \beta + \alpha q^2 + \alpha_{12} q^2)(\beta + \alpha q^2 - \alpha_{12} q^2)]^{1/2}, \quad (7)$$

where $\mathbf{q}$ is the spin-wave momentum. The role of the magnetic energy reduces thus to a lifting of the degeneracy, so that the parameters $u_m^{(n)}$ and $v_m^{(n)}$ can be correctly chosen:

$$u_1^{(1)} = -v_2^{(2)} = \frac{1}{2}\sqrt{\frac{A_q + \omega_q}{\omega_q}},$$

$$v_1^{(1)} = u_2^{(2)*} = \frac{1}{2}\sqrt{\frac{A_q - \omega_q}{\omega_q}} e^{2i\varphi_{\mathbf{q}}},$$

$$u_2^{(1)} = v_1^{(2)*} = -\frac{1}{2}\sqrt{\frac{A_q + \omega_q}{\omega_q}} e^{-2i\varphi_{\mathbf{q}}},$$

$$v_2^{(1)} = -u_1^{(2)} = \frac{1}{2}\sqrt{\frac{A_q - \omega_q}{\omega_q}}, \quad (8)$$

where $\varphi_{\mathbf{q}}$ is the polar angle of the vector $\mathbf{q}$ and

$$A_q = \mu M_0 (\delta + \beta + \alpha q^2). \quad (9)$$

Substituting the expression for the magnetization $\mathbf{M}$ in (1) and recognizing that the nonvanishing matrix elements of the circular spin projections are equal to

$$\langle 1/2 | s_x + is_y | -1/2 \rangle = \langle -1/2 | s_x - is_y | 1/2 \rangle = 1, \quad (10)$$

we obtain the Hamiltonian of the interaction between the electrons and the spin waves

$$H_{int} = I\left(\frac{2\mu}{M_0 V}\right)^{1/2} \sum_{\mathbf{p},\mathbf{q}} \frac{\sqrt{\omega_q}}{(A_q + \omega_q)^{1/2} + (A_q - \omega_q)^{1/2}}$$
$$\times \{a_{\mathbf{p}+\mathbf{q},-}^+ a_{\mathbf{p},+} (b_{\mathbf{q}}^{(2)} + e^{2i\varphi_{\mathbf{q}}} b_{\mathbf{q}}^{(1)}) + a_{\mathbf{p}+\mathbf{q},+}^+ a_{\mathbf{p},-}$$
$$\times (e^{-2i\varphi_{\mathbf{q}}} b_{\mathbf{q}}^{(2)} - b_{\mathbf{q}}^{(1)}) + a_{\mathbf{p}-\mathbf{q},+}^+ a_{\mathbf{p},-} (b_{\mathbf{q}}^{+(2)} + e^{-2i\varphi_{\mathbf{q}}} b_{\mathbf{q}}^{+(1)})$$
$$+ a_{\mathbf{p}-\mathbf{q},-}^+ a_{\mathbf{p}+} (e^{2i\varphi_{\mathbf{q}}} b_{\mathbf{q}}^{+(2)} - b_{\mathbf{q}}^{+(1)})\}. \quad (11)$$

Here $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^+$ are the creation and annihilation operators of an electron with momentum $\mathbf{p}$, while the "plus" and "minus" indices correspond to the spin-projection direction. The Hamiltonian (11) is in our case the analogue of the Froehlich Hamiltonian for the electron-phonon interaction. The major difference, however, is that the electron spin projections change upon absorption or emission of a spin wave.

2. Our next purpose is to obtain the Hamiltonian of the direct interaction between electrons. We eliminate the spin-wave operators in the second-order perturbation theory, defining the matrix elements of the electron-electron interaction Hamiltonian in the following fashion:

$$\langle m | H_{ee} | n \rangle = \langle m | H_{int} (E_n - H_0)^{-1} H_{int} | n \rangle. \quad (12)$$

Here $|n\rangle$ and $|m\rangle$ are purely electronic states, which are the eigenvectors of the electron interaction Hamiltonian H_0: $H_0 |n\rangle = E_n |n\rangle$. As a result we obtain

$$H_{ee} = -\frac{I^2\mu}{M_0 V} \sum_{\mathbf{p},\mathbf{p}',\mathbf{q},\sigma} \frac{\omega_q}{A_q + B_q} \left(\frac{1}{\zeta_{\mathbf{p}-\mathbf{q}} - \zeta_{\mathbf{p}} + \omega_q}\right.$$
$$\left. + \frac{1}{\zeta_{\mathbf{p}'} - \zeta_{\mathbf{p}'+\mathbf{q}} + \omega_q}\right) a_{\mathbf{p}'+\mathbf{q},\sigma}^+ a_{\mathbf{p}',-\sigma} a_{\mathbf{p}-\mathbf{q},-\sigma}^+ a_{\mathbf{p}\sigma}. \quad (13)$$

Here $B_q = \sqrt{A_q^2 - \omega_q^2}$ and $\zeta_{\mathbf{p}}$ is the electron energy measured from the Fermi energy. The electrons interact only if their spins have opposite directions.

We simplify the problem, however, by taking into account, as is customary, only the interaction between electrons with opposite momenta. The Hamiltonian of such an interaction has the form

$$H_{red} = \frac{1}{2V} \sum_{\mathbf{p},\mathbf{p}',\sigma} V(\mathbf{p},\mathbf{p}') a_{\mathbf{p}',\sigma}^+ a_{-\mathbf{p}',-\sigma}^+ a_{\mathbf{p},\sigma} a_{-\mathbf{p},-\sigma}, \quad (14)$$

where

$$V(\mathbf{p},\mathbf{p}') = -\frac{4I^2\mu}{M_0} \frac{\omega_q^2}{A_q + B_q} \frac{1}{\omega_q^2 - (\zeta_{\mathbf{p}} - \zeta_{\mathbf{p}'})^2}, \quad \mathbf{q} = \mathbf{p} + \mathbf{p}'. \quad (15)$$

The interaction "potential" $V(\mathbf{p},\mathbf{p}')$ has a negative sign on the Fermi surface, corresponding to repulsion of the electrons in the S-state.

To investigate the ground state of the system with an interaction described by the Hamiltonian H_{red}, we employ the Gor'kov technique of splitting the chain of equations for the Green's function [3], as generalized in the paper by Gor'kov and Galitskiĭ [7] to include the case of condensation of Cooper pairs with nonzero relative moment. The

difference between our Hamiltonian (14) and that used in [4-6] lies in the fact that only electrons with opposite spin projections interact in an antiferromagnet. Therefore the triplet state of the pair is not degenerate in the total spin projection. The only permissible value of the summary-spin projection of the pair is zero.

Assuming that in the ground state the pairs have a momentum l (even momenta correspond to the singlet state and odd ones to the triplet state), we introduce, following [7], the Green's functions

$$g_{\sigma\sigma'}(p, t-t') = -i\langle Ta_{p\sigma}(t)\, a^+_{p\sigma'}(t')\rangle = g(p, t-t')\,\delta_{\sigma\sigma'},$$
$$F^+_{m\sigma\sigma'}(\mathbf{p}, t-t') = \langle N+2, l, m\,|\,Ta^+_{-\mathbf{p},\sigma}(t)\, a^+_{p\sigma'}(t')\,|\,N\rangle,$$
$$F_{m\sigma\sigma'}(\mathbf{p}, t-t') = \langle N\,|\,Ta_{p\sigma}(t)\, a_{-\mathbf{p},\sigma'}(i')]\,N+2, l, m\rangle, \tag{16}$$

where the state $|N+2, l, m\rangle$ is obtained from the ground state $|N\rangle$ of the system of N particles by adding a pair with relative momentum l and projection m.

The averages of the four electron operators are written in the following form:

$$\langle\psi\psi\psi^+\psi^+\rangle = \sum_m \langle N\,|\,\psi\psi\,|\,N+2, l, m\rangle \times \langle N+2, l, m\,|\,\psi^+\psi^+\,|\,N\rangle. \tag{17}$$

The equations of motion for the Green's functions have the form

$$\left(i\frac{d}{dt} - \zeta_p\right) g_{\sigma\sigma'}(p, t-t') + \frac{i}{(2\pi)^3}\sum_m \int d^3\mathbf{p}' V(\mathbf{p}, \mathbf{p}') \times F_{m\sigma,-\sigma}(\mathbf{p}', 0)\, F^+_{m,-\sigma,\sigma'}(\mathbf{p}, t-t') = \delta_{\sigma\sigma'}\delta(t-t'), \tag{18}$$

$$\left(i\frac{d}{dt} + \zeta_p\right) F^+_{m\sigma\sigma'}(\mathbf{p}, t-t') - \frac{i}{(2\pi)^3}\int d^3\mathbf{p}' V(\mathbf{p}, \mathbf{p}') \times F^+_{m\sigma,-\sigma}(\mathbf{p}', 0)\, g_{-\sigma,\sigma'}(p, t-t') = 0. \tag{19}$$

We can therefore readily conclude that the matrix $F^+_{m\sigma\sigma'}$ (as well as $F_{m\sigma\sigma'}$) does not have diagonal elements in the spinor indices. This means that the total-spin projection of the bound pair of the electrons is equal to zero.

Solving the system of equations for the Fourier components of the Green's functions, we obtain, as in [7], an equation for the energy gap $\Delta(p)$:

$$\Delta(p) = \frac{1}{(2\pi)^2}\int V_l(p, p')\frac{\Delta(p')}{\varepsilon(p')}\, p'^2\, dp'. \tag{20}$$

In this equation $V_l(p, p')$ is the coefficient in the expansion of the interaction "potential" in Legendre polynomials

$$V(\mathbf{p}, \mathbf{p}') = \sum_l (2l+1)\, V_l(p, p')\, P_l(\cos\hat{\mathbf{p}\mathbf{p}}'), \tag{21}$$

and $\epsilon(p) = \sqrt{\Delta^2(p) + \zeta_p^2}$ is the excitation energy.

Equation (20) has a nontrivial solution only if the quantity $V_l(p_0, p_0) \equiv V_l$ is positive on the Fermi surface ($p = p' = p_0$), meaning attraction of the electron pair in the state with momentum l. We must therefore investigate expression (15) for $V(p, p')$, putting $\zeta_p = \zeta_{p'} = 0$ and $q^2 = 2p_0^2(1+x)$, where $x = \cos\hat{\mathbf{p}\mathbf{p}}'$. The value of $P(p_0, p_0, x)$ is [1)]

$$V(p_0, p_0, x) = -2I^2/M_0^2\{[\delta + (\alpha + \alpha_{12})\, p_0^2] + (\alpha + \alpha_{12})\, p_0^2 x\}. \tag{22}$$

We make use here of the fact that the anisotropy constant β, which is determined by the relativistic interactions, is small compared with δ [13].

The coefficients V_l are determined by the integrals

$$V_l = (-1)^{l+1}\frac{1}{2^l}\frac{I^2}{bM_0^2}\int_{-1}^{+1}\frac{(1-x^2)^l}{(x+a/b)^{l+1}}\,dx, \tag{23}$$

where

$$a = \delta + (\alpha + \alpha_{12})\, p_0^2, \quad b = (\alpha + \alpha_{12})\, p_0^2. \tag{24}$$

The odd coefficients V_{2n+1} turn out to be positive, while the even ones V_{2n} are negative. This means that in the triplet state ($l = 2n+1$) the electrons attract and cannot form Cooper pairs, while in the singlet state ($l = 2n$) they are repelled. By determining the sign of the derivative $d|V_l|/dl$, we can readily find that the absolute value of V_l decreases with increasing number: $|V_l| > |V_{l+1}|$. Therefore the maximum attraction corresponds to the P-state ($l = 1$). The coefficient V_1 is equal to

$$V_1 = \frac{I^2}{M_0^2}\frac{1}{b}\left[\frac{a}{b}\ln\frac{a+b}{a-b} - 2\right]. \tag{25}$$

The kernel of the integral equation (20) decreases rapidly with increasing distance from $p = p_0$ over a distance on the order of the characteristic frequency of the spin wave. Equations of this type were investigated in [14]. The asymptotic solution (as $V_l \to 0$) has the form

$$\Delta(p) = 2\omega(p)\exp(-1/N_0V_l), \tag{26}$$

where $\omega(p)$ is a quantity that decreases with in-

[1)] We note that in expression (22) for $V(p_0, p_0, x)$ contributions are made by large spin-wave momenta $q \sim p_0$. In the theory of the thermal properties of antiferromagnets [13] the quantity $(\alpha + \alpha_{12})q^2$ is usually neglected compared with δ. If we had likewise neglected this quantity, then $V(p_0, p_0, x)$ would not depend on x and this would mean in practice the lack of interaction for electrons in pairs with nonzero momenta.

creasing $|p - p_0|$ and having at $p = p_0$ the order of magnitude of the characteristic spin-wave frequency, while $N_0 = p_0^2/2\pi^2 v_0$ is the density of the energy states on the Fermi surface. In our case $V_l = V_1$ in the exponent.

3. So far we have disregarded the influence of the phonons, which also play the role of carriers of interaction between electrons. The "potential" of this interaction, obtained from the Froehlich Hamiltonian, is

$$V_{ph}(\mathbf{p}, \mathbf{p}') = g^2\omega_q^2/[\omega_q^2 - (\zeta_p - \zeta_{p'})^2], \qquad (27)$$

where g is the constant of the coupling with the phonons and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$, while ω_q is the frequency of a phonon with momentum q. On the Fermi surface this "potential" is a constant and makes no contribution to the summary V_l when $l \neq 0$. If g^2 is sufficiently large, namely $g^2 + V_0 > V_1$ (it is assumed here that the Debye and Neel temperatures are of the same order), where

$$V_0 = -\frac{I^2}{bM_0^2}\ln\frac{a+b}{a-b}, \qquad (28)$$

then the produced Cooper pairs will be in the S-state. The isotopic spin may nevertheless be absent here if $|V_0| \gtrsim g^4 N_0$.

An account of the screened Coulomb repulsion (the role of which is analogous in the isotopic effect) can be readily made and leads to the ordinary logarithmic superconductivity criterion [2].

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[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] Bogolyubov, Tolmachev, and Shirkov, Novyĭ metod v teorii sverkhprovodimosti (New Method in Superconductivity Theory), AN SSSR, 1958.

[3] L. P. Gor'kov, JETP **34**, 735 (1958)., Soviet Phys. JETP **7**, 505 (1958).

[4] Geballe, Mathias, Hull, and Corenzwit, Phys. Rev. Lett. **6**, 275 (1961).

[5] P. W. Anderson and P. Morel, Phys. Rev. Lett. **5**, 136 (1960).

[6] P. W. Anderson and P. Morel, Phys. Rev. **123**, 1191 (1961).

[7] L. P. Gor'kov and V. M. Galitskiĭ, JETP **40**, 1124 (1961). Soviet Phys. JETP **13**, 792 (1961).

[8] Vaks, Galitskiĭ, and Larkin, JETP **42**, 1319 (1962). Soviet Phys. JETP **15**, 914 (1962).

[9] T. Kasuya, Progr. Theor. Phys. **20**, 980 (1958).

[10] S. V. Vonsovskiĭ and M. S. Svirskiĭ, DAN SSSR **122**, 204 (1958), Soviet Phys. Doklady **3**, 949 (1959); JETP **37**, 1494 (1959), **39**, 384 (1960), and **40**, 1676 (1961), Soviet Phys. JETP **10**, 1060 (1960), JETP **12**, 272 (1961), and **13**, 1182 (1961).

[11] A. I. Akhiezer and I. Ya. Pomeranchuk, JETP **36**, 859 (1959), Soviet Phys. JETP **9**, 605 (1959).

[12] S. V. Vonsovskiĭ, JETP **16**, 980 (1946).

[13] Akhiezer, Bar'yakhtar, and Kaganov, UFN **71**, 533 (1960) and **72**, 3 (1960). Soviet Phys. Uspekhi **3**, 567 and 661 (1961).

[14] D. N. Zubarev and Yu. A. Tserkovnikov, Nauchn. Dokl. vyssh.shk., ser. fiz.-mat. (Science Notes of the Universities, Phys.-Math. Series), No. 2, 133 (1959).

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