MASSES AND CHARGES OF EXCITED STATES IN THE FERMI-YANG MODEL

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Submitted to JETP editor June 11, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1885-1896 (November, 1962)

The Fermi-Yang equation with a rectangular well of range r_0 is solved for arbitrary angular momenta and parities. Equations are found for the mass eigenvalues of the first excited states with angular momentum 0, 1, 2 for $r_0M = 0.5, 1, 2, 3$; also the charges of the interaction of nucleons with these bosons are obtained. For the values of r_0 considered the density of levels turns out to be high and the charges of many excitation branches are small although they do increase exponentially with r_0. The high level density and the smallness of masses and charges of a number of branches are, apparently, characteristic of the Fermi-Yang equation and are not confirmed by the Bethe-Salpeter equation with an instantaneous interaction in the form of a rectangular well. [1]

1. INTRODUCTION

COMPOSITE models of elementary particles are widely discussed in the literature. [2-6] The interest in them has increased in connection with the discovery of a number of boson resonances; certain models have appeared classifying these resonances in terms of excited states of a baryon-antibaryon system. [5-6] However attempts at quantitative calculations with such models run into mathematical difficulties and difficulties in principle connected with the relativistic bound state problem for strong interactions. Nonetheless one might suppose that some of the most gross characteristics of the excitation spectrum, such as the relative ordering of levels with different quantum numbers, the density of levels, the order of magnitude of the charge relative to the production of such bosons, etc., could be comparatively stable characteristics of an interaction of given strength and range and not too dependent on the interaction details. Thus, in nonrelativistic quantum mechanics a rectangular well potential describes not too badly the qualitative features of quite realistic problems of nuclear physics. Therefore one might suppose that also in the Fermi-Yang model, i.e., in the relativistic problem with a rectangular well, the indicated gross features of the spectrum will not be radically distorted. Then in the absence of anything better this simple model may serve for orientation purposes to judge about the order of magnitude and extent of the "fundamental" interaction, as well as about the existence and density of resonances of given angular momentum and parity that are to be expected.

In the present work the Fermi-Yang equation is solved for arbitrary angular momenta and parities. In Sec. 2 we find the wave functions and the eigenvalue equations; in Sec. 3 we calculate the corresponding charges; the results for the angular momentum values 0, 1, 2 are given in the form of a table. The qualitative features of the results and the important question of the relation of the model to a more rigorous formulation of the problem are discussed in Sec. 4.

2. SOLUTION OF THE EQUATIONS

We begin with the Fermi-Yang equation [2]

$$[(\boldsymbol{\alpha}_p - \boldsymbol{\alpha}_a)\,\mathbf{p} + M_p\beta_p + M_a\beta_a - V\,(1 - \boldsymbol{\alpha}_p\boldsymbol{\alpha}_a)]\,\Psi = E\Psi. \tag{1}$$

Here $\alpha_{p,a}$, $\beta_{p,a}$ are Dirac matrices acting respectively on the spin indices of the particle p —the proton, and the particle a —the antineutron; V(r) is the interaction potential. Writing the 16-component wave function Ψ in the form of a matrix of 4-component entities ψ_i,

$$\psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix},$$

transforming like the product of two-component spinors p and a, we obtain for ψ_i the equations

$$\begin{aligned}
&\mathbf{s}_p\mathbf{p}\psi_3 - \mathbf{s}_a\mathbf{p}\psi_2 + (M_p + M_a)\,\psi_1 = (E + V)\,\psi_1 - V\mathbf{s}_p\mathbf{s}_a\psi_4,\\
&\mathbf{s}_p\mathbf{p}\psi_4 - \mathbf{s}_a\mathbf{p}\psi_1 + (M_p - M_a)\,\psi_2 = (E + V)\,\psi_2 - V\mathbf{s}_p\mathbf{s}_a\psi_3,\\
&\mathbf{s}_p\mathbf{p}\psi_1 - \mathbf{s}_a\mathbf{p}\psi_4 - (M_p - M_a)\,\psi_3 = (E + V)\,\psi_3 - V\mathbf{s}_p\mathbf{s}_a\psi_2,\\
&\mathbf{s}_p\mathbf{p}\psi_2 - \mathbf{s}_a\mathbf{p}\psi_3 - (M_p + M_a)\,\psi_4 = (E + V)\,\psi_4 - V\mathbf{s}_p\mathbf{s}_a\psi_1,
\end{aligned} \tag{2}$$

where $\mathbf{s}_p$, $\mathbf{s}_a$ are Pauli spin matrices acting on the spin indices of the particles p and a. Going over for convenience from the functions ψ_i to their sums and differences we obtain

$$\begin{aligned} \mathbf{sp}\zeta &= [E + V(1 + s_p s_a)]\chi - 2M\varphi, \\ \mathbf{sp}\chi &= [E + V(1 + s_p s_a)]\zeta + \Delta\eta, \\ \boldsymbol{\delta}\mathbf{p}\eta &= [E + V(1 - s_p s_a)]\varphi - 2M\chi, \\ \boldsymbol{\delta}\mathbf{p}\varphi &= [E + V(1 - s_p s_a)]\eta + \Delta\zeta; \end{aligned}$$

$$\varphi = \psi_1 + \psi_4, \quad \chi = \psi_1 - \psi_4,$$

$$\eta = \psi_2 + \psi_3, \quad \zeta = \psi_3 - \psi_2, \quad \mathbf{s} = \mathbf{s}_p + \mathbf{s}_a,$$

$$\boldsymbol{\delta} = \mathbf{s}_p - \mathbf{s}_a, \quad 2M = M_p + M_a, \quad \Delta = M_p - M_a. \tag{3}$$

We are looking for solutions with definite total angular momentum j, its projection m and parity P. Taking into account the different intrinsic parities of particle and antiparticle we obtain for the angle and spin dependence of the wave functions

$P = (-)^{j+1}$:

$$\varphi = \varphi_0\Omega_{jmj0} + \varphi_1\Omega_{jmj1}, \qquad \chi = \chi_0\Omega_{jmj0} + \chi_1\Omega_{jmj1},$$
$$\eta = \eta_+\Omega_{jmj+11} + \eta_-\Omega_{jmj-11}, \qquad \zeta = \zeta_+\Omega_{jmj+11} + \zeta_-\Omega_{jmj-11};$$

$P = (-)^j$:

$$\varphi = \varphi_+\Omega_{jmj+11} + \varphi_-\Omega_{jmj-11}, \qquad \chi = \chi_+\Omega_{jmj+11} + \chi_-\Omega_{jmj-11},$$
$$\eta = \eta_0\Omega_{jmj0} + \eta_1\Omega_{jmj1}, \qquad \zeta = \zeta_0\Omega_{jmj0} + \zeta_1\Omega_{jmj1}. \tag{4}$$

Here the functions φ_i, χ_i, η_i, ζ_i depend only on r, and the spherical spinor Ω_{jmls} is composed in the usual manner of the spherical function $Y_{lm_1}(\mathbf{n})$ and the spin function $u_{s\mu}$ of two particles of total spin s and projection μ by means of the Clebsch-Gordan coefficients $C^{jm}_{lm_1s\mu}$:

$$\Omega_{jmls}(\mathbf{n}) = C^{jm}_{lm_1s\mu} Y_{lm}(\mathbf{n}) u_{s\mu}. \tag{5}$$

For what follows it is convenient to introduce along with the $\Omega_{jmj\pm11}$ the orthonormal combinations

$$\Omega_a = j^{1/2}(2j+1)^{-1/2}\Omega_{jmj+11} + (j+1)^{1/2}(2j+1)^{-1/2}\Omega_{jmj-11},$$
$$\Omega_b = (j+1)^{1/2}(2j+1)^{-1/2}\Omega_{jmj+11} - j^{1/2}(2j+1)^{-1/2}\Omega_{jmj-11}. \tag{6}$$

At that the function f in Eqs. (4), containing $\Omega_{jmj\pm11}$, are rewritten in the form $f = f_a\Omega_a + f_b\Omega_b$, where f_a, f_b are related in an obvious manner to f_+, f_-. Then, making use of the formulas for the differentiation of the spherical functions (see, for example, [7]) and the algebra for the composition of angular momenta one can obtain the relations

$$\frac{i\mathbf{ps}}{2}\begin{cases} f\Omega_a = -(f' + f/r)\,\Omega_{jmj1}, \\ f\Omega_b = -j^{1/2}(j+1)^{1/2}\,\Omega_{jmj1}\,f/r, \\ f\Omega_{jmj1} = -(f' + f/r)\,\Omega_a + j^{1/2}(j+1)^{1/2}\,\Omega_b f/r, \\ f\Omega_{jmj0} = 0; \end{cases}$$

$$\frac{i\mathbf{p}\boldsymbol{\delta}}{2}\begin{cases} f\Omega_a = -j^{1/2}(j+1)^{1/2}\,\Omega_{jmj0}\,f/r, \\ f\Omega_b = -(f' + 2f/r)\,\Omega_{jmj0}, \\ f\Omega_{jmj1} = 0, \\ f\Omega_{jmj0} = j^{1/2}(j+1)^{1/2}\,\Omega_a f/r - f'\Omega_b. \end{cases} \tag{7}$$

Substituting the solution (4) into the Eqs. (3) and making use of the relations (7), we obtain the following equations for the radial functions:

$P = (-)^{j+1}$:

$$2i\,(\zeta_a' + \zeta_a/r + j^{1/2}(j+1)^{1/2}\zeta_b/r)$$
$$= (E + 2V)\chi_1 - 2M\varphi_1, \quad E\varphi_1 = 2M\chi_1,$$
$$2i\,(\chi_1' + \chi_1/r) = (E + 2V)\,\zeta_a + \Delta\eta_a,$$
$$-2ij^{1/2}(j+1)^{1/2}\chi_1/r = (E + 2V)\,\zeta_b + \Delta\eta_b,$$

$$2i\,(\eta_b' + 2\eta_b/r + j^{1/2}(j+1)^{1/2}\eta_a/r)$$
$$= (E + 4V)\,\varphi_0 - 2M\chi_0, \quad (E - 2V)\,\chi_0 = 2M\varphi_0,$$
$$2i\varphi_0' = E\eta_b + \Delta\zeta_b,$$
$$-2ij^{1/2}(j+1)^{1/2}\varphi_0/r = E\eta_a + \Delta\zeta_a; \tag{8a}$$

$P = (-)^j$:

$$2i\,(\zeta_1' + \zeta_1/r) = (E + 2V)\,\chi_a - 2M\varphi_a,$$
$$-2ij^{1/2}(j+1)^{1/2}\zeta_1/r = (E + 2V)\,\chi_b - 2M\varphi_b,$$

$$2i\,(\chi_a' + \chi_a/r + j^{1/2}(j+1)^{1/2}\chi_b/r)$$
$$= (E + 2V)\,\zeta_1 + \Delta\eta_1, \quad E\eta_1 = -\Delta\zeta_1,$$

$$2i\,(\varphi_b' + 2\varphi_b/r + j^{1/2}(j+1)^{1/2}\varphi_a)$$
$$= (E + 4V)\,\eta_0 + \Delta\zeta_0, \quad (E - 2V)\,\zeta_0 = -\Delta\eta_0,$$
$$2i\eta_0' = E\varphi_b - 2M\chi_b,$$
$$-2ij^{1/2}(j+1)^{1/2}\eta_0/r = E\varphi_a - 2M\chi_a. \tag{8b}$$

In the following we consider only the case $M_p = M_a$, $\Delta = 0$. Then, as a consequence of the resultant "charge" symmetry under the exchange of the particles p → a, the spin s (of the "large" components φ and χ) becomes a quantum number, and in Eq. (8a) the equations for φ_0, χ_0, η separate from the equations for φ_1, χ_1, ζ. In Eqs. (8b) η_1 and ζ_0 vanish when $\Delta = 0$. If we further go from an arbitrary potential V(r) to the case of a rectangular well potential

$$V(r) = \begin{cases} V_0, & r < r_0 \\ 0, & r > r_0 \end{cases}, \tag{9}$$

then, as is easy to verify, in the system of equations (8b) the equations containing η_0 also separate from the equations for ζ_1.

The resultant system of equations should be augmented by the condition of continuity at the point r_0. It is obtained directly by integrating equations (8) over a small neighborhood of $r = r_0$: the functions, whose derivatives enter into these equations, should be continuous. As a result we arrive at the following equations and matching conditions:

$$P = (-)^{j+1}, \quad s = 0:$$

$$\varphi_0'' + \frac{2}{r}\varphi_0' - \frac{j(j+1)}{r^2}\varphi_0 + E\left(V + \frac{E}{4} + \frac{M^2}{2V - E}\right)\varphi_0 = 0, \qquad (10a)$$

φ_0 and $\eta_b = 2i\varphi_0'/E$ should be continuous;

$$P = (-)^{j+1}, \quad s = 1:$$

$$\chi_1'' + \frac{2}{r}\chi_1' - \frac{j(j+1)}{r^2}\chi_1 + \left(V + \frac{E}{2}\right)\left(V + \frac{E}{2} - \frac{2M^2}{E}\right)\chi_1 = 0, \qquad (10b)$$

χ_1 and $\zeta_a = 2i(\chi_1 + \chi_1/r)(E + 2V)^{-1}$ should be continuous;

$$P = (-)^j, \quad s = 1:$$

$$\eta_0'' + \frac{2}{r}\eta_0' - \frac{j(j+1)}{r^2}\eta_0 + \left(V + \frac{E}{4}\right)\left(E - \frac{4M^2}{2V + E}\right)\eta_0 = 0,$$

$$\zeta_1'' + \frac{2}{r}\zeta_1' - \frac{j(j+1)}{r^2}\zeta_1 + \left(V + \frac{E}{2}\right)\left(V + \frac{E}{2} - \frac{2M^2}{E}\right)\zeta_1 = 0, \qquad (10c)$$

and the following

$$\eta_0, \zeta_1, \varphi_b = 2i\frac{(2V + E)\eta_0' - 2Mj^{1/2}(j+1)^{1/2}\zeta_1/r}{E^2 + 2EV - 4M^2},$$

$$\chi_a = 2i\frac{E(\zeta_1' + \zeta_1/r) - 2Mj^{1/2}(j+1)^{1/2}\eta_0'}{E^2 + 2EV - 4M^2}$$

should be continuous.

In spite of the fact that Eqs. (10c) for η_0 and ζ_1 are uncoupled, the complete solution turns out to be in this case a certain definite superposition of the particular solutions because of the "mixing" resulting from the matching condition.

As is well known, the solutions of Eqs. (10) are the spherical Bessel functions

$$f_i = \begin{cases} \dfrac{\text{const}}{r^{1/2}} J_{j+1/2}(k_i r), & r < r_0 \qquad (11a) \\ \dfrac{\text{const}'}{r^{1/2}} K_{j+1/2}(\varkappa r), & r > r_0, \qquad (11b) \end{cases}$$

where $\kappa^2 = M^2 - E^2/4$, and the quantities k_i^2 are equal to the coefficients of f_i in the last terms on the left hand sides of Eqs. (10). Substituting these solutions into the matching conditions we arrive at the following equations for the eigenvalues of E, i.e., for the masses of the composite bosons μ:

$$P = (-)^{j+1}, s = 0: \qquad R_j(k_1 r_0) = -Q_j(\varkappa r_0). \qquad (12a)$$

$$P = (-)^{j+1}, \quad s = 1:$$

$$(E + 2V_0)^{-1}[ER_j(k_2 r_0) + 2jV_0] = -Q_j(\varkappa r_0). \qquad (12b)$$

Here

$$R_j(z) = zJ_{j-1/2}(z)/J_{j+1/2}(z), \qquad Q_j(z) = zK_{j-1/2}(z)/K_{j+1/2}(z),$$

$$k_1^2 = E(E/4 + V_0 + M^2/(2V_0 - E)),$$

$$k_2^2 = (E/2 + V_0)(E/2 + V_0 - 2M^2/E). \qquad (13)$$

For parity $P = (-)^j$ we obtain from Eq. (10c) the system of equations

$$(\varkappa^2 - V_0E/2)^{-1}\left[\varkappa^2\left(1 + \frac{2V_0}{E}\right)R_j(k_3 r_0) - V_0 M\left(2\frac{j+1}{E}M + \frac{\sqrt{j(j+1)}}{\xi_j}\right)\right] = -Q_j(\varkappa r_0),$$

$$(\varkappa^2 - V_0E/2)^{-1}[\varkappa^2 R_j(k_2 r_0) - V_0(jE/2 + \xi_j\sqrt{j(j+1)}M)] = -Q_j(\varkappa r_0), \qquad (14)$$

where

$$k_3^2 = (E + 4V_0)[E/4 - M^2/(2V_0 + E)],$$

$$\xi_j = \eta_0(r_0)/\zeta_1(r_0).$$

After elimination of the parameter ξ_j we obtain for the eigenvalues of E the equation

$$P = (-)^j, \quad s = 1:$$

$$jV_0E\left[\varkappa^2(E + 2V_0)R_j(k_3 r_0) + E\left(\varkappa^2 - \frac{V_0E}{2}\right)Q_j(\varkappa r_0)\right] = 2\left[\varkappa^2 R_j(k_2 r_0) + \left(\varkappa^2 - \frac{V_0E}{2}\right)Q_j(\varkappa r_0)\right] \times \left[\varkappa^2(E + 2V_0)R_j(k_3 r_0) + E\left(\varkappa^2 - \frac{V_0E}{2}\right)Q_j(\varkappa r_0) - 2V_0(j+1)\right]. \qquad (15a)$$

For $j = 0$ (scalar) the function $\zeta_1 = 0$ and the eigenvalues are determined by the first of the Eqs. (14):

$$P = +1, \ j = 0, s = 1: \left(\varkappa^2 - \frac{V_0E}{2}\right)^{-1} \times \left[\varkappa^2\left(1 + \frac{2V_0}{E}\right)R_0(k_3 r_0) - \frac{2V_0}{E}\right] = -\varkappa r_0. \qquad (15b)$$

For small $V_0/M \ll 1$ the quantity $E/2M \to 1$ and the Eqs. (12), (15) go over into the usual non-relativistic equations for a rectangular well, with Eq. (15a) becoming the product of the equations for $l = j - 1$ and $l = j + 1$ corresponding to the disappearance of spin-orbit coupling in this limit.

Equations (12), (15) were solved numerically for several values of r_0 with the well depth V_0 fixed by the requirement that the mass of the lowest pseudoscalar state be equal to the mass of the pion: $\mu_\pi/2M = 0.0743$.[1)]

[1)]The values of $\mu_\pi/2M$ and, correspondingly, of V_0 for $r_0M = 1$ differ by a few percent from those used by Fermi and Yang[2] corresponding to the more precise experimental value of μ_π.

r_0M		0,5			1			2			3		
$V_0'M$		90.7			27.6			8.71			4.35		
${}^{2s+1}L_J$	Principal quantum number	$\mu/2M$	$g^2/4\pi$	r	$\mu/2M$	$g^2/4\pi$	r	$\mu/2M$	$g^2/4\pi$	r	$\mu/2M$	$g^2/4\pi$	r
${}^1S_0$ pseudoscalar	1	0.0743	0,19		0.0743	0.71		0.0743	5.7		0.0743	39	
	2	0.507	0.20		0.430	0.80		0.358	7.5		0,325	57	
${}^1P_1$ pseudovector	1	0.224	0,015		0.194	0.10		0.170	1,2		0.160	9.6	
${}^1D_2$ pseudotensor	1	0,447	0.0031		0.372	0.044		0.305	0.87		0.276	8.7	
${}^3P_1$ pseudovector	1	0,011133	$2.2\cdot10^{-12}$		0.0371	$1.7\cdot10^{-8}$		0.121	$1.7\cdot10^{-4}$		0.243	0.058	
	2	0.011135	$7.3\cdot10^{-12}$		0.0392	$6.6\cdot10^{-8}$		0.139	$1.0\cdot10^{-3}$		0.303	0.44	
${}^3D_2$ pseudotensor	1	0.0112	$7.1\cdot10^{-17}$		0.0378	$2.2\cdot10^{-11}$		0.127	$1.0\cdot10^{-5}$		0.261	0.030	
${}^3P_0$ scalar	1	0,440	$1.1\cdot10^{-3}$	—	0.388	0.024	—	0.384	0.17	—	0.443	1.7	—
${}^3(S+D)_1$ vector	1	0.0112	$1.3\cdot10^{-7}$	0.022	0.0378	$4.1\cdot10^{-5}$	0.076	0.126	0.019	0.27	0.252	0.47	0,73
	2	0.0115	$3.6\cdot10^{-7}$	0.023	0.0405	$1.2\cdot10^{-4}$	0,081	0.148	0,052	0.33	0.284	1.1	0.38
${}^3(P+F)_2$ tensor	1	0.0113	$7.9\cdot10^{-12}$	0,023	0.0386	$9.5\cdot10^{-7}$	0,077	0.133	$1.5\cdot10^{-3}$	0,28	0.276	0.42	0.72

Results for low states with angular momentum $j = 0, 1, 2$ are shown in the table; they are rather unexpected.

1) The masses of the vector, tensor, pseudovector, and pseudotensor with spin $s = 1$ turn out to be smaller than the mass of the "pion" if one uses the value $r_0M = 1$ as assumed by Fermi and Yang; these masses rapidly approach zero if r_0 is further decreased. If it is desired that the "pion" be the lightest one must increase r_0 to $r_0M \gtrsim 2$ and decrease correspondingly the well depth V_0.

2) For all values of r_0 considered the levels for the pseudovector and pseudotensor with $s = 1$, as well as for the vector and tensor, are rather dense although the distance between levels increases with increasing r_0. As examples we show in the table two levels differing only in the principal quantum number for the pseudovector with $s = 1$ and for the vector.

3) For all r_0 the lowest pseudovector is lighter than the vector.

Let us note that all these peculiarities are connected with the steep dependence in (12b), (15a) of the function $k_2(E)$, given by Eq. (13), on E. The expression under the square-root sign contains $-2V_0M^2/E$ and turns out to be a very steep function of energy in the region under consideration of large V_0/M and small $E/2M$. We shall return to this point in Sec. 4.

3. CALCULATION OF THE CHARGE OF INTERACTION WITH THE COMPOSITE BOSONS

By the charge for the interaction with the composite boson we shall mean the residue at the pole of the amplitude for scattering of a "proton" by an "antineutron," obtained by analytic continuation of the appropriate partial wave amplitude into the region $E < 2M$. The normalization coefficients will be determined by equating this pole term to the Born term, obtained by introducing phenomenologically the corresponding boson-baryon coupling into the interaction Lagrangian.

The spinor scattering amplitude $G(n)$ is defined, as usual, as the coefficient of the outgoing wave in the asymptotic expression for the "large" component of the wave function:

$$(\psi_1)^{as}_{s\mu} = \tfrac{1}{2}(\varphi + \chi)^{as}_{s\mu}$$

$$= \delta_{s\mu,\, s_0\mu_0}\exp(i\mathbf{k}_0\mathbf{r}) + G_{s\mu,\, s_0\mu_0}(\mathbf{n})\exp(ikr)/r. \qquad (16)$$

Here s_0 and μ_0 are the values of the total spin and its projection for the incident wave, s and μ are the corresponding quantities for the scattered wave. For the equal mass $M_p = M_a$ case being considered the total spin is a constant of the motion so that $s = s_0$.

For the amplitude $G_{s\mu, s_0\mu_0}$ we make use of the formulas of the scattering phase shift theory for spin ½ particles:[8,9]

$$G_{00,00}(\mathbf{n}) = 4\pi \sum_{jm} \alpha_{jj0} \Omega_{jmj0}(\mathbf{n}) \Omega^*_{jmj0}(\mathbf{n}_0),$$

$$G_{1\mu,1\mu_0}(\mathbf{n}) = G^a_{1\mu,1\mu_0}(\mathbf{n}) + G^b_{1\mu,1\mu_0}(\mathbf{n}); \tag{17a}$$

$$G^a_{1\mu,1\mu_0}(\mathbf{n}) = 4\pi \sum_{jm} \alpha_{jj1} [\Omega_{jmj1}(\mathbf{n}) \Omega^*_{jmj1}(\mathbf{n}_0)]_{\mu\mu_0}, \tag{17b}$$

$$G^b_{1\mu,1\mu_0}(\mathbf{n}) = 4\pi \sum_{jm} (\Omega_{jmj-11}(\mathbf{n}) [(\alpha_j \cos^2 \varepsilon_j + \beta_j \sin^2 \varepsilon_j) \times \Omega^*_{jmj-11}(\mathbf{n}_0) + \sin \varepsilon_j \cos \varepsilon_j (\beta_j - \alpha_j) \Omega^*_{jmj+11}(\mathbf{n}_0)] + \Omega_{jmj+11}(\mathbf{n}) [\sin \varepsilon_j \cos \varepsilon_j (\beta_j - \alpha_j) \Omega^*_{jmj-11}(\mathbf{n}_0) + (\alpha_j \sin^2 \varepsilon_j + \beta_j \cos^2 \varepsilon_j) \Omega^*_{jmj+11}(\mathbf{n}_0)])_{\mu\mu_0}. \tag{17c}$$

For notational convenience the amplitude $G_{1\mu,1\mu_0}$ in Eq. (17) is broken up into a sum of terms G^a and G^b, combining respectively harmonics of parity $P = (-)^{j+1}$ and $P = (-)^j$. At that α_{jj0}, α_{jj1} are the conventional partial amplitudes for the given spin and parity, and α_j and β_j are the same amplitudes for the two orthogonal solutions of "type α" and "type β" of Blatt and Biedenharn,[8] corresponding to parity $P = (-)^j$ with "mixing parameter" ϵ_j:

$$\alpha_{jj0} = \frac{\exp(2i\delta_{j0}) - 1}{2ik}, \quad \alpha_{jj1} = \frac{\exp(2i\delta_{j1}) - 1}{2ik}, \quad \alpha_j = \frac{\exp(2i\delta^\alpha_j) - 1}{2ik},$$
$$\beta_j = \frac{\exp(2i\delta^\beta_j) - 1}{2ik}. \tag{18}$$

The phase shifts δ_j are found by solving Eq. (10) for $E > 2M$ subject to the boundary condition of a standing wave at infinity.

In the following we are interested in the amplitudes α, β only at the pole points $E = \mu_i$, corresponding to bound states. The corresponding residues turn out to be simply related to the left parts ω_j of the Eqs. (12) and (14) determining the eigenvalues:

$$\frac{\exp(2i\delta_j) - 1}{2ik} \underset{E \to \mu}{\longrightarrow} (-)^j \frac{\pi}{2\varkappa} \times \left[(E - \mu) K^2_{j+1/2}(\varkappa r_0) \frac{d}{dE} (\omega_j + Q_j(\varkappa r_0)) \big|_{E=\mu} \right]^{-1}. \tag{19}$$

The derivation of Eq. (19) is given in the Appendix. Thus, for known ω_j the problem reduces to the calculation of the Born terms and their transformation into the appropriate term of Eq. (17).

The phenomenological Lagrangian for the interaction of the fermion fields p and n with the boson field Φ_i (where the field Φ_i may be scalar, vector, etc.) will be written in the form

$$L_{int} = g(\bar{\psi}_n O_i \psi_p \Phi_i + \text{H. c.}); \tag{20}$$

the operator O_i contains γ matrices and differential operators acting on ψ_p and $\bar{\psi}_n$. Since in the preceding the state of the antiparticle was described throughout in terms of an ordinary wave function one must before comparing with Eq. (17) go over from $\bar{\psi}_n$ to the charge-conjugate field ψ_a:

$$\psi_n = C\bar{\psi}_a, \qquad \bar{\psi}_n = C^+\psi_a = C\psi_a, \qquad C = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}. \tag{21}$$

Then the Born term, describing the scattering of the particles p and a from the states 1 and 2 respectively into the states 1′ and 2′, is of the following form in the barycentric system

$$-\frac{g^2}{4\pi} (\bar{\psi}_p O_i C \bar{\psi}_a)_{1'2'} (\psi_a C O_l \psi_p)_{21} \frac{E}{2} \frac{d_{il}(p_1 + p_2)}{\mu^2 + (p_1 + p_2)^2}. \tag{22}$$

Here E is the total energy; $p_1 + p_2$ is the total momentum four-vector with components $(0, E)$; $d_{il}(p)/(p^2 + \mu^2)$ is the Green's function of the boson under consideration. The quantities d_{il} are of the following form depending on the boson spin (see, for example, [10])

$$j = 0:\ d = 1; \quad j = 1:\ d_{il}(p) = \delta_{il} + p_i p_l/\mu^2 \equiv h_{il}(p);$$
$$j = 2:\ d_{ik,lm} = \tfrac{1}{2}[h_{il}h_{km} + h_{im}h_{kl} - \tfrac{2}{3} h_{ik}h_{lm}]. \tag{23}$$

At the pole point of interest $E = \mu$, so that the quantity h_{il} is equal to δ_{il} for spatial i, l, and is equal to zero for the remaining cases. The Dirac spinors have the usual form

$$\psi(\mathbf{p}) = (\varepsilon + M)^{1/2} (2\varepsilon)^{-1/2} (u, \boldsymbol{\sigma}\mathbf{p}(\varepsilon + M)^{-1} u),$$

where $\epsilon = E/2$ and u is the unit spinor.

From Eqs. (17)—(23), using standard procedures for composition of angular momenta and using the formulas relating the spin matrices to the Clebsch-Gordan coefficients

$$(\sigma_y)_{\alpha\beta} = -i\sqrt{2}\, C^{00}_{1/2\alpha,\,1/2\beta}, \qquad (\sigma_y \sigma_\mu)_{\alpha\beta} = i\sqrt{2}\,(-)^\mu C^{1-\mu}_{1/2\alpha,\,1/2\beta},$$
$$(\sigma_\mu \sigma_y)_{\alpha\beta} = -i\sqrt{2}\, C^{1\mu}_{1/2\alpha,\,1/2\beta}.$$

(where σ_μ are the spherical components of the vector σ), we obtain the following expressions for the charges $g^2/4\pi$ (the prime on the brackets denotes differentiation with respect to E):

1) $s = 0,\ j = 0$: $\quad L_{int} = g_{PS}(i\bar{\psi}_n \gamma_5 \psi_p \Phi + \text{H.c.}),$

$$\frac{g^2_{PS}}{4\pi} = -\frac{2\exp(2\varkappa r_0)}{[k_1 \operatorname{ctg} k_1 r_0 + \varkappa]'},$$

$j = 1$: $\quad L_{int} = \dfrac{g_{PV}}{2\mu}\left[\left(\bar{\psi}_n \gamma_5 \dfrac{\partial \psi_p}{\partial x_l} - \dfrac{\partial \bar{\psi}_n}{\partial x_l} \gamma_5 \psi_p\right)\Phi_l + \text{H.c.}\right],$

$$\frac{g^2_{PV}}{4\pi} = -\frac{6\mu^2 r_0^3 \exp(2\varkappa r_0)}{(\varkappa r_0 + 1)^2 [R_1(k_1 r_0) + Q_1(\varkappa r_0)]'}; \tag{24a}*$$

*ctg = cot.

$$j=2:\quad L_{int}=\frac{g_{PT}}{4\mu^2}\Big[i\Big(\bar\psi_n\gamma_5\frac{\partial^2\psi_p}{\partial x_l\partial x_m}+\frac{\partial^2\bar\psi_n}{\partial x_l\partial x_m}\gamma_5\psi_p-\frac{\partial\bar\psi_n}{\partial x_l}\gamma_5\frac{\partial\psi_p}{\partial x_m}-\frac{\partial\bar\psi_n}{\partial x_m}\gamma_5\frac{\partial\psi_p}{\partial x_l}\Big)\Phi_{lm}+\text{H.c.}\Big],$$

$$\frac{g_{PT}^2}{4\pi}=-\frac{15\mu^4 r_0^5\exp(2\varkappa r_0)}{(\varkappa^2r_0^2+3\varkappa r_0+3)^2\,[R_2(k_1r_0)+Q_2(\varkappa r_0)]'}.$$

2) $s=1,\quad P=(-)^{j+1}$;

$$j=1:\quad L_{int}=g_{PV}\,[i\bar\psi_n\gamma_l\gamma_5\psi_p\Phi_l+\text{H.c.}],$$

$$\frac{g_{PV}^2}{4\pi}=-\frac{3\mu^2r_0^3\exp(2\varkappa r_0)}{4(1+\varkappa r_0)^2[(ER_1(k_2r_0)+2V_0)/(2V_0+E)+Q_1(\varkappa r_0)]'};$$

$$j=2:\quad L_{int}=\frac{g_{PT}}{4\mu}\Big[\Big(\bar\psi_n\gamma_l\gamma_5\frac{\partial\psi_p}{\partial x_m}-\frac{\partial\bar\psi_n}{\partial x_m}\gamma_l\gamma_5\psi_p+\bar\psi_n\gamma_m\gamma_5\frac{\partial\psi_p}{\partial x_l}-\frac{\partial\bar\psi_n}{\partial x_l}\gamma_m\gamma_5\psi_p\Big)\Phi_{ml}+\text{H.c.}\Big],$$

$$\frac{g_{PT}^2}{4\pi}=-\frac{5}{2}\,\frac{\mu^4r_0^5\exp(2\varkappa r_0)}{(\varkappa^2r_0^2+3\varkappa r_0+3)^2\,[(ER_2(k_2r_0)+4V_0)/(2V_0+E)+Q_2(\varkappa r_0)]'}.\tag{24b}$$

For parity $P=(-)^j$ the charge can be calculated relatively simply for the scalar $j=0$ case. At that the amplitude has the single term form

$$C_{1\mu,1\mu_0}=\frac{\exp(2i\delta_0)-1}{2ik}4\pi\,(\Omega_{00\,11}(\mathbf{n})\,\Omega^*_{0011}(\mathbf{n}_0))_{\mu\mu_0},$$

so that analogously to the preceding we obtain

$$P=+1,\ j=0:\ L_{int}=g_S(\bar\psi_n\psi_p\Phi+\text{H.c.}),$$

$$\frac{g_S^2}{4\pi}=\frac{\mu^2r_0\exp(2\varkappa r_0)}{2\varkappa^2[\omega_0(E)+\varkappa r_0]'},\tag{25a}$$

where $\omega_0(E)$ stands for the left side of Eq. (15b). In the general case $j=0$ we make use of Eq. (17c). At the pole point one of the partial amplitudes, for example α_j, becomes infinite so that all the remaining terms may be ignored and the amplitude becomes

$$G^b_{1\mu,1\mu_0}=\alpha_j4\pi\sum_m[(\Omega_{m-11}(\mathbf{n})\cos\varepsilon_j-\Omega_{jmj+11}(\mathbf{n})\sin\varepsilon_j)\times(\Omega^*_{jmj-11}(\mathbf{n}_0)\cos\varepsilon_j-\Omega^*_{jmj+11}(\mathbf{n}_0)\sin\varepsilon_j)]_{\mu\mu_0}.\tag{26}$$

Into the Eq. (19) for the pole term α_j enters the function ω_j, equal to the left hand side of Eq. (14) and containing the parameter ξ_j. It can be shown that $\xi_j(E)$ can be obtained by setting the left sides of Eqs. (14) equal to each other:

$$\Big(\xi_j-\frac{1}{\xi_j}\Big)j^{1/2}(j+1)^{1/2}MV_0=\varkappa^2\Big[R_j(k_2r_0)-\Big(1+\frac{2V_0}{E}\Big)R_j(k_3r_0)\Big]+V_0\Big(2\frac{(j+1)M^2}{E}-j\frac{E}{2}\Big),\tag{27}$$

and that the ξ_j obtained in this way is related to the mixing parameter ϵ_j by

$$\xi_j=-\operatorname{tg}(\varepsilon_j+\gamma_j),\qquad \operatorname{tg}\gamma_j\equiv j^{1/2}(j+1)^{-1/2}.\tag{28*}$$

In this manner the expression (26) is determined with the help of Eqs. (19), (14), (27), (28). In writing the Born term (22) it should be borne in mind that there always correspond here to a given angular momentum and parity two combinations of Dirac spinors [for example for $j=1$: $i\bar\psi_n\gamma_l\psi_p$ and $i\bar\psi_n\partial\psi_p/\partial x_l-i(\partial\bar\psi_n/\partial x_l)\psi_p$], so that the ratio r_j of the coefficients of this superposition at the pole is determined by the dynamics, in particular by the values of the parameters ϵ_j or ξ_j.

Taking these considerations into account we find by the previous method

$P=(-)^j,\ j=1$:

$$L_{int}=g_V\Big\{i\Big[\bar\psi_n\gamma_l\psi_p+(r_V/2\mu)\Big(\bar\psi_n\frac{\partial\psi_p}{\partial x_l}-\frac{\partial\bar\psi_n}{\partial x_l}\psi_p\Big)\Big]\Phi_l+\text{H.c.}\Big\},$$

$$\frac{g_V^2}{4\pi}=\frac{2\varkappa^2r_0^3\exp(2\varkappa r_0)\,(3/2)^{1/2}(\xi_1^2+1)^{-1/2}}{(\varkappa r_0+1)^2[\omega_1(E)+Q_1(\varkappa r_0)]'},$$

$$r_V=\frac{\mu M}{\varkappa^2}\Big(1+2^{-1/2}\xi_1\frac{\mu}{M}\Big);$$

$$j=2:\quad L_{int}=\frac{g_T}{4\mu}\Big\{\Big[\bar\psi_n\gamma_i\frac{\partial\psi_p}{\partial x_l}-\frac{\partial\bar\psi_n}{\partial x_l}\gamma_i\psi_p+\bar\psi_n\gamma_l\frac{\partial\psi_p}{\partial x_i}-\frac{\partial\bar\psi_n}{\partial x_i}\gamma_l\psi_p+\frac{r_T}{\mu}\Big(\bar\psi_n\frac{\partial^2\psi_p}{\partial x_i\partial x_l}-\frac{\partial\bar\psi_n}{\partial x_i}\frac{\partial\psi_p}{\partial x_l}-\frac{\partial\bar\psi_n}{\partial x_l}\frac{\partial\psi_p}{\partial x_i}+\frac{\partial^2\bar\psi_n}{\partial x_i\partial x_l}\psi_p\Big)\Big]\Phi_{il}+\text{H.c.}\Big\},$$

$$\frac{g_T^2}{4\pi}=\frac{6\mu^2\varkappa^2r_0^5\exp(2\varkappa r_0)\,(5/3)^{1/2}(\xi_2^2+1)^{-1/2}}{(\varkappa^2r_0^2+3\varkappa r_0+3)^2[\omega_2(E)+Q_2(\varkappa r_0)]'},$$

$$r_T=\frac{\mu M}{\varkappa^2}\Big(1+\xi_2 3^{1/2}2^{-3/2}\frac{\mu}{M}\Big).\tag{25b}$$

The expressions for $d\omega_j/dE$ in the denominators in Eqs. (24), (25) may be further simplified by making use of Eqs. (12)–(15), (27), etc.

Numerical values of the quantity $g^2/4\pi$ are given in the table along with the corresponding values of the masses. The following peculiarities are noticeable.

1. The charges increase steeply with increasing r_0. For $s=0$ and for the scalar this increase is essentially determined by the exponential $\exp(2\kappa r_0)$; for $s=1$ and $j\neq0$, in addition, as r_0 increases μ increases and the quantity $dk_2^2/dE\sim V_0M/\mu^2$ decreases. Large values of this quantity at small r_0 lead to extremely small values of the charge.

*tg = tan

2. The charges of spin $s = 0$ bosons for all r_0 are many times larger than for $s = 1$ and increase with the principal quantum number.

3. For a given s the charge decreases with increasing angular momentum j, so that the charge of the pseudoscalar turns out to be largest. The latter is of the order $g^2_{PS}/4\pi \sim 0.1-1$ for $r_0M \sim 0.5-1$, and reaches values $g^2_{PS}/4\pi \sim 10$ for $r_0M \sim 2-3$.

4. DISCUSSION OF RESULTS

In the framework of field theory the bound state problem is formulated in terms of the integral Bethe-Salpeter equation with a kernel representing the sum of all irreducible diagrams. For strong interactions such an equation is at this time symbolic. For the model purposes discussed in the Introduction one might attempt to approximate the properties of the kernel of this equation in some simple way, for example by replacing the sum of all diagrams by one "intermediate boson" line, and try to solve the resultant equation without assuming the coupling constant to be small. Even this problem poses great mathematical difficulties, [11,13] connected in part with the singular nature of the equation, i.e., with the usual divergences of field theory (Goldstein, [11] Polubarinov—see [14]). The problem may be further schematized by replacing the kernel by an instantaneous potential interaction of one or another form. [1] And, finally, as the last step one could go over from this extremely simplified but nevertheless still integral equation, which takes into account the vacuum of the spinor particles, to the differential equation of Breit with a potential V(r), for example the Fermi-Yang equation with a rectangular well.

The present work was undertaken in the hope that there exist certain features of the excitation spectrum that are determined, roughly speaking, by the relativistic kinematics, and that these features would "survive" all the indicated simplifications. The results, given by Eqs. (12)—(15), (24)—(25), apparently fail to confirm this hope. The study of the next step, i.e., of the Bethe-Salpeter equation with an instantaneous interaction in the form of a rectangular well, [1] does not confirm the main results of the Fermi-Yang equation, in particular the singularities of the type V_0/E in Eqs. (12)—(15) for $s = 1$ and the connected with it properties of the excitation spectrum—density of levels, small masses of these branches, etc. Only for large $r_0M > 1$ certain general results are preserved, as for example the exponential growth of the charge with increasing interaction range, the existence for $s = 0$ of an axial vector whose mass is not too far from the mass of the pseudoscalar, etc.

The author is most grateful to A. I. Larkin for numerous suggestions, to V. M. Galitskiĭ for valuable remarks, and to I. Yu. Kobzarev and L. B. Okun' for interesting discussions and cooperation which helped to carry this work to completion.

APPENDIX

DERIVATION OF THE FORMULA FOR THE POLE TERM IN THE AMPLITUDE

The condition for the matching of the outside wave function $f_{out}(r)$ in the region $r > r_0$ with the function in the region $r < r_0$ is, according to Eq. (10), of the form

$$\frac{rf'_{out}(r_0)}{f_{out}(r_0)} = \omega_j(r_0, E) - j - 1, \tag{A.1}$$

where the function ω_j is determined by the internal region and the term $j+1$ is separated out for notational convenience. For bound states f_{out} is given by Eq. (11b) and the relation (A.1) gives the eigenvalue equations (12)—(14):

$$\omega_j = -Q_j(\varkappa r_0) \tag{A.2}$$

with Q_j defined by Eq. (13). For the continuum spectrum f_{out} is of the form

$$f_{out}(r) = \frac{\text{const}}{\sqrt{kr}}(\cos\delta J_{j+1/2}(kr) - \sin\delta N_{j+1/2}(kr)) \tag{A.3}$$

(N being the Neumann function), so that the matching condition (A.1) determines $\tan\delta$. From Eqs. (A.1) and (A.3) we obtain for a given partial wave amplitude

$$\frac{\exp(2i\delta)-1}{2ik} = \frac{i}{k}\,\frac{\omega_j J_{j+1/2}(kr_0) - kr_0 J_{j-1/2}(kr_0)}{H^1_{j+1/2}(kr_0)\,[\omega_j - kr_0 H^1_{j-1/2}(kr_0)/H^1_{j+1/2}(kr_0)]}, \tag{A.4}$$

where $H^1_p(z) = J_p(z) + iN_p(z)$. For the analytic continuation of Eq. (A.4) into the region $E < 2M$ we note that $k \to i(M^2 - E^2/4)^{1/4} = i\varkappa$, $H^1_p(kr_0) \to 2\pi^{-1}(-i)^{p+1}K_p(\varkappa r_0)$, so that the expression in the square brackets in the denominator of (A.4) goes over into $\omega_j + Q_j$, and the poles of Eq. (A.4) are determined by Eq. (A.2), as they should be.

To obtain the residue we replace in the numerator of Eq. (A.4) ω_j by $-Q_j$:

$$\omega_j J_{j+1/2}(kr_0) - kr_0 J_{j-1/2}(kr_0) = \frac{ikr_0}{H^1_{j+1/2}(kr_0)}[N_{j-1/2}(kr_0)J_{j+1/2}(kr_0) - N_{j+1/2}(kr_0)J_{j-1/2}(kr_0)] = \frac{2i}{\pi H^1_{j+1/2}(kr_0)}, \tag{A.5}$$

where we have made use of a formula from the theory of Bessel functions (see, for example, [15]). As a result we find

$$\frac{\exp(2i\delta)-1}{2ik}\xrightarrow[E\to\mu]{}\frac{(-)^j\pi}{2\varkappa(K_{j+1/2}(\varkappa r_0))^2(E-\mu)\,d[\omega_j+Q_j(\varkappa r_0)]/dE|_{E=\mu}}. \quad \text{(A.6)}$$

Let us note that in the derivation of Eq. (A.6) we have not made use of any specific properties of this problem so that this equation is, generally speaking, exact and applicable to realistic systems provided that the radius r_0 is chosen outside the interaction region so that the wave function has the form (A.3). Of course, the formula is of practical value only if some properties of the logarithmic derivative $\omega_j(E, r_0) - j - 1$ are known, for example in the case of an interaction that falls off sharply for $r > r_0$, etc.

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Translated by A. M. Bincer
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