HEAT ANOMALY OF SUPERCONDUCTORS

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The change of the phonon dispersion law at the superconducting transition is calculated. It is shown that this change cannot account for the experimentally observed anomaly of the lattice specific heat of superconductors.

THE present work is devoted to an attempt to obtain from the microscopic theory of superconductivity an explanation for the anomaly of the lattice specific heat of superconductors. According to Ferrell [1] the reason for such an anomaly is the change in the law of phonon dispersion during the superconducting transition1). In the cited paper, Ferrell estimated the magnitude of the effect on the basis of qualitative considerations based on the presence of a gap in the spectrum of the elementary excitations (electrons), and found it to be in good agreement with at least some of the experimental data [22]. We shall show, nonetheless, that a rigorous calculation leads to the deduction that the effect does not exist in the approximation considered.

Using the results of Pokrovskiĭ [3], we write the following expression for the polarization operator of the phonon in the superconductor (we are using a system of units with $\hbar = m = 1$) 2):

$$\Pi = \Pi_1 + \Pi_2; \tag{1}$$

$$\Pi_1(q, \omega) = \frac{1}{(2\pi)^2 q} \int_{-\infty}^{\infty} d\xi \int_{\xi - p_0 q}^{\xi + p_0 q} d\xi_1 \times \left(\frac{u_\xi^2 v_{\xi_1}^2}{\varepsilon_\xi + \varepsilon_{\xi_1} - \omega - i\delta} + \frac{v_\xi^2 u_{\xi_1}^2}{\varepsilon_\xi + \varepsilon_{\xi_1} + \omega - i\delta} \right), \tag{2}$$

$$\Pi_2(q, \omega) = \frac{\Delta^2}{4(2\pi)^2 q} \int_{-\infty}^{\infty} d\xi \int_{\xi - p_0 q}^{\xi + p_0 q} d\xi_1 \frac{1}{\varepsilon_\xi \varepsilon_{\xi_1}} \times \left(\frac{1}{\varepsilon_\xi + \varepsilon_{\xi_1} - \omega - i\delta} + \frac{1}{\varepsilon_\xi + \varepsilon_{\xi_1} + \omega - i\delta} \right); \tag{3}$$

$$\varepsilon_\xi = \sqrt{\Delta^2 + \xi^2},$$
$$u_\xi^2 = \tfrac{1}{2}(1 + \xi/\varepsilon_\xi), \qquad v_\xi^2 = \tfrac{1}{2}(1 - \xi/\varepsilon_\xi) \tag{4}$$

(Δ is the gap; **q** and ω are the momentum and frequency of the phonons; $\delta = +0$).

In the foregoing formulas Π_1 and Π_2 correspond to the contributions of diagrams 1 and 2 respectively (the solid line in Fig. 1 corresponds to $G(\mathbf{p}, \epsilon)$, while the solid line in Fig. 2 corresponds to the function $F(\mathbf{p}, \epsilon)$, where G and F are the functions involved in the Gor'kov theory [5]). We note that for a normal metal the contribution of the diagram of Fig. 2 is generally nil, since it vanishes together with Δ.

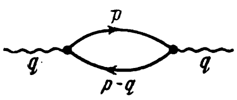

FIG. 1

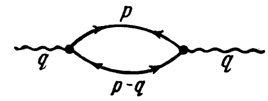

FIG. 2

The correction to the phonon dispersion law, caused by the electron-phonon interaction, is expressed in the following manner in terms of $\Pi(q, \omega)$:

$$\Delta\omega = -\frac{\pi^2 \lambda_0}{p_0} \Pi(q, \omega_q^0)\, \omega_q^0, \tag{5}$$

where p_0 is the Fermi momentum, ω_q^0 the law of phonon dispersion without account of the interaction with the electrons, and λ_0 is the dimensionless electron-phonon interaction constant introduced in [6,7].

1)The thermal anomaly of superconductors, observed by Bryant and Keesom[2], consists in the fact that the lattice specific heat (i.e., the specific heat component proportional to T^3) is smaller in the superconducting state than in the normal state, in spite of the customarily made assumption, by which the lattice does not change in the superconducting transition.

2)It is difficult to confine oneself to the case $T = 0$, since all temperature dependences have an exponential character (see [4]) and can be neglected under the ordinary experimental conditions[2], when the electron specific heat "freezes out."

In a normal metal ($\Delta = 0$) we have on the basis of (2) and (3)

$$\operatorname{Re}\Pi^n(q,\omega) = p_0/2\pi^2,$$

$$\operatorname{Im}\Pi^n(q,\omega) = |\omega|/4\pi q \quad (\omega \ll p_0 q). \tag{6}$$

In this case $\operatorname{Re}\Pi^n$ leads to renormalization of the velocity of sound s, i.e., to a replacement of ω_q^0 by $\omega_q = sq$, whereas $\operatorname{Im}\Pi^n$ corresponds to the Kittel formula [8] (see also the paper by Migdal [6]) for the damping coefficient of sound; the phonon range l, expressed in units of the wavelength λ, is

$$Q = l/\lambda = p_0/\pi^2\lambda_0 s. \tag{7}$$

Proceeding to the effect of interest to us, the change of the phonon frequency in a superconductor, let us calculate the difference $\Pi^s - \Pi^n$. The real correction to the phonon frequency on going from the normal state into the superconducting one is written in the form

$$\Delta\omega_s = -\frac{\pi^2\lambda_0}{p_0}\operatorname{Re}(\Pi^s - \Pi^n)\,\omega_q^0. \tag{8}$$

In calculating the difference $\Pi^s - \Pi^n$ we can extend the integration with respect to ξ_1 in formulas (2) and (3) from $-\infty$ to ∞, since $p_0 q \gg sq$ and $q_0 q \gg \Delta$ (we disregard, just as in [1], wavelengths longer than the correlation radius r_0). Going over in (2) and (3) to the dimensionless variables x and y, namely $\xi = x\Delta$ and $\eta = y\Delta$, we represent $\Delta\omega_s$ with the aid of (7) in the form

$$\Delta\omega_s = \frac{\Delta}{\pi^2 Q}\psi(\omega); \tag{9}$$

$$\psi(\omega) = \frac{1}{2}\int_0^\infty dx\int_0^\infty dy\Big\{\Big(\frac{1}{x+y+\nu} + \frac{1}{x+y-\nu}\Big) - \Big(1 + \frac{1}{\sqrt{1+x^2}\sqrt{1+y^2}}\Big) \times\Big(\frac{1}{\sqrt{1+x^2}+\sqrt{1+y^2}+\nu} + \frac{1}{\sqrt{1+x^2}+\sqrt{1+y^2}-\nu}\Big)\Big\}, \tag{10}$$

where $\nu = \omega/\Delta$.

In the investigation of the question of the anomaly of the lattice specific heat of superconductors, it is sufficient to put in this formula $\omega = 0$, which corresponds to the ordinary setup of the experiment on the observation of the anomaly of the specific heat [2], where the specific heat is first measured in the normal state at low temperature with the magnetic field turned on so as to destroy the superconductivity ($\Delta = 0$), after which the field is turned off and the specific heat of the superconductor is measured at temperature $T \ll T_c$ (which corresponds to $\omega \ll \Delta$). Formula (9) yields then

$$\Delta\omega_s = \alpha\Delta/\pi^2 Q, \tag{11}$$

where α is a numerical constant, equal to

$$\alpha = \int_0^\infty dx\int_0^\infty dy\Big\{\frac{1}{x+y} - \Big(1+\frac{1}{\sqrt{1+x^2}\sqrt{1+y^2}}\Big)\frac{1}{\sqrt{1+x^2}+\sqrt{1+y^2}}\Big\}. \tag{12}$$

Formula (11) corresponds precisely to the results of Ferrell [1], but the calculation of the integral (12) shows that $\alpha = 0$ (see the appendix). We thus arrive at the conclusion that the correction of the type (11) to the phonon frequency and the associated anomaly in the lattice specific heat of superconductors, predicted by Ferrell, do not exist [3)] (this means, of course, that the experimentally observed anomaly [2] is due to other factors, which we did not take into account).

Apparently the reason for the disparity between Ferrell's results [1] and this conclusion is Ferrell's failure to take into account a diagram of the type of Fig. 2. This diagram corresponds to a term $1/\sqrt{1+x^2}\sqrt{1+y^2}$ in the parentheses of (12). Without account of this term, α is no longer equal to zero (and is positive). An attempt to calculate the following terms in the expansion of $\psi(\omega)$ in powers of ω/Δ leads to a (small) negative correction to the phonon frequency, which is in contradiction with the observed sign of the heat anomaly [2].

One might think that the result obtained is a consequence of the isotropic model. It is easy to see, however, that even in the anisotropic case $\Delta\omega_s = 0$. According to Pokrovskiĭ [4], the difference $\Pi^s(\mathbf{q},0) - \Pi^n(\mathbf{q},0)$, which determines the size of the effect, has in an anisotropic superconductor the form

$$\Pi^s(\mathbf{q},0) - \Pi^n(\mathbf{q},0) \approx \frac{1}{2(2\pi)^3}\int d\mathbf{p}\Big[\Big(1 - \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p-q}}}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p-q}}} + \frac{\Delta^2(\mathbf{p})}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p-q}}}\Big)\frac{1}{\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p-q}}} - \Big(1 - \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p-q}}}{|\xi_{\mathbf{p}}||\xi_{\mathbf{p-q}}|}\Big)\frac{1}{|\xi_{\mathbf{p}}|+|\xi_{\mathbf{p-q}}|}\Big]g^2(\mathbf{p}), \tag{13}$$

where $\Delta(\mathbf{p})$ is the gap, which depends on the direction of $\mathbf{n} = \mathbf{p}/p$, and $g(\mathbf{p})$ is a function describing the anisotropy of the electron-phonon interaction. Introducing the variables $\xi_{\mathbf{p}} = \xi$ and $\xi_{\mathbf{p-q}} = \xi_1$ [as in the derivation of formula (2)], we obtain $\xi - \xi_1 = \mathbf{v}\cdot\mathbf{q}$. In the integral of (13) the significant quantities are $\xi \sim \Delta$ and $\xi_1 \sim \Delta$, hence $|\mathbf{v}\cdot\mathbf{q}| \sim \Delta$. By virtue of the condition $v_0 q \gg \Delta$

[3)]Analogous calculations were made by Pokrovskiĭ (private communication) even before the publication of Ferrell's work [1] (the wavelength is much shorter than the correlation radius), this means that the essential region of integration in (13) is a small strip of width $\sim \Delta$ near the line $\mathbf{v} \perp \mathbf{q}$ on the Fermi surface. Therefore all the quantities Δ, g, and D (the Jacobian of the transformation to the new variables) can be set equal to their values on this line. As a result we obtain

$$\Pi^s(\mathbf{q}, 0) - \Pi^n(\mathbf{q}, 0) \approx \frac{1}{2(2\pi)^3 q} \int_0^{2\pi} d\varphi D(\varphi) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi_1$$
$$\times \left[\left(1 + \frac{\Delta^2(\varphi) - \xi\xi_1}{\sqrt{\Delta^2(\varphi) + \xi^2}\sqrt{\Delta^2(\varphi) + \xi_1^2}} \right) \times \frac{1}{\sqrt{\Delta^2(\varphi) + \xi^2} + \sqrt{\Delta^2(\varphi) + \xi_1^2}} - \left(1 - \frac{\xi\xi_1}{|\xi||\xi_1|} \right) \frac{1}{|\xi| + |\xi_1|} \right] g^2(\varphi)$$

(in the isotropic case D = 1).

The internal double integral in this formula reduces to (12) and is equal to zero. Thus, there is likewise no correction to the phonon dispersion law of the type (11) in the anisotropic case.

It is easy to verify that the phonon dispersion law in a superconductor is also unchanged in the other limiting case $p_0 q \ll \Delta$. This corresponds to the experimentally observed very weak change ($\Delta s/s \sim 10^{-4}$) in the velocity of sound (and in the Debye temperature) in the superconducting transition [9].

In conclusion I am grateful to I. M. Lifshitz for a discussion of the work and to V. L. Pokrovskiĭ for valuable remarks.

APPENDIX

Making in (12) the change of variables x = sinh ξ and y = sinh η, we obtain

$$\alpha = \int_0^\infty d\xi \int_0^\infty d\eta \left(\frac{\mathrm{ch}\,\xi\,\mathrm{ch}\,\eta}{\mathrm{sh}\,\xi + \mathrm{sh}\,\eta} - \frac{1 + \mathrm{ch}\,\xi\,\mathrm{ch}\,\eta}{\mathrm{ch}\,\xi + \mathrm{ch}\,\eta} \right). \quad (A.1)^*$$

In this expression we can carry out one integration (for example, with respect to η) by changing over to the variables u and v, using the formulas u = tanh (ξ/2) and v = tanh (η/2) [in the second term of (A.1)]. Returning then from u, v to ξ, η we obtain

*sh = sinh, ch = cosh.

$$\alpha = \int_0^\infty d\xi \left\{ \mathrm{ch}\,\xi \left[\ln(\mathrm{sh}\,\xi + \mathrm{sh}\,\eta) - \ln\frac{1 + \mathrm{th}(\eta/2)}{1 - \mathrm{th}(\eta/2)} \right] \Bigg|_{\eta=0}^{\eta=\infty} + \mathrm{sh}\,\xi \ln \frac{1 + \mathrm{th}(\xi/2)\,\mathrm{th}(\eta/2)}{1 - \mathrm{th}(\xi/2)\,\mathrm{th}(\eta/2)} \Bigg|_{\eta=0}^{\eta=\infty} \right\} \quad (A.2)^*$$

(the correctness of this formula can be readily verified by differentiating with respect to η).

Substitution of the limits of integration with respect to η in formula (A.2) yields

$$\alpha = \int_0^\infty [\xi\,\mathrm{sh}\,\xi - \mathrm{ch}\,\xi \ln(2\mathrm{sh}\,\xi)]\, d\xi. \quad (A.3)$$

The expression obtained is reduced by means of the substitution sinh ξ = x to the form

$$\alpha = \int_0^\infty \left[\frac{x}{\sqrt{1+x^2}} \ln(x + \sqrt{1+x^2}) - \ln(2x) \right] dx. \quad (A.4)$$

This integral can be readily evaluated and is equal to zero.

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*th = tanh.

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