SOME PROPERTIES OF THE ELASTIC SCATTERING AMPLITUDE AT HIGH ENERGIES

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It is shown that the imaginary part of the scattering amplitude $A_1(s,t)$ in the channel where s is the energy and in the nonphysical region of momentum transfer $t > 0$ is positive, and all its derivatives with respect to t are positive, up to its first singularity determined by the Landau curve $t = t_0(s)$. It then follows that for the Regge pole with maximum Re l one has in that region of t values $dl/dt > 0$. The dependence on t of $l(t)$ for $t \rightarrow 4\mu^2$ is investigated [Eqs. (13a) and (13b)]. It is proved that for $t > 4\mu^2$ the curve $l(t)$ moves into the upper half of the l plane. All these results are obtained without assuming the existence of a Hamiltonian. Various possibilities for the t dependence of l as $t \rightarrow \infty$ are discussed. A discussion is presented of the question whether or not the poles corresponding to "elementary particles" are continuous functions of l. In the Appendix it is demonstrated that in the calculation of the spectral density function $\rho(s,t)$ the condition for neglecting all singularities in the l plane except for the pole with maximum Re l is $s(t-4\mu^2) \gg \mu^4$.

In recent years the development of the theory of strong interactions has proceeded mainly along the lines of studying the analytic properties of amplitudes for various processes as functions of energy and momentum transfer. The main achievement in this field has been the discovery by Mandelstam [1] of the double representation for amplitudes for processes corresponding to the transformation of two particles into two.

The Mandelstam representation made it possible, in particular, to investigate in more detail the question of the asymptotic behavior of scattering amplitudes at large energies.

It became clear from such investigations [2] that the usual diffraction picture for high energy scattering could not be made in a simple way to agree with the Mandelstam representation. In a simple way diffraction agrees with a slowly falling cross section. At the same time, thanks to the work of Regge, [3] it became clear that in the nonrelativistic theory the scattering amplitude is for large momentum transfers a rapidly varying function of the form $f(t)\,s^{l(t)}$ (t = energy, s = momentum transfer). Here $l = l(t)$ is the position of the pole in the partial wave amplitude as a function of the angular momentum l.

With the help of the Mandelstam representation it was then shown [4] that also in field theory the partial wave amplitudes are analytic functions of the angular momentum l and may have poles. [5] The asymptotic behavior of the scattering amplitude for large momentum transfers can then have the same form as in the nonrelativistic theory. However in the relativistic theory the region of negative energies t and large momentum transfers s is at the same time the physical region of another reaction at high energy s and finite momentum transfer t (the "diffraction peak" region). In this way the possibility arises that the asymptotic behavior of the scattering amplitude at large energies is of the form $f(t)\,s^{l(t)}$. Such behavior differs substantially from the usual diffraction scattering. As was discussed in detail in [6] it corresponds to the scattering by a system whose radius increases with energy. This asymptotic behavior has also been discussed in the recent note of Chew and Frautschi. [7]

In this paper we first establish (Sec. 1) a certain exact property of the scattering amplitude, valid at all energies. In Sec. 2 we discuss in detail the possible behavior of $l(t)$ —the position of the pole in the partial wave as a function of t. In Sec. 3 we show that the experimental study of the cross section for the scattering of pions on nucleons at large angles and of the cross section for two-meson annihilation can decide the question whether or not the position of the partial wave pole corresponding to the neutron is a continuous function of the angular momentum. The existence of such a possibility was indicated in the note of Chew and Frautschi. [7]

1. GENERAL PROPERTIES OF THE IMAGINARY PART OF THE ELASTIC SCATTERING AMPLITUDE

Let us show that the imaginary part of the scattering amplitude $A_1(s,t)$ in the channel where s is the energy and in the nonphysical region of momentum transfers $t > 0$, is positive and has all its derivatives positive, up to the first singularity determined by the Landau curve $t = t_0(s)$. As $s \to \infty$ $t_0(s) \to 4\mu^2$ (μ = mass of the pion).

We write $A_1(s,t)$ as a sum over partial waves and ignore at first the spin variables:

$$A_1(s,t) = \sum_{n=0}^{\infty} a'_n(s)(2n+1)P_n(z), \tag{1}$$

$$z = 1 + 2st/[s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2]. \tag{2}$$

As a consequence of unitarity

$$a'_n \equiv \operatorname{Im} a_n > 0.$$

The series (1) converges uniformly up to the first singularity of $A_1(s,t)$. The proof is based on the fact that for $t > 0$, i.e., $z > 1$,

$$P_n(z) > 0. \tag{3a}$$

$$P'_n(z) = \sum_{k=1}^{N} (2n - 4k + 3) P_{n-2k+1}(z) > 0, \tag{3b}$$

where $N = n/2$ for n even, $N = (n+1)/2$ for n odd. Differentiating Eq. (1) an arbitrary number of times and making use of Eq. (3) we conclude that $A_1(s,t)$ is positive and has all its derivatives positive in the region $t > 0$ up to the first singular point.

Let us consider now the amplitude for the elastic scattering of pions on nucleons. In the barycentric frame it may be written in the form

$$f_1(s,t) + i(\sigma[\mathbf{k}_2\mathbf{k}_1])k^{-2}f_2(s,t), \tag{4*}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of the pions before and after scattering and $k_1^2 = k_2^2 = k^2$. The partial wave expansion of $f_1(s,t)$ and $f_2(s,t)$ is of the form

$$f_1(s,t) = \frac{1}{2}\sum_{n=0}^{\infty} [f_{n+}(s)2(n+1) + f_{n-}(s)2n]P_n(z), \tag{5a}$$

$$f_2(s,t) = \sum_{n=0}^{\infty} (f_{n-} - f_{n+})P'_n(z), \tag{5b}$$

where $f_{n\pm}$ are the partial wave amplitudes corresponding to orbital angular momentum n and total angular momentum $(n + 1/2)$ and $(n - 1/2)$ respectively.

Since $\operatorname{Im} f_{n\pm} > 0$, by virtue of the unitarity condition, we can go through the same considerations as above and arrive at the conclusion that $\operatorname{Im} f_1(s,t)$ —the imaginary part of the scattering amplitude without spin flip—is positive and has all its derivatives positive in the nonphysical region $0 < t < t_0(s)$.

It is easy to derive the corresponding statement for the elastic scattering of particles with arbitrary spin. If the scattering amplitude f is viewed as a matrix with respect to the spin variables then it can be shown that

$$F = \operatorname{Sp} f = \sum_l P_l(z) \sum_{JS} (2J+1) f_{Sl}^J, \tag{6}$$

where f_{Sl}^J is the amplitude for the scattering without changing orbital and spin angular momentum in the state with total angular momentum J, orbital angular momentum l, and total spin S. $\operatorname{Im} f_{Sl}^J > 0$ and consequently $\operatorname{Im} F(z)$ has the above described properties. To verify the validity of Eq. (6) it is sufficient to write the matrix element of f between states with spin projections λ_1, λ_2 and λ'_1, λ'_2 in the form

$$\langle \lambda'_1, \lambda'_2 | f | \lambda_1, \lambda_2 \rangle = \sum_{\substack{ll'SS' \\ mm'}} C(S_1, S_2, S'; \lambda'_1, \lambda'_2) C(S_1, S_2, S; \lambda_1, \lambda_2)$$
$$\times C(S', l', j; \mu - m', m') C(S, l, j; \mu - m, m) Y_{l'm'}(\mathbf{p}')$$
$$\times Y^*_{lm}(\mathbf{p}) \langle j, S', l' | j, S, l \rangle.$$

Setting in the above expression $\lambda_1 = \lambda'_1$ and $\lambda_2 = \lambda'_2$, summing over λ_1 and λ_2, and taking into account the orthogonality of the Clebsch-Gordan coefficients we obtain Eq. (6).

We have stopped to give such a detailed presentation of this simple theorem because we shall make use of it below, and also because it may turn out to be useful in the analysis of possible asymptotic behaviors of the scattering amplitude. It should be noted that the expression for the scattering amplitude obtained by Lovelace[8] violates the theorem and is therefore incorrect.

2. THE TRAJECTORY OF THE PARTIAL WAVE POLE IN THE COMPLEX l PLANE

It was proposed in [6] that the asymptotic behavior of the scattering amplitude at large energies is determined, in analogy with nonrelativistic theory,[3] by the pole in the partial wave in the channel where the momentum transfer is the energy. For the case of pion-pion scattering this asymptotic behavior is of the form

$$A_1(s,t) = f(t)s^{l_0(t)}, \qquad f(t) = \frac{\pi\Gamma(2l+1)}{\Gamma^2(l+1)}(t - 4\mu^2)^{-l_0(t)} r_0(t). \tag{7}$$

Here $l_0(t)$ is the position of the pole in the partial wave $f_l(t)$ in the t channel, and $r_0(t)$ is the

* $\sigma[\mathbf{k}_2\mathbf{k}_1] = \boldsymbol{\sigma} \cdot \mathbf{k}_1 \times \mathbf{k}_2$.

residue of the partial wave at that pole.

Let us discuss in more detail the properties of the function $l_0(t)$. Since $A_1(s,t)$ and its first derivative are positive, $f(t) > 0$ and $dl_0/dt > 0$ in the interval $0 < t < 4\mu^2$. In what follows we assume that the only t-dependent singularities of $f_l(t)$ in the l plane are poles. The likelihood of this assumption being correct is argued in detail in [5]. Then the l plane may be split into two parts by the line $\mathrm{Re}\, l = \nu$ in such a way that for $\mathrm{Re}\, l > \nu$ $f_l(t)$ has only poles. $\nu < 1$, if it is assumed that the asymptotic behavior of the scattering amplitude is of the form, Eq. (7).

In order to clarify the behavior of $l_0(t)$ in the neighborhood of $t = 4\mu^2$ we make use of the unitarity condition, which, as was shown in [4,5], must be satisfied by the partial wave amplitude in the t channel for $16\mu^2 > t > 4\mu^2$:

$$(1/2i)\,[f_l(t) - f^*_{l^*}(t)] = (k/\omega)\, f_l(t)\, f^*_{l^*}(t). \tag{8}$$

$f_l(t)$ coincides for even l with the usual partial waves of the symmetric part of the amplitude for pion-pion scattering. The quantity $f_l^{(-)}$, which interpolates the partial waves of the antisymmetric part, has analogous properties but we shall not be interested in it.

As a consequence of the unitarity condition (8) $f_l(t)$ cannot have poles on the real axis for $t > 4\mu^2$, and therefore for $t > 4\mu^2$ the function $l_0(t)$ must become complex

$$l_0(t) = l_0'(t) + i l_0''(t).$$

Let us find the general expression for a function that satisfies the condition (8) and has a pole in the vicinity of the real axis for $l = l_0(t)$. This problem is equivalent to the problem of finding a general expression for a partial wave in the vicinity of a resonance (the Breit-Wigner formula). Going through the usual considerations that result in the Breit-Wigner formula we obtain

$$f_l = \frac{\omega}{2ik}\left\{e^{2i\eta_l(t)}\,\frac{l - l_0'(t) + i l_0''(t)}{l - l_0'(t) - i l_0''(t)} - 1\right\}, \tag{9}$$

where $\eta_l(t)$ is real for real l and has no pole at $l = l_0(t)$.

As was shown in [4] when $t \to 4\mu^2$, $f_l(t) \sim (t - 4\mu^2)^l$. If for fixed $l \neq l_0(4\mu^2)$ t approaches $4\mu^2$ then in order that $f_l(t)$ approach zero like $(t - 4\mu^2)^l$ it is necessary that

$$\eta_l(t) = n\pi + \gamma\,(t - 4\mu^2)^{l+1/2}. \tag{10}$$

Therefore for small $t - 4\mu^2$ the quantity $e^{2i\eta l}$ may be approximated by unity and $f_l(t)$ may be written in the form

$$f_l(t) = \frac{\omega}{k}\,\frac{l_0''(t)}{l - l_0(t)}. \tag{11}$$

In order that f_l should behave for l near to l_0 like $(t - 4\mu^2)^l$ it is necessary that

$$l_0''(t) = \alpha\,(t - 4\mu^2)^{l_0(t)+1/2}. \tag{12}$$

Assuming that the point $t = 4\mu^2$ is an isolated singular point of the function $l_0(t)$ we may, obviously, with the help of Cauchy's formula obtain the real part $l_0'(t)$ in the vicinity of $t = 4\mu^2$ from the imaginary part l_0''. In that way we obtain, setting $l(4\mu^2) = \lambda$,

$$l_0(t) = \lambda + \sum_{n=0}^{n_0} c_n (t - 4\mu^2)^n - \alpha\,\frac{(4\mu^2 - t)^{\lambda+1/2}}{\cos\pi\lambda}, \tag{13a}$$

$\lambda - 1/2 < n_0 < \lambda + 1/2$, $\lambda + 1/2$ not equal to an integer, $\lambda + 1/2 > 0$;

$$l_0(t) = \lambda + \sum_{n=0}^{\lambda+1/2} c_n (t - 4\mu^2)^n - \frac{\alpha}{\pi}(t - 4\mu^2)^{\lambda+1/2}\ln(4\mu^2 - t), \tag{13b}$$

$\lambda + 1/2$ an integer, $\arg(4\mu^2 - t) = 0$ for $t < 4\mu^2$, $\lambda + 1/2 > 0$.

From Eq. (11) it follows easily that α in Eq. (12) must be larger than zero. Indeed, continuing Eq. (11) into the region $t < 4\mu^2$ we find that near $t = 4\mu^2$ the residue at the pole is

$$r(t) = 2\mu\alpha\,(t - 4\mu^2)^\lambda. \tag{14}$$

Thus the positive function $f(t)$ entering Eq. (7) is given by $f = 2\mu\alpha\pi\Gamma(2\lambda+1)/\Gamma^2(\lambda+1)$, i.e., $\alpha > 0$, and consequently the pole moves into the upper half-plane for $t > 4\mu^2$.

For the considerations that follow it is of exceptional importance to know whether the pole remains in the upper half-plane for arbitrary real $t > 4\mu^2$, as is the case in the nonrelativistic theory.[3] To this end it is necessary to clarify whether the trajectory can cross the real axis for $t > 4\mu^2$. In view of the unitarity condition $f_l(t)$ cannot become infinite for real l and t and consequently the pole can cross the real axis only if simultaneously the residue vanishes.

In order to understand what such a possibility means let us investigate $f_l(t)$ as a function of t for real l. $f_l(t)$ has a pole in the t plane of t close to $4\mu^2$ when l is close to λ. In order to clarify the behavior of this pole in the t plane one must solve Eqs. (13a) and (13b) for t. For $\lambda + 1/2 > 1$ the position of the pole is given in first approximation by the condition

$$l = \lambda + c_1 (t - 4\mu^2),$$

i.e., $t - 4\mu^2 = (l - \lambda)/c_1$. If $\lambda + 1/2 < 2$ we obtain in the next approximation

$$t - 4\mu^2 = \frac{l-\lambda}{c_1} + \frac{\alpha}{c_1 \cos \pi\lambda}\left(\frac{\lambda - l}{c_1}\right)^{\lambda+1/2}. \qquad (15)$$

On passing above the point $l = \lambda$ (i.e., $l = \lambda + i\epsilon$) we find that $(\lambda - l)^{\lambda+1/2}/c_1 = e^{-i\pi(\lambda+1/2)}(l-\lambda)^{\lambda+1/2}/c_1$ for $l > \lambda$, i.e.,*

$$t - 4\mu^2 = \frac{l-\lambda}{c_1} - i\frac{\alpha}{c_1}\left(\frac{l-\lambda}{c_1}\right)^{\lambda+1/2} + \frac{\alpha}{c_1}\operatorname{ctg}\pi\lambda\left(\frac{l-\lambda}{c_1}\right)^{\lambda+1/2}.$$

Because α and c_1 are positive the pole moves to the lower half-plane, crossing the real axis to the right of the point $t = 4\mu^2$ (Fig. 1); this means that the pole moves onto a nonphysical sheet in the t plane. On passing below the point $l = \lambda$ (i.e., $l = \lambda - i\epsilon$) the pole moves into the upper half-plane but also on a nonphysical sheet.

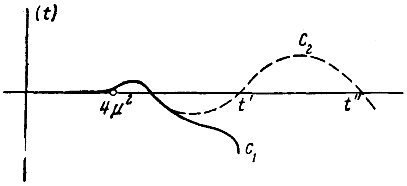

FIG. 1. c_1, c_2, $t = t_0(l)$ for real l.

When $\lambda + 1/2 \gg 2$ the term $c_2(t - 4\mu^2)^2$ must be taken into account, which however does not change the results since this term is not singular at $t = 4\mu^2$.

When $0 < \lambda + 1/2 < 1$ then

$$4\mu^2 - t = \left(\frac{\lambda - l}{\alpha}\cos\pi\lambda\right)^{1/(\lambda+1/2)}.$$

In this case, too, the pole moves to the right of the singular point $t = 4\mu^2$, i.e., moves onto a nonphysical sheet. If $\lambda + 1/2$ is an integer Eq. (13b) leads to the same results.

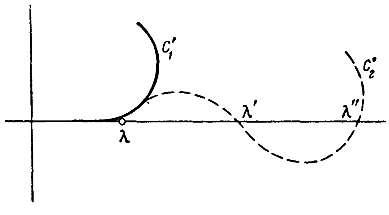

FIG. 2. c_1, c_2, $l = l_0(t)$ for real t.

The fact that for real $t > 4\mu^2$ the pole moves in the l plane (Fig. 2) into the upper half-plane ($\alpha > 0$) thus leads to the result that for real $l > \lambda$ the pole moves in the t plane off to a nonphysical sheet. If as t increases $l_0(t)$ does not cross the real axis then the singularities of f_l in the t plane lie on nonphysical sheets for arbitrary real l, i.e., the partial wave amplitudes have the same properties (no complex singularities) for integer and noninteger l. In the nonrelativistic theory[3] this property is insured by the hermiticity of the Hamiltonian for real l.

If for some $l = \lambda'$ (Fig. 2) the pole crosses the real axis, then in the t plane the pole reenters the physical sheet at $t = t'$ (Fig. 1). Since for sufficiently large l, determined by the number of subtractions in the dispersion relation in s, $f_l(t)$ has no complex singularities (see [4,5]) for some $l = \lambda''$ the pole must return to a nonphysical sheet in the t plane (Fig. 1) and, consequently, for some $t = t''$ into the upper half of the l plane (Fig. 2).

We remark that if the interval $\lambda' < l < \lambda''$ contains $l = 2n$ then the fact that the physical partial amplitude $f_{2n}(t)$ has no complex singularities is not obvious and results from involved compensations.

So far we have been considering the pole which determines the asymptotic behavior of $A_1(s, t)$ in the region of the diffraction peak, and which is, by hypothesis, first in the sense that $l = l_0(t)$ for $t < 4\mu^2$ has the largest value in comparison with other poles. If we were to consider other poles $l_i(t)$ we would, obviously, arrive at the conclusion that they behave in the same way as $l_0(t)$ in the vicinity of $t = 4\mu^2$, if $\lambda_i = l_i(4\mu^2) > \nu$. In Eqs. (13a) and (13b) it would only be necessary to replace λ by λ_i. We would not, however, be able to prove that these poles move for $t > 4\mu^2$ into the upper half of the l plane, since these poles do not contribute substantially to $A_1(s, t)$. We would only be able to verify that if it is required that $f_l(t)$ have no complex singularities for real l then all the poles $t > 4\mu^2$ must move into the upper half of the l plane and must remain there.

In connection with what has been said above it becomes most attractive to hypothesize that the poles of $f_l(t)$ for $t > 4\mu^2$ and $\operatorname{Re} l > -1/2$ may lie only in the upper half of the l plane. It would be most interesting to investigate the consequences of such a hypothesis. One of the consequences is as follows. Let us consider the properties of $l_0(t)$ in the complex t plane. This function has singularities on the real axis $t > 4\mu^2$. It cannot have singularities connected with the left cut of the function $f_l(t)$ $(t < 0)$ since the jump in $f_l(t)$ across the left cut has in general no singularities.[5] Generally speaking $l_0(t)$ may have singularities at values of t which are not singular points of $f_l(t)$. Such points could be those values of t for which $l_0(t)$ coincides with the position of some other pole (intersection of poles).

*ctg = cot.

The behavior of $l_0(t)$ for $t \to \infty$ is substantially restricted by the condition that $s^{l_0(t)}$ has no essential singularities at infinity in the complex plane cut for $t > 4\mu^2$. This means that Re l_0 is bounded. Re $l_0 < c$ is determined by the number of subtractions in the Mandelstam representation. This condition gives rise to the inequality $|l_0(t)| < B|t|^q$ for $|t| \to \infty$, $q < 1$. The latter means that the dispersion relation for $l_0(t)$ requires no more than one subtraction. If we assume in addition that no intersection of singularities occurs, i.e., that $l_0(t)$ has singularities only on the real axis $t > 4\mu^2$, then we arrive at the following dispersion relation

$$l_0(t) = l_0(0) + \frac{t}{\pi}\int_{4\mu^2}^{\infty i} \frac{dt' l_0''(t')}{t'_i(t'-t)}. \qquad (16)$$

Assuming that $l_0''(t) = \text{Im}\, l_0(t) > 0$ (pole lies in the upper half plane) it then follows that for $t < 4\mu^2$ all the derivatives of $l_0(t)$ are larger than zero.

Since a bound state of two pions with angular momentum $l = 2$ does not exist it follows that $l_0(4\mu^2) < 2$. Therefore

$$l_0'(0) < [l(4\mu^2) - l(0)]/4\mu^2 \leqslant 1/4\mu^2,$$

since $l(0)$ is assumed equal to unity. Consequently, the index in the exponent that determines the angular distribution within the diffraction peak [6]

$$\gamma t \ln(s/4\mu^2) < (t/4\mu^2)\ln(s/4\mu^2) \qquad (\gamma = l_0'(0)).$$

Another consequence of Eq. (16) is the fact that if $l_0''(t) \to 0$ as $t \to \infty$ so that $\int l''(t)\,dt/t < \infty$, then as $t \to \pm\infty$ $l_0(t)$ tends to the constant limit

$$l_0(0) - \frac{1}{\pi}\int_{4\mu^2}^{\infty} l_0''(t)\frac{dt}{t}$$

—the pole "freezes" as $t \to \pm\infty$. We remark that in the nonrelativistic theory the pole behaves in precisely this way if the potential goes at small distances like r^k ($k > -2$, k not equal to an even integer) and the value that $l_0(t)$ approaches is given by $-3/2 - k/2$. If $\int l_0''(t')\,dt'/t' = \infty$, then $l_0(t)$ tends to $-\infty$ as $t \to -\infty$ —the pole disappears. Such a behavior of $l_0(t)$ in the nonrelativistic theory takes place if at small distances the potential is of the form $\sum_{n=0}^{\infty} c_n r^{2n}$, i.e., $r = 0$ is not a singular point of the potential and small distances play no role in the scattering.

To conclude this section we discuss the behavior of $l_0(t)$ for $t < 0$, without assuming the validity of Eq. (16). If $l_0(t)$ continues to decrease with decreasing t for $t < 0$ we have the following possibilities for the behavior of $l_0(t)$ and, consequently, for the asymptotic behavior of $A(s,t)$ and $A_1(s,t)$ as $s \to \infty$.

1. $l_0(t) \to C \geq 0$ as $t \to -\infty$. $A(s,t)$ tends to infinity for any finite t.

2. $l_0(t)$ goes through zero for some finite value of $t = t_1$. If the residue of the pole does not vanish at $t = t_1$ it is necessary, in order that the physical amplitude for s-wave scattering have no pole at $t = t_1$ (this would mean the existence of a particle with imaginary mass), that this amplitude not coincide with $f_l(t)$ for $l = 0$. In order for this to be possible it is necessary, at least, that the real part of $A(s,t)$ not decrease as $s \to \infty$ for arbitrary t. If $A(s,t)$ were to approach zero as $s \to \infty$ we could write for it a dispersion relation in s without subtractions and show that the amplitude for s-wave scattering coincides with $f_0(t)$. If we compare the behavior of $l_0(t)$ in these two cases with the possible behaviors of $l_0(t)$ in the nonrelativistic theory we arrive at the conclusion that in the first case the interaction has singularities at small distances with no nonrelativistic analog, since the pole freezes for $l > 0$, and in the second case the singularity must be at least of a delta-function like nature.

3. If $l_0(t)$ passes through zero in such a way that at $t = t_1$ the residue at the pole vanishes, then $A(s,t)$ may decrease as $s \to \infty$ arbitrarily rapidly for sufficiently large t. This case corresponds to the situation when small distances play no role in the scattering even at high energies. Let us note that in nonrelativistic quantum mechanics the residue at the pole cannot vanish. Therefore this case of course has no nonrelativistic analog.

3. ARE THE POLES, CORRESPONDING TO "ELEMENTARY" PARTICLES, CONTINUOUS FUNCTIONS OF l?

In a recent note Chew and Frautschi have remarked on the experimental possibility of clarifying the question whether the properties of the poles corresponding to "elementary" particles are the same as of the poles corresponding to bound states of nonrelativistic quantum mechanics. We would like to discuss this question in more detail.

We consider the amplitude for the scattering of pions on nucleons. The invariant amplitude has the form

$$a(u,t) + \tfrac{1}{2}(\hat{k}_1 + \hat{k}_2)b(u,t), \qquad (17)$$

where k_1 and k_2 are the four-momenta of the pion before and after scattering. We have used u to denote the square of the energy in the barycentric frame, and t —the momentum transfer—is related to the scattering angle by

$$t = -\tfrac{1}{2} u^{-1} [u^2 - 2u(m^2 + \mu^2) + (m^2 - \mu^2)^2](1 - z).$$

To analyze the partial wave amplitudes it is convenient to write the scattering amplitude in the barycentric frame in two-component form, Eq. (4):

$$f_1(u, t) + i(\sigma [\mathbf{k}_1, \mathbf{k}_2]) k^{-2} f_2(u, t), \tag{18}$$

$f_1(u,t)$ and $f_2(u,t)$ are simply related to the functions $a(u,t)$ and $b(u,t)$ and have, as functions of s, the same analytic properties:

$$f_2 = \tfrac{1}{32}\pi^{-1}[(w - m)^2 - \mu^2][a - (w + m) b],$$

$$f_1 - zf_2 = -\tfrac{1}{32}\pi^{-1}[(w + m)^2 - \mu^2][a + (w - m) b],$$

where $w = \sqrt{u}$. The partial wave expansion of $f_1(u,t)$ and $f_2(u,t)$ differs from the Eqs. (5a) and (5b) only in the replacement of s by u.

In a manner analogous to that used in [4] partial waves with complex l may be introduced by making use of the dispersion relation in s. If a dispersion relation in t for $f_{1,2}(u,t)$ is written, for example with one subtraction, in the form

$$f_{1,2}(u, z) = f_{1,2}(u, 0) + \frac{z}{\pi}\int_{z_0}^{\infty} \frac{dz' f_{1,2}^{(1)}(u, z')}{z'(z' - z)} + \frac{z}{\pi}\int_{-z_0'}^{\infty} \frac{dz' f_{1,2}^{(2)}(u, z')}{z'(z' + z)}, \tag{19}$$

then it is easy to show that

$$F_n(u) = f_{n+}(u)(n + 1) + f_{n-}(u) n = \frac{1}{\pi}\int_{z_0}^{\infty} Q_n(z) f_1^{(1)}(u, z)\, dz + (-1)^n \frac{1}{\pi}\int_{z_0'}^{\infty} Q_n(z) f_1^{(3)}(u, z)\, dz, \tag{20a}$$

$$H_n(u) = f_{n-} - f_{n+} = \frac{1}{\pi}\int_{z_0}^{\infty} Q_n(z)\, z dz \int_{z_0}^{z} f_2^{(1)}(u, z') \frac{dz'}{z'} + (-1)^n \frac{1}{\pi}\int_{z_0'}^{\infty} Q_n(z)\, z dz \int_{z_0'}^{\infty} f_2^{(3)}(u, z') \frac{dz'}{z'}, \tag{20b}$$

where $Q_n(z)$ are Legendre functions of the second kind. No formulas corresponding to Eqs. (20a) and (20b) exist for $n = 0$ and $n = 1$. If a larger number of subtractions is necessary, then Eqs. (20a) and (20b) exist starting with larger values of n.

Equations (20a) and (20b) make it possible to introduce analytic functions $F_l^{\pm}(u)$, $H_l^{\pm}(u)$, which coincide with Eqs. (20a) and (20b) for even and odd n (see [4,5]). Explicit expressions for $F_l^{\pm}$ and $H_l^{\pm}$ are obtained from Eqs. (20a) and (20b) by replacing $Q_n(z)$ by Q_l —the Legendre function of the second kind defined for arbitrary values of l —and by replacing $(-1)^n$ by ± 1 respectively. By repeating the same considerations as in [4,5] one sees easily that $F_l^{\pm}$, $H_l^{\pm}$ satisfy the same unitarity condition as F_n, H_n, for real values of l. $F_l^{\pm}$ and $H_l^{\pm}$ possess all the properties as do the partial waves with complex l for spinless particles discussed in [4,5].

The functions $f_1(u,z)$ and $f_2(u,z)$ may be expressed in terms of the functions $F_l^{\pm}(u)$ and $H_l^{\pm}(u)$ in the form

$$f_1^{\pm}(u, z) = \frac{1}{2}\sum_{n=0}^{m_1} F_n^{\pm}(u)[1 \pm (-1)^n] P_n(z) + \frac{i}{4}\int_{a_1 - i\infty}^{a_1 + i\infty} \frac{dl F_l^{\pm}(u)}{\sin l\pi}[P_l(-z) \pm P_l(z)], \tag{21a}$$

$$f_2^{\mp}(u, z) = \frac{1}{2}\sum_{n=1}^{m_2} H_n^{\pm}(u)[1 \pm (-1)^n] P_n'(z) + \frac{i}{4}\int_{a_2 - i\infty}^{a_2 + i\infty} \frac{dl H_l^{\pm}(u)}{\sin l\pi}[-P_l'(-z) \pm P_l'(z)], \tag{21b}$$

where $f_1^{\pm}(u,z)$, $f_2^{\pm}(u,z)$ are the symmetric and antisymmetric parts of $f_1(u,z)$ and $f_2(u,z)$; m_1 and m_2 are determined by the number of subtractions, $a_1 > m_1$, $a_2 > m_2$.

We have given these formulas to be able to formulate more precisely the question raised in the title of this section. The partial wave $f_{1-}(u)$ has a pole at $u = m^2$, corresponding to the nucleon. The functions $F_1(u)$ and $H_1(u)$ have the same pole.

If the functions F_l^- and H_l^-, when analytically continued in l to the point $l = 1$, where generally speaking Eqs. (20a) and (20b) are not applicable, coincide with the true partial waves then one may assert that F_l^- and H_l^- should have for fixed l a pole in u at $u = u(l)$, such that $u(1) = m^2$. In that case we would say that the nucleon pole is a continuous function of l. If the functions F_l^- and H_l^-, when analytically continued in l to the point $l = 1$, do not coincide with the true partial waves then they cannot have a pole corresponding to the nucleon, and in that case we would say that the nucleon pole is not a continuous function of l.

A continuous dependence of the pole on l arises naturally if the pole corresponds to a bound state (the centrifugal barrier changes the binding energy). In application to a pole due to a particle which is thought to be elementary such a dependence seems somewhat unexpected, however since the concept of elementarity has no precise meaning in the presence of strong interactions it is possible that all particles are elementary to the same degree and that their corresponding poles are continuous functions of l.

The question whether the poles depend on l continuously may be decided experimentally. This has to do with the fact that if in a certain range of variation of u the amplitude $f_2(u, z)$ tends to zero as $z \to \pm\infty$, and $f_1(u, z)$ increases slower than z, then for such u we may write a dispersion relation with no subtractions for $f_2(u, z)$ and with one subtraction for $f_1(u, z)$. At that Eqs. (20a) and (20b) are valid for $n = 1$, i.e., in that interval of u the functions F_l^- and H_l^-, when analytically continued in l, coincide with F_1^- and H_1^-. In view of analyticity in u they coincide for any u, and consequently the position of the nucleon pole varies continuously with l.

The interval of variation of u, within which the functions $f_2(u, z)$ and $f_1(u, z)$ have the indicated properties, may turn out to be the interval $u < 0$. For $s \to \infty$ and $u < 0$ we enter into the physical region for large angle scattering of pions on nucleons (s being the energy). For $s \to -\infty$ we enter the physical region for two-meson annihilation (t being the energy). It is not difficult to see that if the differential cross sections for these reactions tend to zero as $s \to \infty$, $u = \text{const}$ and $t \to \infty$, $u = \text{const}$ respectively, then f_1 and f_2 have the above described properties and, consequently, the neutron pole is a continuous function of l.

Other processes may be investigated in an analogous manner and conclusions arrived at about poles corresponding to other elementary particles. In particular, the processes enumerated in [9] could be so utilized.

Let us emphasize that even if the position of the neutron pole is a continuous function of l and the differential cross sections for annihilation and large angle scattering tend to zero this does not mean that

$$f_1^-(u,s) \sim s^{l_n(u)}, \qquad f_2^+(u,s) \sim s^{l_n(u)-1} \tag{22}$$

as $s \to \infty$ and $u < 0$, where $l_n(u)$ is the position of the neutron pole, since for $\operatorname{Re} l < 0$ the functions F_l^- and H_l^- may have singularities other than poles.

If it is assumed that the imaginary part of $l_n(u)$ is larger than zero for $u > (m+\mu)^2$ and that a dispersion relation of the form of Eq. (16) is valid, then it is easy to obtain an inequality for $l_n(u)$ in the physical region $u < 0$, by making use of the fact that in that case all the derivatives of $l_n(u)$ are larger than zero. Since there does not exist a stable particle with $l = 3$ and baryon number equal to unity it follows that $l[(m+\mu)^2] < 3$. Consequently

$$l_n'(u) < 2/(2m\mu + \mu^2) < 1/m\mu.$$

Since $l_n'' > 0$, $l_n' < 1/m\mu$ also for $u < m^2$. Hence

$$l_n(0) = -m/\mu + 1.$$

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APPENDIX

Let us consider the problem of evaluating the spectral function $\rho(s, t)$ starting from an asymptotic form of $A_1(s, t)$ of the type

$$r(t)\left(\frac{s}{t-4\mu^2}\right)^{l(t)}.$$

To that end it is necessary to calculate the difference $A_1(t+i0) - A_1(t-i0)$ for $t > 4\mu^2$. Starting from Eqs. (7), (13a), and (13b) it is easy to show that for $\lambda + 1/2$ not equal to an integer

$$\rho \sim s^{\lambda}\left[s^{i\alpha(t-4\mu^2)^{\lambda+1/2}} - s^{-i\alpha(t-4\mu^2)^{\lambda+1/2}}\right]. \tag{A.1}$$

When $(t-4\mu^2)^{\lambda+1/2} \ln s \ll 1$, Eq. (A.1) goes over into

$$\rho = \text{const}\cdot[s(t-4\mu^2)]^{\lambda}\sqrt{t-4\mu^2}\ln s. \tag{A.2}$$

λ is the value of the right-most pole for $t = 4\mu^2$. Suppose the next pole occurs for $\lambda_1 < \lambda$. Then the contribution to ρ, due to this pole, will be in analogy to Eq. (A.2) given by

$$\text{const}\cdot[s(t-4\mu^2)]^{\lambda_1}\sqrt{t-4\mu^2}\ln s. \tag{A.3}$$

On comparing Eqs. (A.2) and (A.3) we see that whereas the condition for ignoring all singularities in A_1 (and A) except $l = \lambda$ is the statement $s \gg \mu^2$, in evaluating ρ an entirely different condition is involved:

$$s(t-4\mu^2) \gg \mu^4. \tag{A.4}$$

For $t - 4\mu^2 = \text{const}\cdot\mu^2$ Eq. (A.4) reduces to $s \gg \mu^2$. The condition (A.4) apparently means that an asymptotically pole-like regime serves to determine ρ only if a large number of Landau singularities contribute to ρ, as was to be expected. An analogous result is obtained also when $\lambda + 1/2$ is equal to an integer.

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48