THEORY OF THE SPONTANEOUS HALL EFFECT IN FERROMAGNETS

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The kinetic equation for scattering of electrons by phonons is obtained by a density-matrix method similar to that of Kohn and Luttinger;[1,2] higher-order terms in the interaction with the scatterers are taken into account. In contrast to the impurity case,[2] only the field term in the second approximation contains a term linear in the spin-orbit interaction; this leads to a proportionality of the spontaneous Hall coefficient R_S to the square of the electrical resistivity ρ. Possible reasons for the deviation of experimental data from the $R_S \sim \rho^2$ law are discussed.

THE occurrence of an anomalous Hall effect in ferromagnets is commonly attributed to spin-orbit interaction. However, because of its periodic character in a crystal lattice, this interaction cannot lead to the appearance of any resultant force of the Lorentz-force type, originating from an external magnetic field or from an induction field. As was first shown by Kohn and Luttinger,[1,2] the appearance of the resultant electromotive force of the anomalous Hall effect is connected with a change in the nature of the scattering and in the action of the external electric field on the current carriers under the influence of spin-orbit interaction. This influence appears only in higher-order terms in an expansion in the scattering potential; consequently, in order to calculate the anomalous Hall effect it is necessary to use a kinetic equation that takes account of such terms.

In [1,2] the case considered was that of scattering by randomly distributed impurities. Two types of terms were obtained, corresponding to $R_S \sim \rho$ and to $R_S \sim \rho^2$; the second term, as compared with the first, was obtained in the next order of smallness and is in fact always appreciably smaller.* Thus the result of the work cited signifies essentially that scattering by impurities gives a linear relation between R_S and the impurity part of the electrical resistivity. With regard to the temperature dependence $R_S(T)$, especially in the high-temperature region, it is quite evidently necessary to give special consideration to scattering by phonons and also by magnetic inhomogeneities, which are known to play a basic role in the dependence $\rho(T)$.

*At impurity concentration ~ 1%, the ratio of the second term to the first is ~10^{-3} to 10^{-4} (estimated by formulas of [2]).

In the present work we shall obtain the kinetic equation for scattering by phonons by a method similar to the method of Kohn and Luttinger,[1] and we shall examine the dependence $R_S(T)$ thus obtained.

1. DERIVATION OF THE KINETIC EQUATION

Interaction with phonons is conveniently treated in the second-quantization representation. We write the original Hamiltonian $\hat{\mathcal{H}}$ in the form

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}' + \hat{\mathcal{H}}^F; \tag{1}$$

$$\hat{\mathcal{H}}_0 = \sum_l \varepsilon_l a_l^+ a_l + \sum_q \varepsilon_q b_q^+ b_q,$$

$$\hat{\mathcal{H}}' = \sum_{ll'q} (Q_{ll'q} a_l^+ a_{l'} b_q + Q_{ll'q}^* a_{l'}^+ a_l b_q^+),$$

$$\hat{\mathcal{H}}^F = eF_\alpha \sum_{ll'} r_{ll'}^\alpha a_l^+ a_{l'};$$

$$Q_{ll'q} = \tfrac{2}{3} i (2NM\varepsilon_q)^{-1/2} q C_{ll'} \delta_{\mathbf{k}-\mathbf{k}',\mathbf{q}}. \tag{2}$$

Here $\epsilon_l = \epsilon_{n\mathbf{k}}$ is the energy of an electron in band n with wave vector $\mathbf{k}$; ϵ_q is the energy of a phonon with wave vector $\mathbf{q}$; N is the number of ions in a fundamental volume of the crystal; M is the mass of the ions; $C_{ll'}$ is the familiar Bloch constant,* in the theory of electrical conductivity; F_α and r^α are components, respectively, of the external electric field and of the position vector; finally, a_l and b_q are the second-quantization operators of the electrons and of the phonons, l being a representation in which the Hamiltonian $\hat{\mathcal{H}}_0$, including the spin-orbit interaction, is diagonal.

We further introduce the second-quantization

*We retain the indices l and l' but relinquish the possibility of taking account of transitions between bands ($n \neq n'$) in our model.

density matrix $\hat{\rho} = K \exp(-\beta\hat{\mathcal{H}})$, which satisfies the equation ($\beta = 1/kT$)

$$i\partial\hat{\rho}/\partial t = [\hat{\mathcal{H}}, \hat{\rho}] = \hat{\mathcal{H}}\hat{\rho} - \hat{\rho}\hat{\mathcal{H}}. \quad (3)$$

We seek a solution of (3), linear in F_α, in the form

$$\hat{\rho} = \hat{\rho}_0 + \hat{f}, \quad (4)$$

where $\hat{f}$ satisfies the equation

$$i\partial\hat{f}/\partial t = [\hat{\mathcal{H}}^F, \hat{\rho}_0] + [\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}', \hat{f}]. \quad (5)$$

We transform, in (5), from the second-quantization representation to the l-representation by multiplying (5) by $a_l^\dagger a_{l'}$ and taking the trace in accordance with the occupation numbers:

$$i\partial f_{l'l}/\partial t = \mathrm{Sp}\{[\hat{\mathcal{H}}^F, \hat{\rho}_0] a_l^\dagger a_{l'}\} + \mathrm{Sp}\{[\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}', \hat{f}] a_l^\dagger a_{l'}\}$$
$$= \mathrm{Sp}\{\hat{\rho}_0 [a_l^\dagger a_{l'}, \hat{\mathcal{H}}^F]\} + \mathrm{Sp}\{\hat{f}[a_l^\dagger a_{l'}, \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}']\}. \quad (6)$$

On calculating the commutators, we get (the infinitesimal parameter s, $s \to +0$, insures adiabatic application of the external electric field)

$$(\varepsilon_{l'} - \varepsilon_l - is) f_{l'l} = eF_\alpha C^\alpha_{l'l} + \sum_{l_1 q} [(Q_{l_1 l q} f_{l' l_1 q} + Q^*_{l l_1 q} f^+_{l' l_1 q}) - (Q_{l' l_1 q} f_{l_1 l q} + Q^*_{l_1 l' q} f^+_{l_1 l q})]; \quad (7)$$

$$C^\alpha_{l'l} = [\hat{\rho}_0, r^\alpha]_{l'l}, \quad f_{l'lq} = \mathrm{Sp}\{\hat{f} a_l^\dagger a_{l'} b_q\}, \quad f^+_{l'lq} = \mathrm{Sp}\{\hat{f} a_l^\dagger a_{l'} b_q^+\}. \quad (8)$$

The values of $f_{l'lq}$ and $f^+_{l'lq}$ must also be found from (5). In analogy to Eq. (7) we obtain

$$(\varepsilon_{l'} - \varepsilon_l + \varepsilon_q - is) f_{l'lq} = eF_\alpha C^\alpha_{l'lq} + \sum_{l_1 q_1} [(Q_{l_1 l q_1} f_{l' l_1} \rho^-_{q q_1} + Q^*_{l l_1 q_1} f_{l' l_1} \rho^-_{q q_1}) - (Q_{l' l_1 q_1} f_{l_1 l} \rho^-_{q q_1} + Q^*_{l_1 l' q_1} f_{l_1 l} \rho_{q q_1})]$$
$$- \sum_{l_1 l_1'} Q^*_{l_1' l_1 q} [f_{l' l} \rho_{l_1' l_1} + \rho_{l' l} f_{l_1' l_1} + f_{l_1' l} (\delta_{l' l_1} - \rho_{l' l_1}) - \rho_{l_1' l} f_{l' l_1}]. \quad (9)$$

$$(\varepsilon_{l'} - \varepsilon_l - \varepsilon_q - is) f^+_{l'lq} = eF_\alpha C^{+\alpha}_{l'lq} + \sum_{l_1 q_1} [(Q_{l_1 l q_1} f_{l' l_1} (\rho_{q_1 q} + \delta_{q q_1}) + Q^*_{l l_1 q_1} f_{l' l_1} \rho^+_{q q_1}) - (Q_{l' l_1 q_1} f_{l_1 l} (\rho_{q_1 q} + \delta_{q q_1}) + Q^*_{l_1 l' q_1} f_{l_1 l} \rho^+_{q q_1})]$$
$$- \sum_{l_1 l_1'} Q_{l_1 l_1' q} [f_{l' l} \rho_{l_1' l_1} + \rho_{l' l} f_{l_1' l_1} + f_{l_1' l} (\delta_{l' l_1} - \rho_{l' l_1}) - \rho_{l_1' l} f_{l' l_1}]. \quad (10)$$

Here

$$C^\alpha_{l'lq} = \sum_{l_1} (r_{l' l_1 q} r^\alpha_{l_1 l} - r^\alpha_{l' l_1} r_{l_1 l q}), \quad \rho_{l l' q} = \mathrm{Sp}\{\hat{\rho}_0 a_{l'}^\dagger a_l b_q\},$$
$$\rho^-_{q q_1} = \mathrm{Sp}\{\hat{\rho}_0 b_{q_1} b_q\}, \quad \rho^+_{q q_1} = \mathrm{Sp}\{\hat{\rho}_0 b_{q_1}^+ b_q^+\},$$
$$\rho_{q q_1} = \mathrm{Sp}\{\hat{\rho}_0 b_{q_1}^+ b_q\} \quad (11)$$

etc. In (9) and (10) we have performed a separation by representation of the two-particle density matrices through single-particle ones:

$$\rho_{l l_1 q q_1} = \rho_{l l_1} \rho_{q q_1}, \quad \rho_{l_1 l_1', l l'} = \rho_{l_1 l_1'} \rho_{l l'} + \rho_{l_1 l'} (\delta_{l l_1'} - \rho_{l l_1'}). \quad (12)$$

On substituting (9) and (10) in (7), we get a closed system of equations in the unknown functions $f_l \equiv f_{ll}$ and $f_{ll'}$. In particular, we get to the lowest order in the interaction Q with the phonons (remembering that the f_l series begins with $f_l^{(-2)}$),*

$$f^{(-1)}_{l'lq} = (\varepsilon_{l'} - \varepsilon_l + \varepsilon_q - is)^{-1} Q^*_{l l' q} [\rho^0_q (f^{(-2)}_{l'} - f^{(-2)}_l) - f^{(-2)}_l (1 - \rho^0_{l'}) + \rho^0_l f^{(-2)}_{l'}]. \quad (13)$$

Now on substituting (13) and the analogous expression for $f^{(-1)+}_{l'lq}$ in (7) for $l = l'$, we get in the limit $s \to 0$

$$eF_\alpha C^{(0)\alpha}_l + 2\pi i \sum_{l_1 q} \{|Q_{l l_1 q}|^2 \delta(\varepsilon_{l_1} - \varepsilon_l + \varepsilon_q) [\rho^0_q (f^{(-2)}_{l_1} - f^{(-2)}_l) - f^{(-2)}_l (1 - \rho^0_{l_1}) + \rho^0_l f^{(-2)}_{l_1}] + |Q_{l_1 l q}|^2 \delta(\varepsilon_{l_1} - \varepsilon_l - \varepsilon_q) [(\rho^0_q + 1) \times (f^{(-2)}_{l_1} - f^{(-2)}_l) + f^{(-2)}_l (1 - \rho^0_{l_1}) - \rho^0_l f^{(-2)}_{l_1}]\} = 0; \quad (14)$$

this agrees with the usual kinetic equation, since $C^{(0)\alpha}_l = i\partial\rho^0_l/\partial k_\alpha$.

Equation (14) cannot contain terms linear in the spin-orbit interaction (i.e., in the magnetization), which are necessary for calculation of the spontaneous Hall effect, for the same reason as in the case of impurity scattering.[2] Therefore we must seek higher-order terms in the expansion of $f_{ll'}$ in powers of Q.

2. THE SPONTANEOUS HALL EFFECT TO THE SECOND APPROXIMATION IN THE INTERACTION WITH PHONONS

For what follows, we need to know the expansions of the functions that occur in Eqs. (7), (9), and (10)— $\rho_{qq'}$, $\rho^-_{qq'}$, $\rho^+_{qq'}$, $\rho_{ll'}$, $\rho_{ll'q}$, and $\rho^+_{ll'q}$ —in powers of Q. Such an expansion is easily obtained by application of Feynman's theorem to the matrix elements that are obtained when the equations for these quantities, (11), are expanded. Calculation to the second order gives

$$\rho_{qq} = \rho^0_q + \sum_{ll'} \rho^0_q (1 + \rho^0_q) |Q_{ll'q}|^2 \beta^2 \rho^0_l (1 - \rho^0_{l'}),$$

$$\rho_{q,-q} = \sum_{ll'} \rho^0_q \rho^0_{-q} |Q_{ll'q}|^2 \rho^0_l (1 - \rho^0_{l'}) \frac{1 - e^{2\beta\varepsilon_q}(1 - 2\beta\varepsilon_q)}{2\varepsilon_q^2},$$

*Here and hereafter, the upper index corresponds to the degree of the coupling constant.

$$\rho^{+}_{q,-q} = \sum_{ll'} (\rho^0_q + 1)(\rho^0_{-q} + 1) |Q_{ll'q}|^2 \rho^0_l (1 - \rho^0_{l'}) \times \frac{1 - e^{-2\beta\varepsilon_q}(1 + 2\beta\varepsilon_q)}{2\varepsilon_q^2},$$

$$\rho_{ll'} = \rho^0_{ll'}\delta_{ll'}, \quad \rho_{l'lq} = -\beta Q^*_{ll'q}\rho^0_{l'}(1 - \rho^0_l)\rho^0_q,$$

$$\rho^{+}_{l'lq} = -\beta Q_{l'lq}\rho^0_{l'}(1 - \rho^0_l)(\rho^0_q + 1). \quad (15)$$

It is easily seen that the lowest approximation for $f_{l'l}$, when $l = l'$, is obtained by substitution of (13) in (7):

$$f^{(0)}_{l'l} = (\varepsilon_{l'} - \varepsilon_l - is)^{-1}\left\{eF_\alpha C^{(0)\alpha}_{l'l} - \sum_{l_1 q}\left[Q_{l'l_1q}Q^*_{ll_1q}\frac{\varphi^{(-2)}_{l_1lq}}{\Delta^{+}_{l_1lq}} - Q^*_{l_1l'q}Q_{l_1lq}\frac{\varphi^{(-2)}_{l_1l'q}}{\Delta^{-}_{l_1lq}} - Q_{ll_1q}Q^*_{l_1l'q}\frac{\varphi^{(-2)}_{l'l_1q}}{\Delta^{+}_{l'l_1q}} + Q^*_{ll_1q}Q_{l'l_1q}\frac{\varphi^{(-2)}_{l_1l'q}}{\Delta^{-}_{l'l_1q}}\right]\right\}. \quad (16)$$

Here we have introduced the notation

$$\Delta^{\pm}_{l_1lq} = \varepsilon_{l_1} - \varepsilon_l \pm \varepsilon_q - is, \quad (17)$$

$$\varphi^{(-2)}_{l_1lq} = [\rho_q(f^{(-2)}_{l_1} - f^{(-2)}_l) + f^{(-2)}_l\rho^0_{l_1} + f^{(-2)}_{l_1}\rho^0_l - f^{(-2)}_{l_1}]. \quad (18)$$

To obtain the diagonal matrix elements f^0_l in the zeroth approximation,* we substitute (9) and (10) in (7) and use (5). If we restrict ourselves to consideration of the scattering of electrons within a single band (n = n' for all indices l = n, **k** and l' = n', **k'**) and use the law of conservation of momentum **k'** = **k** ± q, we get for s → 0

$$\sum_{l'q}\{|Q_{ll'q}|^2\delta(\varepsilon_{l'} - \varepsilon_l + \varepsilon_q)\varphi^{(0)}_{l'lq} - |Q_{l'lq}|^2\delta(\varepsilon_{l'} - \varepsilon_l - \varepsilon_q)\varphi^{(0)}_{ll'q}] + \sum_{l'q}[|Q_{ll'q}|^2\delta(\varepsilon_{l'} - \varepsilon_l + \varepsilon_q)(f^{(-2)}_{l'} - f^{(-2)}_l) \times (\rho^{(2)}_q + \rho^{-(2)}_{q,-q}) + |Q_{l'lq}|^2\delta(\varepsilon_{l'} - \varepsilon_l - \varepsilon_q) \times (f^{(-2)}_{l'} - f^{(-2)}_l)(\rho^{(2)}_q + \rho^{+(2)}_{q,-q})] + eF_\alpha\sum_{l'q}[\beta\rho^0_q|Q_{ll'q}|^2 \times \delta(\varepsilon_{l'} - \varepsilon_l + \varepsilon_q)\rho^0_{l'}(1 - \rho^0_l)(r^\alpha_{l'} - r^\alpha_l) + \beta(\rho^0_q + 1)|Q_{l'lq}|^2 \times \delta(\varepsilon_{l'} - \varepsilon_l - \varepsilon_q)\rho^0_{l'}(1 - \rho^0_l)(r^\alpha_{l'} - r^\alpha_l)] = 0. \quad (19)$$

The second sum in (19) contains no terms linear in the spin-orbit interaction, since $f^{(-2)}_l$, as obtained by solving (14), contains none.

In the case of high temperatures, $\beta\epsilon_q \ll 1$, we have

$$\beta\rho^0_{l'}(1 - \rho^0_l) = \beta\rho^0_l(1 - \rho^0_l) = -\partial\rho^0_l/\partial\varepsilon_l, \quad (20)$$

and therefore the solution of (19), linear in the spin-orbit interaction, will have the form

$$f^{(0)}_l = ieF_\alpha J^\alpha_l\frac{\partial\rho^0_l}{\partial\varepsilon_l}, \quad (21)$$

where J^α_l is the diagonal part of the coordinate r^α_l in a Bloch wave-function representation containing the spin-orbit interaction:

$$r^\alpha_{ll'} = i\delta_{ll'}\partial/\partial k_\alpha + iJ^\alpha_{ll'}\delta_{kk'}. \quad (22)$$

By means of (17) and (21) we can now calculate the mean velocity $\bar{v}_\beta$ of the electrons to the zeroth order in Q:

$$\bar{v}_\beta = \sum_l f^{(0)}_l v^\beta_l + \sum_{l\neq l'} f^{(0)}_{l'l}v^\beta_{ll'} = \bar{v}^d_\beta + \bar{v}'_\beta, \quad (23)$$

where

$$v^\beta_l = \partial\varepsilon_l/\partial k_\beta, \qquad v^\beta_{ll'} = -(\varepsilon_l - \varepsilon_{l'})J^\beta_{ll'}\delta_{kk'} \quad \text{for } l \neq l'.$$

Therefore

$$\bar{v}^d_\beta = ieF_\alpha\sum_l\frac{\partial\rho^0_l}{\partial\varepsilon_l}v^\beta_l J^\alpha_l. \quad (24)$$

In the calculation of $\bar{v}'_\beta$ we note that the terms in the expression (16) that contain products of the type $Q_{l'l,q}Q^*_{ll,q}$ with $l \neq l'$, i.e., with n ≠ n' (**k** = **k'** by virtue of the law of conservation of quasimomentum), correspond to interband transition in scattering. On neglecting these and using the expression for $C^{(0)\alpha}_{l'l}$ obtained from (8),

$$C^{(0)\alpha}_{l'l} = (\rho^0_{l'} - \rho^0_l)r^\alpha_{l'l} = iJ^\alpha_{l'l}\delta_{kk'}(\rho^0_{l'} - \rho^0_l), \quad (25)$$

we have

$$\bar{v}'_\beta = ieF_\alpha\sum_{l\neq l'}J^\beta_{ll'}J^\alpha_{l'l}(\rho^0_{l'} - \rho^0_l)\delta_{kk'} = -ieF_\alpha\sum_l\rho^0_l\sum_{n'}(J^\beta_{nn'}(\mathbf{k})J^\alpha_{n'n}(\mathbf{k}) - J^\alpha_{nn'}(\mathbf{k})J^\beta_{n'n}(\mathbf{k})).$$

Hence, using the relation [3]

$$\sum_{n'}(J^\beta_{nn'}J^\alpha_{n'n} - J^\alpha_{nn'}J^\beta_{n'n}) = \frac{\partial J^\beta_n}{\partial k_\alpha} - \frac{\partial J^\alpha_n}{\partial k_\beta}, \quad (26)$$

we get

$$\bar{v}'_\beta = -ieF_\alpha\sum_l\rho^0_l\left(\frac{\partial J^\beta_n(\mathbf{k})}{\partial k_\alpha} - \frac{\partial J^\alpha_n(\mathbf{k})}{\partial k_\beta}\right). \quad (27)$$

On integrating (27) by parts and combining it with (24), we find the final expression for $\bar{v}_\beta$:

$$\bar{v}_\beta = ieF_\alpha\sum_l\frac{\partial\rho^0_l}{\partial\varepsilon_l}v^\alpha_l J^\beta_l. \quad (28)$$

Our result (28), obtained by systematic treatment of the equations for the density matrix to the second approximation in the interaction with the phonons, agrees with the result of the earlier work of Karplus and Luttinger [3] and differs from the

*We remark that in scattering by phonons, in contrast to the case of scattering by impurities,[2] there are no terms $f^{(-1)}_l$ and $f^{(-1)}_{ll'}$ inversely proportional to the first power of the scattering potential.

results of Luttinger [2] for the case of scattering by impurities. The reason for this lies in the fact that in scattering by phonons, there is no correction to the velocity of the electrons because of a change of the scattering potential under the influence of spin-orbit interaction. The entire effect of spin-orbit interaction in this case is contained in the field terms of the kinetic equation.

The diagonal elements $J_n^\beta(k)$ differ from zero only if the Hamiltonian contains an imaginary part. In our case the spin-orbit interaction $\mathcal{H}^{SO}$ is such an imaginary part. In the first approximation of perturbation theory,

$$J_n^\beta(\mathbf{k}) = 2\sum_{n'(\neq n)} \frac{I^\beta_{nn'}(\mathbf{k})\,\mathcal{H}^{so}_{n'n}(\mathbf{k})}{\varepsilon_n^0(\mathbf{k}) - \varepsilon^0_{n'}(\mathbf{k})}, \tag{29}$$

where all the matrix elements on the right are taken in the representation of the Hamiltonian $\mathcal{H}_0 - \mathcal{H}^{S0}$. We transform (29) in a manner similar to that of Karplus and Luttinger, [3] recalling that

$$I^\beta_{nn'} = -I^\beta_{n'n} = -p^\beta_{nn'}/m\omega_{nn'}(\mathbf{k}), \qquad \mathcal{H}^{so}_{n'n} = -\mathcal{H}^{so}_{nn'}, \tag{30}$$

where m is the mass and p the momentum of the electron, and $\omega_{nn'} = \epsilon_n^{(0)} - \epsilon_{n'}^{(0)}$. Thus

$$J_n^\beta(\mathbf{k}) = \frac{1}{m}\sum_{n'} \frac{\mathcal{H}^{so}_{nn'}p^\beta_{n'n} - p^\beta_{nn'}\mathcal{H}^{so}_{n'n}}{\omega^2_{nn'}}. \tag{31}$$

Assuming that $\omega_{nn'} \cong \Delta = \text{const}$ and substituting (31) in (28), we get for the electrical conductivity

$$\sigma_{\beta\alpha} = \frac{en\bar{v}_\beta}{F_\alpha} = \frac{e^2n}{m\Delta^2}\sum_l \frac{\partial\rho^0_l}{\partial\varepsilon_l} v^\alpha_{li}\,[\mathcal{H}^{so}, \mathbf{p}]^\beta_l, \tag{32}$$

where n is the number of current carriers in unit volume.

Since

$$\mathcal{H}^{so} = \frac{1}{4m^2c^2}\nabla V\,[\mathbf{p}\sigma], \tag{33}*$$

where the spin σ may be replaced by the mean relative magnetization $\mathbf{M}/M_S$, it follows that

$$[\mathcal{H}^{so}, \mathbf{p}]^\beta = \frac{i}{4m^2c^2M_s}\left([\mathbf{pM}]^\gamma \frac{\partial}{\partial r_\gamma}\right)\frac{\partial V}{\partial r_\beta}.$$

Then on calculating the Bloch diagonal matrix element, we have

$$[\mathcal{H}^{so}, \mathbf{p}]^\beta_l = -\frac{i}{4m^2c^2M_s}[\mathbf{kM}]^\gamma \int u^*_{nk}(\mathbf{r})\frac{\partial^2 V}{\partial r_\gamma \partial r_\beta} u_{nk}(\mathbf{r})\,d\mathbf{r}_0 + \frac{1}{4m^2c^2M_s}\int u^*_{nk}(\mathbf{r})\frac{\partial^2 V}{\partial r_\gamma \partial r_\beta}\left[\frac{\partial}{\partial \mathbf{r}}\mathbf{M}\right]^\gamma u_{nk}(\mathbf{r})\,d\mathbf{r}_0. \tag{34}$$

The second integral in (34) vanishes if there is a center of symmetry, but the first differs from zero when $\gamma = \beta$. Using Poisson's equation, we get after substitution of (34) in (32)

$$\sigma_{\beta\alpha} = -\frac{\pi}{3}\frac{e^4n}{m^3c^2\Delta^2M_s}\sum_l \nu_l \frac{\partial\rho^0_l}{\partial\varepsilon_l} v^\alpha_l\,[\mathbf{kM}]^\beta, \tag{35}$$

where

$$\nu_l = \int u^*_{nk}(\mathbf{r})\,\rho(\mathbf{r})\,u_{nk}(\mathbf{r})\,d\mathbf{r}_0 \tag{36}$$

and where $\rho(\mathbf{r})$ is the density of the charge responsible for the potential V. If $M = M_z$, we get for σ_{yx}:*

$$\sigma_{yx} = \frac{\pi}{3}\frac{e^4nM_z}{m^3c^2\Delta^2M_s}\sum_l \nu_l \frac{\partial\rho^0_l}{\partial\varepsilon_l} v^x_l k_x. \tag{37}$$

On integrating over k and on averaging the result over all bands n, as in [3], we get from (37)

$$\sigma_{yx} = \frac{\pi}{3}\frac{e^4n\bar{\nu}}{m^2c^2\Delta^2}\frac{M_z}{M_s}\,\delta\left\langle\frac{1}{m^*}\right\rangle, \tag{38}$$

where $\delta \cong 5$ is the number of bands whose electrons take part in the conduction, and where $\langle 1/m^*\rangle$ and $\bar{\nu}$ are the averages over these bands of the inverse effective mass and of the effective density of the electrons.

When there is no current in the y direction, the anomalous Hall coefficient R_S, according to Ohm's law and (38), is

$$R_s = -\rho^2\sigma_{yx}\frac{1}{4\pi M_z} = -\frac{1}{12}\frac{e^4n\bar{\nu}}{m^2c^2\Delta^2}\frac{\delta}{M_s}\left\langle\frac{1}{m^*}\right\rangle\rho^2. \tag{39}$$

Setting $\Delta \sim 10^{-12}$ erg, $n \sim 10^{22}$ cm^{-3}, $m^* \sim 10\,m$, $M_S \sim 10^3$ Oe, and $\rho(300°K) \approx 7 \times 10^{-6}$ Ω-cm, we find that for agreement of R_S with the experimental values at room temperature ($\sim 10^{-11}$ V-cm/A-Oe [4]) it is necessary to take $\bar{\nu} \sim 10^{27}$ cm^{-3}. Such a value of $\bar{\nu}$ evidently corresponds to appreciable localization of the electron density in small regions of linear dimensions 10^{-9} cm.

On comparing (38) with the corresponding formulas from the work of Karplus and Luttinger, [3] we get for the effective spin-orbit field introduced there

$$H^{so} = -\pi e\bar{v}/3mc \sim 10^7 \text{ Oe}.$$

It is interesting to compare the value of R_S for scattering by phonons with the results of the work of Luttinger [2] for the impurity case. If we calculate the quantity $(\partial J^\alpha_l/\partial k_\beta)_0$ that occurs in the impurity conductivity σ^i_{yx} by means of formulas (31) and (34), we get

$$\sigma^i_{yx}/\sigma_{yx} = 2m^*\varepsilon_F/m3\bar{\Phi}_i, \tag{40}$$

where the Fermi energy $\epsilon_F \sim 10^{-12}$ erg and the mean potential impurity scattering $\bar{\Phi}_i \sim 10^{-14}$ to

*$[\mathbf{p}\sigma] = \mathbf{p}\times\sigma$.

*Equation (35) satisfies Onsager's relation $\sigma_{xy}(M_z) = -\sigma_{yx}(-M_z)$.

10^{-15} erg; this gives $\sigma_{yx}^{i}/\sigma_{yx} \sim 10^3$ to 10^4. This result means that at high temperatures the contribution of impurity scattering in the total $(\sigma_{yx}^{T})^{-1} = (\sigma_{yx})^{-1} + (\sigma_{yx}^{i})^{-1}$ is very small and that the total Hall constant $R_S^T = -\rho_T^2\sigma_{yx}^T/4\pi M_z \cong R_S$ remains, as before, proportional to ρ^2.

3. DISCUSSION OF RESULTS

The experimental data on the spontaneous Hall effect in the high-temperature range are known to give, for Fe and Ni, a dependence $R_S \sim \rho^n$, where n = 1.9 and 1.4 respectively. At low temperatures no such relation between R_S and ρ is usually found.[4] Thus we see that even in the high-temperature range, the mechanism of phonon scattering that we have considered, which gives n = 2 exactly, is incapable of explaining the experiments. In our treatment of scattering by phonons we neglected interband transitions, which can in general play a role for d-bands in the transition metals because of their considerable degeneracy. Consideration of such transitions, however, can apparently not affect the basic result $R_S \sim \rho^2$ but can only change the values of the coefficients.

One of the possible reasons for a deviation from the $R_S \sim \rho^2$ law is scattering by magnetic inhomogeneities, which at high temperatures can be treated like statistical ones. Elastic scattering by such inhomogeneities, by analogy with the results of Luttinger's work,[2] can lead to the appearance of a linear term in the dependence $R_S(\rho)$. Another reason, which complicates the whole phenomenon, may be the presence of several types of carriers. Thus, for example in Ni the normal Hall effect indicates that at least two types of carriers take part in electrical conduction. It is possible that this is also connected with the experimentally observed change of sign of R_S in Co.[4]

In the low-temperature range, our treatment of scattering should be modified by taking account of the deflection of the electrons by the induction field; this becomes important when the length of the free path becomes comparable with the radius of curvature of the electron trajectories. Inclusion of the corresponding terms in the kinetic equation completely changes the expansion of the scattering functions in powers of the scattering potential.

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Note added in proof (April 17, 1962). By separating as in formula (12), we have omitted correlation functions proportional to high powers of the phonon-interaction constant. Calculation of such correlation effects may give additional terms in the equations we are studying. They do not, however, give a contribution to the anomalous Hall effect, because the correlators obviously contain the interaction constant only in the form of even powers of its modulus.

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203