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CLOTHED OPERATORS AND THE COMMUTATIVITY CONDITION

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The causal properties of relativistic field theory operators, including the clothing condition, are studied. It is proved that no satisfactory relativistic field theory including the clothing condition and the weakened local commutativity condition can be constructed.

THE difficulties of the Hamiltonian method have led to the development of an axiomatic approach, in which the theory is based on several general physical requirements. At the present time there is only one trivial example of a field satisfying the field-theoretical axioms of Lorentz invariance, spectrality, and local commutativity; this is the free field. The question of the existence of the non-trivial example remains moot.

A new approach to the study of the field-theory axioms was pointed out in [1-3]. Additional limitations imposed on field theory lead to a sufficient criterion for the equivalence of this theory to the theory of the free field. In [5] the author established a sufficient criterion for the equivalence of a theory with field operators that obey the clothing condition.[3,7,8] A similar result was obtained by Federbush and Johnson[6] for a theory with canonical commutation relations.

In the present note we generalize the author's results[5] and also the results of Federbush and Johnson,[6] and establish a stronger equivalence criterion for the theory with the clothing condition. An important factor is in this case the admission of the commutativity condition in a certain arbitrary region of three-dimensional **x**-space. The criterion obtained answers in the negative the question whether a satisfactory relativistic field theory with the clothing condition and with the commutativity condition in a certain arbitrary region of **x**-space is possible. An attempt to answer this question was also made by Braun and Novozhilov.[9]

We use for our proof the analytic properties of the vacuum expectations of field-operator products.[2,10,11] It turns out that although the relaxed commutativity condition does not lead to all the properties of the vacuum expectations which follow from the local commutativity condition, nevertheless the commutativity condition is sufficient to prove the theorem with conditions that are formulated below.

We consider a relativistic field theory described by a Hermitian operator $A(x)$. The following conditions are assumed:

1) there exists a representation of the inhomogeneous Lorentz group $U(L)$ in a specified Hilbert space, relative to which the field operators satisfy the relation $U(L)\,A(x)\,U^{-1}(L) = A(Lx)$;

2) there exists a unique normalized vacuum state Ψ_0 and the spectrum of the energy momentum operator is positive;

3) the clothing condition is satisfied:

$$A(x)\,\Psi_0 = A^{in}(x)\,\Psi_0, \tag{1}$$

where the operator $A^{in}(x)$ is defined as follows:[12-14]

$$A^{in}(x) = A(x) + \int \Delta_R(x - x')\, j(x')\, dx', \tag{2}$$

$$j(x) = (\Box - m^2)\, A(\lambda). \tag{3}$$

It follows from the definition (2) that the operator $A^{in}(x)$ satisfies the Klein-Gordon equation

$$(\Box - m^2)\, A^{in}(x) = 0. \tag{4}$$

The following theorem holds true:

A relativistic field theory satisfying conditions 1)—3) and the relaxed commutativity condition[12]

$$[A(x^0, \mathbf{x}), A(x^{0\prime}, \mathbf{x}')] = 0, \quad x^0 = x^{0\prime},\ \mathbf{x} \neq \mathbf{x}',\ (\mathbf{x} - \mathbf{x}') \in D, \tag{5a}$$

where D is a certain open domain* in three-dimensional **x**-space, is equivalent to the free field theory.

In addition, we assume that the relaxed commutativity condition admits of the condition†

*A domain in three dimensional space is defined here as a a set of points with nonzero measure.

†A detailed investigation of the causality condition is given by Faĭnberg[15].

$$[A(x^0, \mathbf{x}), j(x^{0'}, \mathbf{x}')] = 0, \quad x^0 = x^{0'}, \quad \mathbf{x} \neq \mathbf{x}', \quad (\mathbf{x} - \mathbf{x}') \in D. \tag{5b}$$

Let us prove the theorem. Wightman[10] has shown that any relativistic field theory which contains conditions (1) and (2) is completely determined by specifying an infinite series of vacuum expectations of the field operator product. Two relativistic field theories are considered equivalent if all the vacuum expectations of the product of the field operators of one theory coincide with the corresponding vacuum expectations of the second field theory. Consequently it is sufficient for our purpose to demonstrate the equality

$$(\Psi_0, A(x_1) A(x_2) \ldots A(x_n) \Psi_0) = (\Psi_0, A^{in}(x_1) A^{in}(x_2) \ldots A^{in}(x_n) \Psi_0) \tag{6}$$

in all x-space and for all positive integers n.

Let us set up with the aid of (2) the expression

$$(\Psi_0, A(x_1) A(x_2) \ldots A(x_n) \Psi_0) = (\Psi_0, A^{in}(x_1) A^{in}(x_2) \ldots A^{in}(x_n) \Psi_0) + \Sigma, \tag{7}$$

where Σ denotes all the remaining terms. The vacuum expectations contained in Σ depend only on the operators $A^{in}(x)$ and $j(x)$. Vacuum expectations in which the current operator $j(x)$ is in the n-th place vanish by virtue of (1). By repeated application of (2), the remaining vacuum expectations can be expressed in terms of the vacuum expectations that depend only on the operators $A(x)$ and $j(x)$. To investigate such vacuum expectations we can already use the relaxed condition of local commutativity.

Let us prove that all these vacuum expectations vanish in all x-space. Analogous arguments have been developed in detail in [5]. Let us consider one of these vacuum expectations, for example

$$\begin{aligned} f(x_1, \ldots, x_i, \ldots, x_n) &= (\Psi_0, A(x_1) A(x_2) \ldots A(x_{i-1}) j(x_i) A(x_{i+1}) \ldots A(x_n) \Psi_0) \\ &= (\Box_{x_i} - m^2)(\Psi_0, A(x_1) \ldots A(x_{i-1}) A(x_i) A \\ &\quad \times (x_{i+1}) \ldots A(x_n) \Psi_0) \\ &= (\Box_{x_i} - m^2) F(x_1, \ldots, x_n). \end{aligned} \tag{8}$$

By virtue of the translational invariance, the function $F(x_1, \ldots, x_n)$ depends only on the differences $x_1 - x_2, \ldots, x_i - x_{i+1}, \ldots, x_{n-1} - x_n$. Let us denote the Fourier transform of $F(x_1 - x_2, \ldots, x_{n-1} - x_n)$ by $G(p_1, \ldots, p_{n-1})$, i.e.,

$$F(x_1 - x_2, \ldots, x_{n-1} - x_n) = \int \exp\left\{-i \sum_{k=1}^{n-1} (x_k - x_{k+1}) p_k\right\} \times G(p_1, \ldots, p_{n-1})\, dp_1 \ldots dp_{n-1}. \tag{9}$$

By virtue of assumption 2), the function $G(p_1, \ldots, p_{n-1})$ vanishes if the condition $p_k^2 \geq 0$, $p_k^0 \geq 0$ is not fulfilled for at least one of the p_k. It follows from this property of $G(p_1, \ldots, p_{n-1})$ that the function $F(x_1 - x_2, \ldots, n_{n-1} - x_n)$ admits of analytic continuation in tubular region T_1 of the complex variables $z_k = x_k - x_{k+1} - i\eta_k$;

$$-\infty < x_k^\mu - x_{k+1}^\mu < \infty, \quad \eta_k^2 > 0, \quad \eta_k^0 > 0 \quad (k = 1, 2, \ldots, n-1; \ \mu = 0, 1, 2, 3). \tag{10}$$

Hall and Wightman[2] have shown that the function $F(z_1, \ldots, z_{n-1})$ defined by the equality

$$F(z_1', \ldots, z_{n-1}') = \int \exp\left\{-i \sum_{k=1}^{n-1} z_k' p_k\right\} G(p_1, \ldots, p_{n-1})\, dp_1 \ldots dp_{n-1}, \tag{11}$$

is analytic in the expanded tubular domain T_1' and invariant relative to the complex proper Lorentz group $L_+(C)$. The region of the complex variables $z_1', \ldots, z_{m-1}'$, which contains all the points $z_k' = z_k$, $z_k \in T_1$, $\Lambda \in L_+(C)$, is called the expanded tubular region T_1'.

The function $F(x_1 - x_2, \ldots, x_{n-1} - x_n)$ is the limiting value of the function $F(z_1', \ldots, z_{n-1}')$, defined by Eq. (11) under the condition that the limiting transition from z_k' to $x_k - x_{k+1}$ is made in the domain $(\mathrm{Im}\, z_k')^2 > 0$, $(\mathrm{Im}\, z_k^{0'}) > 0$.

Jost[11] has shown that the domain T_1' contains also real points $x_k - x_{k+1}$, defined by the following condition: the vector

$$\sum_{k=1}^{n-1} \lambda_k (x_k - x_{k+1}), \quad \sum_{k=1}^{n-1} \lambda_k = 1, \quad \lambda_k \geqslant 0$$

should be space-like. This domain will be denoted by

$$I = \left\{x_1 - x_2, \ldots, x_{n-1} - x_n \,\middle|\, \left[\sum_{k=1}^{n-1} \lambda_k (x_k - x_{k+1})\right]^2 < 0, \ \sum_{k=1}^{n-1} \lambda_k = 1, \ \lambda_k \geqslant 0\right\}. \tag{12}$$

The points from this domain will be called Jost points. By virtue of assumption 1), we have

$$F(x_1 - x_2, \ldots, x_{n-1} - x_n) = F(\Lambda(x_1 - x_2), \ldots, \Lambda(x_{n-1} - x_n)), \quad \Lambda \in L_+, \tag{13}$$

where L_+ is the proper real Lorentz group.

By virtue of the commutativity condition (5a), the equality

$$\begin{aligned} F(x_1 - x_2, \ldots, x_{n-1} - x_n) &- F'(x_1 - x_2, \ldots, x_{i-1} - x_{i+1}, \ x_{i+1} - x_i, x_i - x_{i+2}, \ldots, x_{n-1} - x_n) \\ &= (\Psi_0, A(x_1) \ldots A(x_{i-1}) [A(x_i), A(x_{i+1})] \times A(x_{i+2}) \ldots A(x_n) \Psi_0) = 0 \end{aligned} \tag{14}$$

holds true in the domain

$$s = \{x_1 - x_2, \ldots, x_{n-1} - x_n \mid x_i^0 = x_{i+1}^0, \quad \mathbf{x}_i \neq \mathbf{x}_{i+1}, \quad (\mathbf{x}_i - \mathbf{x}_{i+1}) \in D\}. \tag{15}$$

Using relation (13), we find that equality (14) holds true in a broader domain

$$s' = \{x_1 - x_2, \ldots, x_{n-1} - x_n \mid (x_i - x_{i+1}) \in D'\}, \tag{16}$$

where the four-dimensional domain D′ is obtained from the three-dimensional domain D by applying the proper real Lorentz group L_+ to the vectors $\mathbf{x}_i - \mathbf{x}_{i+1} \in D$. From the definition of the domain D we see that it is located inside the cone $(\mathbf{x}_i - \mathbf{x}_{i+1})^2 < 0$ and the condition $(\mathbf{x}_i - \mathbf{x}_{i+1})^2 > 0$ is satisfied for the vectors $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ from this domain. After carrying out the transformations $\Lambda \in L_+$ on the vectors from D, we obtain the domain D′, which is characterized by the fact that the vectors from this domain will satisfy the inequality $(\mathbf{x}_i - \mathbf{x}_{i+1})^2 - (x_i^0 - x_{i+1}^0)^2 > 0$. For example, let the domain D have the form $0 < a < |\mathbf{x}_i - \mathbf{x}_{i+1}| < b$. Then the domain D′ will be characterized by the inequality

$$a^2 < (\mathbf{x}_i - \mathbf{x}_{i+1})^2 - (x_i^0 - x_{i+1}^0)^2 < b^2.$$

It is obviously located inside the cone.

Let us show that domains I and s′ contain a certain general $4(n-1)$-dimensional domain M_1. In the domain D′ we can always separate a domain D_1' which contains all the vectors satisfying the following condition: the difference $\mathbf{x}_i^\mu - \mathbf{x}_{i+1}^\mu$ has only one sign for each μ ($\mu = 1, 2, 3$). In other words, we require that the corresponding spatial components of the vectors $x_i - x_{i+1} \in D_1'$ lie on one side of zero on the corresponding axis. The domain of all vectors from D′, the spatial components of which have signs opposite those of the corresponding components of the vectors from D_1', will be denoted by D_2'. It is obvious that $D' = D_1' \cup D_2'$.

We note that if certain two space-like vectors ξ_1 and ξ_2 have the same signs in ξ^μ ($\mu = 1, 2, 3$) as the corresponding components of the vectors $x_i - x_{i+1} \in D_1'$, then the sum $\xi_1 + \xi_2$ is also a space-like vector. Indeed, by virtue of the inequalities $\xi_1^{02} - \xi_1^2 < 0$ and $\xi_2^{02} - \xi_2^2 < 0$, and by virtue of the property that for each value of μ the components ξ_1^μ and ξ_2^μ have the same sign, we have the inequality $\xi_1^0 \xi_2^0 - \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 < 0$. This means that $(\xi_1 + \xi_2)^2 < 0$. An analogous statement holds true also for the domain D_2'.

Let us show that the domain

$$s_1' = \{x_1 - x_2, \ldots, x_{n-1} - x_n \mid (x_i - x_{i+1}) \in D_1'\},$$

which is contained in s′, is also contained in I provided the vectors $x_1 - x_2, \ldots, x_{n-1} - x_n$ are space-like and their spatial components have the same signs as the corresponding components of the vectors from domain D_1'. The set of such vectors from s_1' forms a $4(n-1)$-dimensional set M_1, which satisfies all the requirements under which the domain I is defined. Indeed, the vector $\sum_{k=1}^{n-1} \lambda_k (x_k - x_{k+1})$, by virtue of the remark made above, will be space-like for all $\lambda_k \geq 0$, $\sum_{k=1}^{n-1} \lambda_k = 1$.

The domain s′ and the Jost domain for the function $F'(x_1 - x_2, \ldots, x_{n-1} - x_n)$

$$I' = \Big\{x_1 - x_2, \ldots, x_{n-1} - x_n \mid [\lambda_1' (x_1 - x_2) \ldots + \lambda_i' (x_{i+1} - x_i) \ldots + \lambda_{n-1}' (x_{n-1} - x_n)]^2 < 0, \quad \lambda_k' \geqslant 0, \quad \sum_{k=1}^{n-1} \lambda_k' = 1\Big\} \tag{17}$$

also contain a certain common $4(n-1)$-dimensional domain M_2. Indeed, in this case, as in the preceding one we can show that the domain

$$s_2' = \{x_1 - x_2, \ldots, x_{n-1} - x_n \mid (x_{i+1} - x_i) \in D_2\},$$

contained in s′, is also contained in I′ provided the vectors $x_1 - x_2, \ldots, x_1 - x_n$, $x_2 - x_3, \ldots, x_{n-1} - x_n$ are space-like and their spatial components have signs that coincide with those of the corresponding components of the vectors from the domain D_2'.

The equality (14) means that the values of the functions $F(z_1, \ldots, z_{n-1})$ and $F'(z_1, \ldots, -z_i, \ldots, z_{n-1})$ coincide in the doman s′. We have shown the domain s′ contains a domain M_1 in which the function $F(z_1, \ldots, z_{n-1})$ is analytic and single valued, and a domain M_2 which is the analyticity and uniqueness domain of the function $F'(z_1, \ldots, z_{n-1})$. By virtue of (14), our functions coincide in the domain M_2. This means that the function $F(z_1, \ldots, z_{n-1})$ is likewise analytic in the domain M_2. Consequently, the function $F(z_1, \ldots, z_{n-1})$ is analytic in the region $M_1 \cup M_2$. Thus, the domain $M_1 \cup M_2$ lies inside the intersection of the analyticity domains of the functions $F(z_1, \ldots, z_{n-1})$ and $F'(z_1, \ldots, z_{n-1})$, (see the appendix), the values of which coincide within this domain. This means [16,17] that $F(z_1, \ldots, z_{n-1})$ and $F'(z_1, \ldots, z_{n-1})$ are one and the same analytic function.*

It is seen from our reasoning that the condition

*This conclusion can also be obtained with the help of the "edge of the wedge" theorem.[18,19]

(5a) can be used to extend the region of analyticity of the function $F(z_1, \ldots, z_{n-1})$.

For our purpose it is necessary to extend our conclusions to include a more general case. Let us show that the equality

$$\begin{aligned} &F(x_1 - x_2, \ldots, x_{n-1} - x_n) \\ &\quad -\Phi(x_1 - x_2, \ldots, x_{i-1} - x_{i+1}, \quad x_{i+1} \\ &\quad - x_{i+2}, \ldots, x_{n-1} - x_n, x_n - x_i) \\ &\quad = (\Psi_0, A(x_1) \ldots A(x_i) \ldots A(x_n)\Psi_0) \\ &\quad - (\Psi_0, A(x_1) \ldots A(x_{i-1}) A(x_{i+1}) \\ &\quad \ldots A(x_n) A(x_i) \Psi_0) = 0 \end{aligned} \tag{18}$$

takes place in a certain $4(n-1)$-dimensional domain located inside the intersection of the analyticity regions of the functions

$F(z_1, \ldots, z_{n-1})$ and $\Phi(z_1, \ldots, z_{i-1} + z_i, \ldots, z_i + \ldots + z_{n-1})$.

Performing $n-i$ steps similar to those used to derive (14) and (16) with the aid of condition (5a), we arrive at the conclusion that Eq. (18) holds in the domain

$$S' = \{x_1 - x_2, \ldots, x_{n-1} - x_n \mid (x_i - x_\alpha) \in D', \quad \alpha = i+1, \ldots, n\}. \tag{19}$$

From the definition of the domain D' it follows that the vectors $(x_i - x_\alpha) \in D'$ are located inside the corresponding cones $(x_i - x_\alpha)^2 < 0$. Obviously, the domain S' contains a subdomain

$$M = \left\{x_1 - x_2, \ldots, x_{n-1} - x_n \mid (x_i - x_\alpha) \in D', \quad (x_k - x_j)^2 < 0, \begin{matrix} k, j = 1, 2, \ldots n; \\ \alpha = i+1, \ldots n \end{matrix}\right\}. \tag{20}$$

It follows from the condition (5a) that weak local commutativity takes place on the set of spatially separated vectors $x, x_2, \ldots, x_n$ (see the appendix). Consequently, weak local commutativity takes place also on M. Then, by virtue of Dyson's theorem, [20,21] the functions

$F(z_1, \ldots, z_{n-1})$ and $\Phi(z_1, \ldots, z_{i-1} + z_i, \ldots, z_i + \ldots + z_{n-1})$

represent the same analytic function. Thus, the functions

$$F(x_1 - x_2, \ldots, x_{n-1} - x_n),$$
$$\Phi(x_1 - x_2, \ldots, x_{i-1} - x_{i+1}, \ldots, x_n - x_i)$$

are the limiting values of the same analytic function.

Let us consider another function $F(x_1 - x_2, \ldots, x_{n-1} - x_n)$ and

$$\begin{aligned} &f'(x_1 - x_2, \ldots, x_{n-1} - x_n) \\ &\quad = (\Psi_0, A(x_1) \ldots A(x_{i-1}) A(x_{i+1}) \ldots A(x_n) j(x_i) \Psi_0). \end{aligned} \tag{21}$$

By virtue of the commutativity condition (5b), the equality

$$\begin{aligned} &f(x_1 - x_2, \ldots, x_{n-1} - x_n) \\ &\quad - f'(x_1 - x_2, \ldots, x_n - x_i) = 0 \end{aligned} \tag{22}$$

is satisfied in the domain S'. Repeating the arguments given above, we arrive at the conclusion that (22) is satisfied in a certain $4(n-1)$-dimensional domain, which is contained in the intersection of the analyticity domains of the functions $f(z_1, \ldots, z_{i-1})$ and $f'(z_1, \ldots, z_{i-1} + z_i, \ldots, z_i + \ldots + z_{n-1})$. These functions are analytic continuations of the functions $f(x_1 - x_2, \ldots, x_{n-1} - x_n)$ and $f'(x_1 - x_2, \ldots, x_n - x_i)$ respectively into the tubular region. In addition, from the conclusions drawn it follows that the functions $f'(z_1, \ldots, z_{n-1})$ and $f'(z_1, \ldots, z_i + \ldots + z_{i-1})$ are the same function, $\Psi(z_1, \ldots, z_{n-1})$.

By virtue of the clothing condition (1), the function $f'(x_1 - x_2, \ldots, x_i - x_i) \equiv 0$ in all x-space. It follows therefore that its analytic continuation $\Psi(z_1, \ldots, z_{n-1}) \equiv 0$ in all z-space. This in turn signifies that all the boundary values of the function $f(z_1, \ldots, z_{n-1})$ are also equal to zero. Consequently, $f(x_1 - x_2, \ldots, x_{n-1} - x_n) \equiv 0$ in all x-space.

Analogous arguments apply to each term in Σ. This completes the proof of the theorem.

In conclusion, I am grateful to V. Ya. Faĭnberg for a discussion and valuable remarks and to V. I. Kolomytsev and D. Ya. Petrina for discussions.

APPENDIX

We examine the connection between the relaxed local commutativity condition (5) and the weak local commutativity (WLC).*

We say that WLC holds on a set of real 4-vectors $\xi_1, \ldots, \xi_{n-1}$ if the following condition is satisfied on this set

$$F(\xi_1, \ldots, \xi_{n-1}) = F(-\xi_{n-1}, \ldots, -\xi_1), \quad \xi_j = x_j - x_{j+1}. \tag{A.1}$$

We shall show that conditions 1), 2), and (5) imply WLC on all real spatially separated points $x_1, \ldots, x_n$, i.e., on points satisfying the conditions and $(x_k - x_j)^2 < 0$; $k \neq j$, $k, j = 1, 2, \ldots, n$.

The function $F(\xi_1, \ldots, \xi_{n-1})$, by virtue of as-

*This question arose in a discussion with V. Ya. Faĭnberg.

sumptions 1) and 2), admits of an analytic continuation $F(z_1, \ldots, z_{i-1})$ into the tubular region T. In accordance with the Hall-Wightman theorem $F(z_1, \ldots, z_{n-1})$ is a function of the scalar products $z_k \cdot z_j$ $(k, j = 1, 2, \ldots, n)$, analytic on the complex manifold $\mathfrak{M}$ described by the scalar products $z_k \cdot z_j$, as the vectors $z_1, \ldots, z_{n-1}$ vary in the tubular region T. Hall and Wightman have also shown that $F(\xi_1, \ldots, \xi_{i-1})$ is an analytic function of the variables $Z_{kj} = \xi_k \cdot \xi_j$ in a certain subset of the set of space-like vectors ξ_j $(j = 1, 2, \ldots, n-1)$. In addition, the function $F(\xi_1, \ldots, \xi_{n-1})$ is uniquely defined on the manifold $Z_{kj} = \xi_k \cdot \xi_j$ for the spatially separated vectors $x_1, x_2, \ldots, x_n$. We denote the set of all the spatially separated vectors $x_1, x_2, \ldots, x_n$ by I_1.

We have already shown that the equality

$$F(\xi_1, \ldots, \xi_{n-1}) = F'(\xi_1, \ldots, \xi_{i-1} + \xi_i, -\xi_i, \xi_i + \xi_{i+1}, \ldots, \xi_{n-1}) \qquad \text{(A.2)}$$

is satisfied in the domain $s' = \{\xi_1, \ldots, \xi_{n-1} | \xi_i \in D'\}$ where the 4-dimensional domain D' is contained in the cone $\xi_i^2 < 0$. We denote by N the set of vectors satisfying the definition of the domain I_1 and the condition $\xi_i \in D'$. Since the function $F'(\xi_1, \ldots, -\xi_i, \ldots, \xi_{n-1})$ is also analytic and single-valued in I_1, we arrive at the conclusion that the functions $F(\xi_1, \ldots, \xi_{n-1})$ and $F'(\xi_1, \ldots, -\xi_i, \ldots, \xi_{n-1})$ have a common domain, in which they are analytic and single-valued, and in this domain the equality (A.2) is satisfied.

It is known[17] that such functions are analytic continuations of one another. In other words, if R and R′ denote the regularity domains of the functions $F(\ldots, z_i \cdot z_j, \ldots)$ and $F'(\ldots, z_i \cdot z_j, \ldots)$ then our conclusion signifies that there exists one and only one function $\Phi(\ldots, z_i \cdot z_j, \ldots)$ analytic in the domain $R \cup R'$ and coinciding with $F(\ldots, z_i \cdot z_j, \ldots)$ in the domain R and with $F'(\ldots, z_i \cdot z_j, \ldots)$ in the domain R′.

On the other hand, $F(z_1, \ldots, z_{n-1})$ and $F'(z_1, \ldots, z_{n-1})$, as functions of 4-vectors, are regular in the expanded tubular domains T_1' and T_2'. Consequently, by virtue of commutativity condition (5), the analyticity domain of $F(z_1, \ldots, z_{n-1})$ will be $T_1' \cup T_2'$. Continuing this process, we find that by virtue of condition (5) the function $F(z_1, \ldots, z_{n-1})$ is analytic within the holomorphy envelope $\bigcup_{j=1}^{m} T_j'$, where m is the number of commutations of the indices 1, 2, . . . , and n.

By virtue of Ruelle's theorem,[22] the holomorphy envelope encloses all the real spatially separated points $x_1, \ldots, x_n$. Consequently, in our case Dyson's theorem holds true:[20] "If weak local commutativity takes place for all spatially separated points, then the function $F(\xi_1, \ldots, \xi_{n-1})$ is analytic on the same points, and vice versa." This means that WLC follows from the commutativity condition (5).

The Jost theorem[11] states that for the CPT invariance of the function $F(\xi_1, \ldots, \xi_{n-1})$ it is necessary and sufficient for WLC to hold at least on one point in T_1'. The conditions of this theorem are fulfilled. Consequently, condition (5) leads also to the CPT invariance of the theory.

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