ANALYTIC PROPERTIES OF THE BOUND STATE WAVE FUNCTION FOR A SUPERPOSITION OF YUKAWA POTENTIALS

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It is proved that the wave function of the bound S state for a superposition of Yukawa potentials in momentum space is an analytic function of the square of the momentum t throughout the complex plane except for a pole at the point corresponding to the bound state energy and a cut on the negative half-axis. As $|t| \to \infty$ the wave function decreases at least as rapidly as $|t|^{-2}$. An equation is set up from which the discontinuity of the wave function can be determined in a simple manner. The eigenvalues of the Schrödinger equation are determined from the law of decrease of the discontinuity at infinity.

1. INTRODUCTION

THE analytic properties of the wave function of the bound state for a superposition of Yukawa potentials are of interest in two respects. First, they can lead to a new method for a practical determination of the wave function, which may be useful in nuclear physics problems. Second, they are closely related to the quantum theory of fields. It is known that the problem of two nonrelativistic particles interacting through a potential of the above-mentioned type leads to an S matrix with analytic properties which are very similar to what one obtains in quantum field theory. In the latter one encounters quantities which are the analog of the wave function of a bound state.[1] Their analytic properties are possibly also close to the properties of the corresponding quantities in quantum mechanics.

For simplicity we consider spinless particles and restrict our discussion to the S state. We set $\hbar = 2M = 1$, where M is the reduced mass of the particles. The Schrödinger equation has the form

$$-\nabla^2\tilde{\psi}(r) + V(r)\tilde{\psi}(r) = -\varkappa^2\tilde{\psi}(r),$$
$$V(r) = \frac{1}{r}\int_{\mu^2}^{\infty} e^{-r\sqrt{z}}\rho(z)\,dz. \tag{1}$$

We assume that $\rho(z)$ is a continuous function except, perhaps, for δ-type singularities, and decreases faster than z^{-1}. In momentum space we obtain (after integrating by parts)

$$(k^2+\varkappa^2)\psi(k^2) = \frac{1}{2\pi}\int_0^{\infty}\frac{dq^2}{q}\varphi(q^2)\int_{\mu^2}^{\infty}\frac{z+k^2-q^2}{[z+(q+k)^2][z+(q-k)^2]}\rho(z)\,dz, \tag{2}$$

where $\varphi(k^2) \equiv \int\psi(k^2)\,dk^2$ is assumed real for $k^2 \geq 0$. For $k^2 \to \infty$ the function $\psi(k^2)$ decreases faster than k^{-3}.

Formula (2) is valid for $k^2 \geq 0$. We shall be interested in the analytic properties of the function $\psi(k^2)$, the solution of Eq. (2), for all complex values $k^2 \equiv t$.

2. ANALYTIC PROPERTIES OF $\psi(t)$

We shall show in this section that $\psi(t)$ is an analytic function in the entire complex plane except for a pole at the point $t = -\kappa^2$ and a cut along the line $t \leq -\mu^2$. Later on we shall show that this cut actually starts somewhat further away from the origin, namely at the point $t \leq -(\mu+\kappa)^2$. The function $\psi(t)$ vanishes more strongly than t^{-2} as $|t| \to \infty$.

Let $\psi(k^2)$ be a solution of Eq. (2). We substitute $\varphi(q^2)$ in the right-hand side and consider the resulting equation for complex $k^2 \equiv t$. It is convenient to begin with the discussion of the simpler function

$$f(z,\,t) \equiv \int_0^{\infty}\varphi(q^2)\frac{z+t-q^2}{(z+t+q^2)^2-4tq^2}\frac{dq^2}{q}. \tag{3}$$

The denominator vanishes at two points:

$$q_1^2 = a^2-(b-\sqrt{z})^2+i2a(b-\sqrt{z}),$$
$$q_2^2 = a^2-(b+\sqrt{z})^2+i2a(b+\sqrt{z}), \tag{4}$$

where a and b are the real and imaginary parts of $\sqrt{t}$:

$$t = a^2-b^2+2iab, \qquad b \geqslant 0.$$

As long as the singular points (4) do not fall on the positive real half-axis, the function $f(z,t)$ is an analytic function of the variable t. A singularity occurs at $b = \sqrt{z}$, when $q_1^2 = a^2$. The singular curve of the function $f(z,t)$ in the t plane has the form of a parabola:

$$\mathrm{Re}\, t = \frac{1}{4z} (\mathrm{Im}\, t)^2 - z. \tag{5}$$

The function $f(z,t)$ is therefore analytic in the t plane except for a cut along the curve (5).

The function $\psi(t)$ can be written in the form

$$\psi(t) = \frac{1}{2\pi (t + \varkappa^2)} \int_{\mu^2}^{\infty} f(z, t)\, \rho(z)\, dz \tag{6}$$

and will be analytic to the right of the parabola (5) with $z = \mu^2$, except for a pole at $t = -\kappa^2$. The function $\varphi(t)$ will be analytic in the same region with an additional cut along the line $t \le -\kappa^2$.

We note further that formula (3) and the law of decrease of $\psi(k^2)$ as $k^2 \to \infty$ imply that $f(z,t)$ vanishes within the region of analyticity at least as rapidly as t^{-1} as $|t| \to \infty$ and $\psi(t)$ at least as rapidly as t^{-2}.

Let us now show that the function $\psi(t)$ can be continued analytically out of the above-mentioned region of analyticity into the whole complex plane except for a cut along the negative real half-axis.

Let $a > 0$ and $b = 0$. The positions of the two singularities (4) in the complex q^2 plane are as shown in the figure. As b increases with fixed a, these singularities move along the parabola B,

$$\mathrm{Re}\, q^2 = a^2 - \frac{1}{4a^2} (\mathrm{Im}\, q^2)^2$$

in the directions indicated by the arrows. When the point 1 meets the integration contour $q^2 > 0$, $f(z,t)$ becomes singular. Using the proven analyticity of $\varphi(t)$, we deform the integration contour of the integral (3) in the upward direction so as to avoid the point 1. A singularity will then occur for values of b larger than $\sqrt{z}$, i.e., we continue $f(z,t)$ analytically across the cut. The limiting position of the integration contour C is, of course, fixed by the above-shown region of analyticity of $\varphi(t)$, and is given by the upper half of the parabola (5) with $z = \mu^2$ and the strip $(-\mu^2, 0)$. A singularity now occurs when the point 1 crosses the new contour of integration (i.e., when the point 1 reaches the point 1′). This gives $b = \sqrt{z} + \mu$.

We have thus been able to continue $f(z,t)$ analytically beyond the parabola (5) into the upper half-plane up to the parabola

$$\mathrm{Re}\, t = \frac{1}{4(\sqrt{z} + \mu)^2} (\mathrm{Im}\, t)^2 - (\sqrt{z} + \mu)^2. \tag{7}$$

The analytic continuation of the function $f(z,t)$ in the new region of analyticity will, as before, vanish at least as rapidly as t^{-1} for $|t| \to \infty$ (by reason of its integral representation). Formula (6) enables us to continue the function $\psi(t)$ analytically into the upper half-plane up to the parabola (5) with $z = (2\mu)^2$ and to determine its behavior at infinity. This again makes it possible to deform the contour of integration in (3) even further, thus allowing a further analytic continuation of $f(z,t)$. Let us assume that we have, at some stage, continued $\psi(t)$ into the upper half-plane up to the parabola (5) with $z = (n\mu)^2$ and that the function $\psi(t)$ vanishes in this region at least as rapidly as t^{-2} for $|t| \to \infty$. We place the integration contour in the q^2 plane on this parabola and on the strip on the negative half-axis. A singularity of $f(z,t)$ occurs when the point reaches the parabola, i.e., when simultaneously

$$\mathrm{Re}\, q^2 = a^2 - \frac{1}{4a^2} (\mathrm{Im}\, q^2)^2,$$
$$\mathrm{Re}\, q^2 = \frac{1}{4(n\mu)^2} (\mathrm{Im}\, q^2)^2 - (n\mu)^2, \tag{8}$$

where $\mathrm{Im}\, q^2 = 2a(b - \sqrt{z})$. Solving this system of equations for b with fixed $a > 0$, we find $b = n\mu + \sqrt{z}$. The function $f(z,t)$ can then be continued analytically up to the parabola

$$\mathrm{Re}\, t = \frac{1}{4(n\mu + \sqrt{z})^2} (\mathrm{Im}\, t)^2 - (n\mu + \sqrt{z})^2,$$

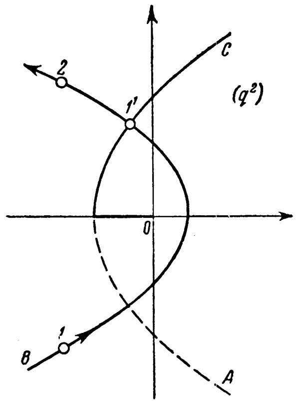
and $\psi(t)$ can be continued up to the parabola (5) with $z = ((n+1)\mu)^2$. It follows from the integral representation that in the new region the function $\psi(t)$ also decreases at least as rapidly as t^{-2} when $|t| \to \infty$.

We have thus proved by induction that $\psi(t)$ can be continued analytically into the entire upper half-plane, and we have determined its behavior as $|t| \to \infty$. By deforming the contour in the downward direction, we can prove in an analogous manner that the function can be continued into the lower half-plane. The negative half-axis coincides with the cut for $t \le -\mu^2$ (we have made essential use of of the fact that $a \ne 0$).

3. DISCONTINUITY OF THE FUNCTION $\psi(t)$ ON THE NEGATIVE HALF-AXIS*

The discontinuities of the functions $\psi(t)$, $\varphi(t)$, and $f(z,t)$ on the negative half-axis are pure imaginary and equal to twice the imaginary parts of the corresponding functions. Let us begin the discussion with the discontinuity of the function $f(z,t)$. We make use of the proven analyticity of $\varphi(t)$. We select the branch of the root appearing

*See also the paper of Blankenbecler and Cook.[1]



A – parabola Re $q^2 = (1/4\mu^2)(\mathrm{Im}\ q^2)^2 - \mu^2$, B – parabola Re $q^2 = a^2 - (1/4a^2)(\mathrm{Im}\ q^2)^2$, C – contour of integration.

under the integral sign in (3) such that the cut lies on the negative half-axis $(-\infty, 0)$.

Let $a > 0$ and $b < \sqrt{z}$. As we have seen, the singularities of the integrand as a function of q^2 consist of a cut on the negative half-axis and the two poles q_1^2 and q_2^2 shown in the figure. If $a < 0$, the points q_1^2 and q_2^2 interchange places. Let us now replace the integration along the positive half-axis by one along the upper or lower branches of the negative half-axis. We use the notation

$$F(q^2) \equiv \frac{\varphi(q^2)}{q}\,\frac{z+t-q^2}{(z+t+q^2)^2-4tq^2}. \tag{9}$$

It is clear that for Im $t > 0$ $(a > 0)$

$$\begin{aligned} f(z,t) &= 2\pi i \operatorname*{Res}_{q^2=q_2^2} F(q^2) - \int_{-\infty}^{0} F^+(q^2)\,dq^2 \\ &= -2\pi i \operatorname*{Res}_{q^2=q_1^2} F(q^2) - \int_{-\infty}^{0} F^-(q^2)\,dq^2, \end{aligned} \tag{10}$$

and for Im $t < 0$ $(a < 0)$

$$\begin{aligned} f(z,t) &= 2\pi i \operatorname*{Res}_{q^2=q_1^2} F(q^2) - \int_{-\infty}^{0} F^+(q^2)\,dq^2 \\ &= -2\pi i \operatorname*{Res}_{q^2=q_2^2} F(q^2) - \int_{-\infty}^{0} F^-(q^2)\,dq^2. \end{aligned} \tag{11}$$

Here F^+ and F^- are the values of F above and below. The integrals along the large semicircles vanish, since we have shown that $\varphi(q^2)$ does not increase as $|q^2| \to \infty$. The integrals on the right-hand side are analytic functions of t with a cut along the real axis. The first two expressions therefore continue $f(z,t)$ into the entire upper half-plane, and the last two expressions, into the entire lower half-plane. The discontinuity of $f(z,t)$ on the negative half-axis is the sum of the discontinuities of the integrals and of the remaining terms. It is easily seen that for $b < \sqrt{z}$, $t < 0$

$$f^+(z,t) - f^-(z,t) = 0, \tag{12}$$

and for $b > \sqrt{z}$, $t < 0$

$$f^+(z,t) - f^-(z,t) = -2\pi i \left\{ \operatorname*{Res}_{q^2=q_1^2} F^+(q^2) + \operatorname*{Res}_{q^2=q_1^2} F^-(q^2) \right\}. \tag{13}$$

Let us denote Im $\varphi^+(t) \equiv \eta(v)$, where $v = +\sqrt{-t}$, and define $\eta(v)$ for $v < 0$ as being equal to zero. Then it follows from (12) and (13) that

$$\mathrm{Im}\, f^+(z,t) = \pi v^{-1}\eta(v - \sqrt{z}). \tag{14}$$

The discontinuity of $\psi(t)$ is found from this formula with the help of (6).

Let us use this relation for an actual construction of the function $\psi(t)$. Differentiating η with account of the definition of $\varphi(t)$ and formulas (6) and (14), we obtain

$$\eta'(v) = \frac{1}{v^2-\varkappa^2}\int_{\mu^2}^{\infty} \eta(v-\sqrt{z})\,\rho(z)\,dz + d\delta(v-\varkappa), \tag{15}$$

$$d = \frac{1}{2}\int_{\mu^2}^{\infty} \mathrm{Re}\, f(z, -\varkappa^2)\,\rho(z)\,dz. \tag{16}$$

We have thus arrived at an integro-differential equation (15) for the determination of η. It is solved rather easily. Let us assume first that $0 < v < \min(\mu, \kappa)$. Then (15) implies that $\eta'(v) = 0$. Furthermore, it is clear from (15) that $\eta(v)$ is continuous except at the point $v = \kappa$. For the chosen values of v we thus have $\eta = 0$. If $\kappa > \mu$, we consider the interval $\mu < v < \min(2\mu, \kappa)$ and show in an analogous manner that also here $\eta = 0$, etc. The final result will be the following:

$$\eta(v) = 0; \qquad v < \varkappa \tag{17}$$

(this also confirms our earlier assertion that Im $\psi^+(t) = 0$ for $t > -(\mu+\kappa)^2$ except for the contribution from the pole).

In the point $v = \kappa$ the function discontinuously becomes equal to d. Let us consider the interval $\kappa < v < \kappa + \mu$. Here $v - \sqrt{z} < v - \mu < \kappa$ and, according to (15), $\eta'(v) = 0$. Hence η is here also constant and equal to d. For $\kappa + \mu < v < \kappa + 2\mu$ we have $v - \sqrt{z} < \kappa + \mu$, and the function under the integral sign in (15) is then either zero or equal to d. Knowing $\eta'(v)$ and using continuity, we find η also in this interval. The next interval, $\kappa + 2\mu < v < \kappa + 3\mu$, is treated in the same way, etc. Proceeding thus in steps of μ, we determine the function $\eta(v)$ for all values of μ.

Having found $\eta(v)$, we write $\varphi(t)$ as an integral of the Cauchy type:

$$\varphi(t) = \frac{1}{\pi}\int_{-\infty}^{-\varkappa^2} \frac{\eta(\sqrt{-t'})\,dt'}{t'-t} + C \tag{18}$$

and obtain the desired function $\psi(t)$ by differentiation of $\varphi(t)$.

We must now investigate the convergence of the integral (18), which is closely connected with the behavior of our functions φ and ψ at infinity. We have seen that the desired solution $\psi(t)$ must vanish at infinity at least as rapidly as t^{-2}, which implies that $\eta(v) \to 0$ as $v \to \infty$. Conversely, if $\eta(v) \to 0$ as $v \to \infty$, then $\psi(t)$ behaves no worse than t^{-2} as $|t| \to \infty$. Indeed, if $\eta(v) \to 0$ as $v \to \infty$, then $\operatorname{Im} f^{+}(z,t)$, according to (14), vanishes more rapidly than $t^{-1/2}$ as $|t| \to \infty$, so that $f(z,t)$ decreases more rapidly than $t^{-1/2}$. Hence $\psi(t)$ is better-behaved than $t^{-3/2}$. It follows at once that $\eta(v)$ vanishes more rapidly than v^{-1} as $v \to \infty$. This means that $f(z,t)$ vanishes at least as rapidly as t^{-1} and $\psi(t)$ at least as rapidly as t^{-2}, q.e.d. Therefore

$$\eta(v) \to 0 \quad \text{for } v \to \infty \tag{19}$$

is the necessary and sufficient condition for the correct behavior of our φ and ψ at infinity.

It would seem that, besides condition (19), we should also satisfy Eq. (16). It turns out, however, that (16) and (19) are completely equivalent. This follows from the fact that any function $\eta(v)$ which satisfies Eq. (15) also satisfies the relation

$$\frac{1}{2}\int_{\mu^2}^{0} \rho(z)\,dz \int_{-\infty}^{\infty} \frac{\eta(\sqrt{-t'}-\sqrt{z})}{t'+\varkappa^2}\,\frac{dt'}{\sqrt{-t'}} = d - \eta(\infty). \tag{20}$$

In order to construct the solution $\psi(t)$, we must therefore find η by successive integration of Eq. (15) and then require that (16) or (19) be satisfied. The functions η obtained in this fashion provide us with the required eigenfunctions $\psi(t)$, and conditions (16) or (19) determine the eigenvalues $-\kappa^2$.

4. CONCLUSION

Let us summarize the results obtained. We have shown that the wave function $\psi(k^2)$ of the bound S state for a superposition of Yukawa potentials is an analytic function of the complex variable $k^2 \equiv t$ in the entire complex plane except for a pole at $t = -\kappa^2$ and a cut for $t \leq -(\mu+\kappa)^2$. We have established Eq. (15) for the discontinuity of the function $\varphi(t) \equiv \int \psi(t)\,dt$. The discontinuity can be determined from this equation in successive steps of μ in the direction of increasing $|t|$. This corresponds to a successive determination of the wave function at ever closer ranges in coordinate space. The condition that the discontinuity vanish at infinity (19) or the equivalent condition (16) provide equations for the determination of the eigenvalues $-\kappa^2$.

We hope that a further study of the field theoretic quantities related to bound states will reveal a picture which is not too different from the one obtained here.

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[1] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

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