ANISOTROPY OF ABSORPTION OF ULTRASOUND IN SUPERCONDUCTORS

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The peculiarities of ultrasonic absorption in anisotropic superconductors are investigated. It is shown that the absorption coefficient in the body of the metal is sharply anisotropic at low temperatures. The absorption coefficient is found for a superconducting plate whose thickness is much smaller than the range of the excitations. The absorption in this case is determined by the size of the energy gap of the quasiparticles moving in the plane of the plate perpendicular to the direction of sound propagation. Detailed experimental investigation of the ultrasonic absorption in superconductors makes it possible to obtain the angular and temperature dependence of the energy gap.

THE absorption of ultrasound in isotropic superconductors has been investigated theoretically by Bardeen, Cooper, and Schrieffer;[1] it was shown that the ratio of the coefficients of absorption in superconducting and normal states of a metal is determined by the size of the energy gap.

Clearly expressed anisotropies of the energy gap of real metals have been established experimentally by Zavaritskiĭ,[2] and also by Bezuglyĭ, Galkin and Korolyuk.[3,4] It is therefore of interest to explain how the data on the energy spectrum of superconductors can be obtained by means of measurement of ultrasonic absorption.

The mechanism af the absorption of ultrasound in normal metals has been studied in a number of researches.[5-9] If the length of the sound wave is small in comparison with the free path of the electrons, then the basic contribution to the absorption coefficient is made by electrons moving in planes of constant phase of the sound wave. It is clear that these considerations are also valid for superconductors. Therefore, as a consequence of the Boltzmann distribution of the energy excitations, the absorption coefficient at low temperatures depends (in the case of an infinite free path) only on the minimum value of the energy gap along the line $\mathbf{k}\cdot\mathbf{v} = 0$ on the Fermi surface ($\mathbf{k}$ is the sound propagation constant and $\mathbf{v}$ is the velocity of the electron).*

Ultrasonic absorption in the body of superconductors has been studied by Pokrovskiĭ[10] with the aid of the method of Green's functions, and it has been made clear that the experimental study of this phenomenon can serve as a method for establishing the anisotropy of the energy gap. However, the effect of collisions of electrons with impurities was not taken into account in this work. The presence of collisions materially changes the results in the region of low temperatures. The absorption coefficient at low temperatures is sharply anisotropic and reaches a maximum when the absolute minimum of the gap is located on the line $\mathbf{k}\cdot\mathbf{v} = 0$. A sharp anisotropy of the absorption coefficient at low temperatures takes place also for infinitely long free paths. However, the presence of collisions leads to a number of specific peculiarities in the angular dependence of the absorption coefficient.

In particular, for comparatively high temperatures, the width of the maximum in the angular diagram of the absorption coefficient can be much less in certain cases than in the absence of collisions. On the other hand, upon further decrease in the temperature, the collisions of electrons with impurities lead to a broadening of the maxima.

The temperature dependence of the absorption coefficient can be established from data on the attenuation of the ultrasound in the body of the specimen only for temperatures much less than critical, i.e., exactly in the region where this dependence is of little importance.

The study of ultrasonic absorption in a superconducting plate whose thickness is small in comparison with the path length of the excitations between the collisions with impurities, offers greater prospects. In diffusion scattering on the boundaries, the real free path in the direction perpendicular to

*Precisely this gap was measured by Bezuglyĭ, Galkin, and Korolyuk,[3,4] who mistakenly assumed that the absorption coefficient is determined by the size of the gap in the direction of propagation of the sound.

the plane of the plate falls off sharply. Therefore, for wavelengths that are small in comparison with the path length, and at not very low temperatures, the principal contribution to the absorption coefficient is made by excitations moving in the plane perpendicular to the direction of sound propagation. The absorption coefficient depends on the size of the gap only for selected directions. This makes it possible to establish completely the temperature and angular dependence of the energy gap for not too complicated Fermi surfaces.

1. We turn to the quantitative consideration of the problem. Following Khalatnikov,[11] we assume that the excitation energy in the undeformed metal is equal to

$$\varepsilon(\mathbf{p}) = \sqrt{\Delta^2(\mathbf{p}) + \xi^2(\mathbf{p})}, \tag{1}$$

where $\Delta(\mathbf{p})$ is the gap, $\xi(\mathbf{p}) = E(\mathbf{p}) - \mu$ is the energy of the electron, measured from the Fermi energy, and $\mathbf{p}$ is the quasimomentum.

As a result of the lattice deformation, brought about by the propagation of the sound wave in the crystal, the energy of the electron and the Fermi energy are renormalized.

Taking into account the results of Akhiezer, Kaganov and Lyubarskiĭ,[5,6] we assume

$$\xi(\mathbf{p}, \mathbf{r}) = \xi(\mathbf{p}) + \xi_1(\mathbf{p}, \mathbf{r}) = \xi(\mathbf{p}) + (\lambda_{ik} - \langle\lambda_{ik}\rangle)\, u_{ik}. \tag{2}$$

Here $u_{ik} = (1/2)(\partial u_i/\partial x_k + \partial u_k/\partial x_i)$, $\mathbf{u} = \mathbf{u}_0 e^{i(\omega t - kx)}$ is the displacement vector,

$$\langle\lambda\rangle = \frac{\int d\tau_p \lambda \delta(\xi(\mathbf{p}))}{\int d\tau_p \delta(\xi(\mathbf{p}))}, \qquad d\tau_p = dp_x dp_y dp_z = \frac{2d\varepsilon dS}{|\partial\varepsilon/\partial\mathbf{p}|}, \tag{3}$$

$\lambda_{ik} \sim \mu$, and dS is the element of area of the constant-energy surface.

The energy of the excitation gains the amount [see (1), (2)]

$$\varepsilon_1(\mathbf{p}, \mathbf{r}) = \xi_1(\mathbf{p}, \mathbf{r})\, \xi(\mathbf{p})/\varepsilon(\mathbf{p}). \tag{4}$$

The renormalization of the energy gap can be neglected. The absorption coefficient is equal to (see[5-9])

$$\alpha_s = \frac{T\dot{S}}{W} = -\frac{2}{\hbar^3 W}\int \nu' |\chi|^2 \frac{\partial n_0}{\partial\varepsilon}\frac{d\varepsilon dS}{|\partial\varepsilon/\partial\mathbf{p}|}. \tag{5}$$

Here $T\dot{S}$ is the rate of energy dissipation in the sound wave, W is the acoustic energy density, $(\partial n_0/\partial\epsilon)\chi$ is the nonequilibrium contribution to the local-equilibrium distribution function of the excitations

$$n_0(\varepsilon) = (e^{\varepsilon/T} + 1)^{-1},$$

$\nu' = \nu|\xi|/\epsilon$ is the collision frequency of excitations with impurities, ν is the collision frequency of the electrons (see[12]), and h is Planck's constant. The possibility of the introduction of the collision frequency can be motivated in the same way as in the paper by Gurevich.[7]

To determine the absorption coefficient, we shall solve the classical kinetic equation, which is permissible upon satisfaction of the conditions $\hbar k \ll p$, $\hbar\omega \ll T$, $\hbar\nu \ll \Delta$ (the latter inequality means that the "pair radius" $\xi_0 \sim \hbar v/\Delta$ is small in comparison with the path length of the electron $l = v/\nu$). All the enumerated conditions are satisfied in the experiment (see[3,4]).

The case in which the free path of the electron is comparable with the "pair radius" deserves separate consideration.*

The linearized kinetic equation has the form

$$\left(i\omega - ik\frac{\partial\varepsilon}{\partial p_x} + \nu'\right)\chi + \frac{\partial\varepsilon}{\partial p_z}\frac{\partial\chi}{\partial z} = \dot{\varepsilon}_1 \tag{6}$$

(the z axis is perpendicular to the plane of the plate) with boundary conditions

$$\chi_+\left|\frac{\partial\varepsilon}{\partial p_z}\right|_{z=0} = R\chi_-\left|\frac{\partial\varepsilon}{\partial p_z}\right|_{z=0}, \qquad \chi_-\left|\frac{\partial\varepsilon}{\partial p_z}\right|_{z=d} = R\chi_+\left|\frac{\partial\varepsilon}{\partial p_z}\right|_{z=d}; \tag{7}$$

R is the reflection coefficient, d is the thickness of the plate and the indices + and − correspond to excitations moving in the positive ($\partial\epsilon/\partial p_Z > 0$) and negative ($\partial\epsilon/\partial p_Z < 0$) directions of the z axis, respectively.

In the case of a bulk metal ($d \gg l = v/\nu$), one may discard the last term on the left side of Eq. (6) and not take the boundary conditions into account. Therefore, the solution of Eq. (6) for a bulky specimen is represented in the form

$$\chi_0 = i\varepsilon_1/(k\partial\varepsilon/\partial p_x - \omega + i\nu'). \tag{8}$$

It is easy to see that for specular reflection of the excitations from the boundaries ($R = 1$) the function (8) satisfies Eq. (6) with the boundary conditions (7). This means that the specular reflection does not "spoil" the motions of the excitations along the acoustic wave vector $\mathbf{k} = \{k, 0, 0\}$. Therefore, a lower ultrasonic absorption will be observed in both the bulky specimen and in the plate with specularly reflecting boundaries.

In the general case ($R \neq 1$) the solution of the kinetic equation (6) can be expressed in terms of the function χ_0:

$$\chi = \chi_0\varphi, \tag{9}$$

where

*It should be noted that the simultaneous satisfaction of the inequalities $kl \gg 1$ and $\xi_0 > l$ still presents significant difficulties for the experimenters in connection with the necessity of obtaining very high frequencies ($\omega \sim 10^{10}$-10^{11} sec^{-1}).

$$\varphi_+ = 1 - \frac{1-R}{1-Re^{-iPd}} e^{-iPz},$$

$$\varphi_- = 1 - \frac{1-R}{1-Re^{-iPd}} e^{-iP(d-z)}. \qquad (10)$$

Here

$$iP = \frac{i\omega - ik\partial\varepsilon / \partial p_x + \nu'}{|\partial\varepsilon / \partial p_z|}. \qquad (11)$$

For arbitrary "reasonable" assumptions on the values of k, l, d, μ and T, the function $\partial n_0/\partial\epsilon$ is of maximum "sharpness." Therefore, one can integrate with respect to ϵ in (5). As a result, we get

$$\alpha_s = \frac{2}{h^3 W} \int \nu' |\chi|^2 n_0(\Delta) \frac{dS}{|\partial\varepsilon / \partial p|}, \qquad (12)$$

where the integration is carried out over the Fermi surface.*

For $\Delta = 0$, (the temperature is equal to the critical temperature) the absorption coefficient has the same form as in the normal metal.

2. If the presence of the boundaries can be neglected (bulky specimen or a plate with specularly reflecting boundaries), then the integrand in (12) does not depend on z ($\chi = \chi_0$). For not very low temperatures ($\delta\Delta = \Delta_{max} - \Delta_{min} \lesssim T$) the function $n_0(\Delta)$ is smooth. Setting

$$\nu' |\chi_0|^2 = \frac{\pi}{k} \delta\left(\frac{\partial\varepsilon}{\partial p_x}\right) |\dot{\varepsilon}_1|^2,$$

we get

$$\alpha_s = \frac{2\pi}{kh^3 W} \int \delta\left(\frac{\partial\varepsilon}{\partial p_x}\right) n_0(\Delta) |\dot{\varepsilon}_1|^2 \frac{\partial S}{|\partial\varepsilon / \partial p|}. \qquad (13)$$

Taking it into consideration that $\partial\epsilon/\partial\mathbf{p} = \xi\epsilon^{-1}\partial\xi/\partial\mathbf{p}$, we put (13) in the form

$$\alpha_s = \frac{2\pi}{kh^3 W} \int \delta(v_x) n_0(\Delta) |\dot{\xi}_1|^2 \frac{dS}{v}. \qquad (14)$$

The temperature dependence of α_S is determined by the function $n_0(\Delta)$. In an isotropic model, $\alpha_S = 2n_0(\Delta)\alpha_n$, where α_n is the absorption coefficient in the normal state of the metal. In the anisotropic case, Δ stands in this formula for the value of the gap, averaged over the line $v_x = 0$. Therefore, measurement of the ratio α_S/α_n cannot give information on the anisotropy of the temperature dependence of the energy gap.

For low temperatures ($\exp\{(\Delta_{max} - \Delta_{min})/T\} \gg 1$), the absorption coefficient is extremely anisotropic. The integrand in (12) is the product of two "sharp" functions. The function χ differs materially from zero near the line $v_x = 0$. The function $n_0(\Delta)$ has a sharp maximum at the point of the Fermi surface where the gap has a minimum ($\Delta = \Delta_{min}$). The maximum of the absorption coefficient corresponds to those directions of the vector k for which the minimum of the gap is located on the line $v_x = 0$. If the vector k rotates in the plane $\mathbf{k}\cdot\mathbf{v}_{\Delta=\Delta_{min}} = 0$, the anisotropy of the ultrasonic absorption is insignificant. The plane $\mathbf{k}\cdot\mathbf{v}_{\Delta=\Delta_{min}} = 0$ should coincide with one of the planes of symmetry of the crystal, which considerably simplifies the determination of the minimum gap in the experiment.

When the vector k lies in the plane shown, the integration along the line $v_x = 0$ is carried out only in the vicinity of the point $\Delta = \Delta_{min}$ (the relative size of the region of integration is of the order of $\sqrt{T/\delta\Delta}$). For integration in the perpendicular direction one can assume smoothness of one of the functions $n_0(\Delta)$ or χ, depending on the ratio of the parameters $\sqrt{T/\delta\Delta}$ and $1/kl$.

In the case $1/kl < \sqrt{T/\delta\Delta}$, the absorption coefficient is equal to

$$\alpha_s^{max} \sim e^{-\Delta_{min}/T} \sqrt{T/\delta\Delta}\, \alpha_n. \qquad (15)$$

If $\sqrt{T/\delta\Delta} < 1/kl$, then

$$\alpha_s^{max} \sim e^{-\Delta_{min}/T} kl\, T\alpha_n / \delta\Delta. \qquad (16)$$

When the vector k is inclined to the plane $\mathbf{k}\cdot\mathbf{v}_{\Delta=\Delta_{min}} = 0$ at an angle $\theta \sim 1$, one must compare (for the determination of the absorption coefficient) the contributions to the integral (12) of the vicinity of the point $\Delta = \Delta_{min}$ and of the point at which Δ has a minimum along the line $v_x = 0$ ($\Delta = \Delta^0_{min}$). The contribution to the absorption coefficient of the vicinity of the point $\Delta = \Delta_{min}$ is inversely proportional to the free path

$$\alpha_s' \sim (kl)^{-1} e^{-\Delta_{min}/T} \alpha_n T/\delta\Delta. \qquad (17)$$

The contribution of the point $\Delta = \Delta^0_{min}$ need only be taken into account for not very low temperatures ($1/kl < T/\delta\Delta$). In this case, integrating along the line $v_x = 0$ close to the values $\Delta = \Delta^0_{min}$, we get, with account of (14),

$$\alpha_s'' \sim e^{-\Delta^0_{min}/T} \sqrt{T/\delta\Delta}\, \alpha_n. \qquad (18)$$

The angular dependence of the absorption coefficient is different in three cases:

a) $\frac{1}{kl} < \sqrt{\frac{T}{\delta\Delta}}$, $\exp\left(\frac{\Delta^0_{min} - \Delta_{min}}{T}\right) < kl\left(\frac{\delta\Delta}{T}\right)^{1/2}$.

The behavior of α_S/α_n as a function of the angle θ between the vector k and the plane $\mathbf{k}\cdot\mathbf{v}_{\Delta=\Delta_{min}} = 0$ is determined by the dependence of $\Delta^0_{min}(\theta)$

*If $l = \infty$, then maximum sharpness is achieved by the function $\nu'|\chi_0|^2$ which can be replaced by $\pi\delta(\mathbf{k}\partial\varepsilon/\partial\mathbf{p} - \omega)$. In this case, one can get the formulas of Pokrovskii for the absorption coefficient in the bulky specimen.[10]

[see (15), (18)]. A sharp decrease takes place in the absorption coefficient in the range of angles $\delta\theta \sim \sqrt{T/\delta\Delta}$.

b) $\frac{1}{kl} < \sqrt{\frac{T}{\delta\Delta}}$, $\exp\left(\frac{\Delta^0_{min} - \Delta_{min}}{T}\right) > kl\left(\frac{\delta\Delta}{T}\right)^{1/2}$.

In this case, the absorption coefficient increases sharply with increase in the angle θ (by a factor of $kl\sqrt{\Delta/T} \gg 1$). the half-width of the maximum on the polar plot $\alpha_S(\theta)$ is of the order of $1/kl$.

c) $1/kl > \sqrt{T/\delta\Delta}$.

The ratio α_S/α_n increases sharply [by a factor $(kl)^2 \gg 1$] in the narrow range of angles $\delta\theta \sim 1/kl$. The qualitative angular dependence of the absorption coefficient is described by the formula

$$\alpha_s/\alpha_s^{max} \sim [1 + (kl\theta)^2]^{-1}, \qquad (19)$$

where α_S^{max} is determined by the relation (16).

It should be noted that the cases considered above are experimentally possible.

3. We proceed to the consideration of the ultrasonic absorption in a plate with diffusely reflecting boundaries (R = 0). From the general formula (10) it follows that in this case

$$\varphi_+ = 1 - e^{-lPz}, \qquad \varphi_- = 1 - e^{-iP(d-z)}. \qquad (20)$$

The function $\chi = \chi_0\varphi$ has a sharp maximum at the point $v_X = v_Z = 0$ on the Fermi surface.

a) If the temperature is not small ($\delta\Delta \lesssim T$), then the expression $\nu' |\chi|^2$ can be replaced by

$$\frac{\pi d\nu' I}{k} |\dot{\xi}_1|^2 \delta\left(\frac{\partial\varepsilon}{\partial p_x}\right) \delta\left(\frac{\partial\varepsilon}{\partial p_z}\right),$$

where

$$I = \int_0^\infty (1 - e^{-1/x})^2 dx = 0.47\ldots .$$

Substituting this expression in (12), we get

$$\alpha_s = 2n_0(\Delta_0)\alpha_n. \qquad (21)$$

Here Δ_0 is the value of the gap at the point $v_X = v_Z = 0$, α_n is the absorption coefficient for $\Delta = 0$. The absorption α_n in the plate is less than the absorption in the bulky specimen by a factor $l/d \gg 1$.

In contrast with the approximate equalities (15) — (19), the relation (21) is asymptotically exact (for $l/d \gg 1$).

The inequality $\delta\Delta \lesssim T$ is satisfied in practice in the entire temperature interval in which the change in Δ is significant. Therefore, the measurement of the ultrasonic absorption in the superconducting plate can be used to establish the energy gap as a function of the direction and temperature for all values of the arguments.

b) For low temperatures ($\delta\Delta > T$) all three functions χ_0, φ, and $n_0(\Delta)$ entering into the integrand of (12) are "sharp."

We consider the most important limiting case, in which the parameters $1/kl$ and d/l are small in comparison with $T/\delta\Delta$. At the point $v_X = v_Z = 0$, (maximum of the function $\chi = \chi_0\varphi$), the function $n_0(\Delta)$ can be regarded as small while at the point $\Delta = \Delta_{min}$ χ is a smooth function (it is assumed that the locations of the maxima of χ and $n_0(\Delta)$ do not coincide).

Integration near the point $v_X = v_Z = 0$ leads to Eq. (21). The contribution to the absorption coefficient of the vicinity of the point $\Delta = \Delta_{min}$ is equal (in order of magnitude) to

$$\alpha_1 \sim \frac{1}{kd}\frac{T}{\delta\Delta} e^{-\Delta_{min}/T} \alpha_n. \qquad (22)$$

If $kd \gg \exp\{(\Delta_0 - \Delta_{min})/T\} T/\delta\Delta$, then the absorption coefficient is determined by the expression (21) and $\ln(\alpha_S/\alpha_n) = -\Delta_0/T$. When the inverse inequality holds, the ratio α_S/α_n is determined by the quantity α_{min}.

If the locations of the maxima of $n_0(\Delta)$ and χ coincide (which is entirely probable when the sound is propagated along the symmetry axis of the crystal), the ratio α_S/α_n falls off exponentially with increase in Δ_0 [see (21)].*

c) Finally, without going into the details of the calculation, we write down the obvious interpolation formula for the absorption coefficient for an arbitrary reflection law ($R \neq 0; 1$):

$$\alpha_s = (1 - R)\alpha_{diff} + R\alpha_{spec}. \qquad (23)$$

Here α_{diff} and α_{spec} are the ultrasonic absorption coefficients at R = 0 and R = 1, respectively.

Under the experimental conditions, the second term on the right side of (21) is always small in comparison with the first. Therefore, for an analysis of the experiment, it is necessary to use Eqs. (21) and (22).

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*When the ratio $T/\delta\Delta$ is small in comparison with even one of the parameters $1/kl$ or d/l, the absorption coefficient must be computed by formula (22).

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