PARTIAL WAVES WITH COMPLEX ORBITAL ANGULAR MOMENTA AND THE ASYMPTOTIC BEHAVIOR OF THE SCATTERING AMPLITUDE

V. N. GRIBOV

Leningrad Physico-technical Institute, Academy of Sciences, U.S.S.R.

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It is shown that in relativistic theory the partial wave amplitudes f_l are analytic functions of the angular momentum l. The asymptotic behavior of the scattering amplitude as a function of momentum transfer is determined by the nearest singularities of f_l. An expression is obtained for the scattering amplitude for arbitrary momentum transfer in terms of the f_l, which satisfies the Mandelstam equation relating the spectral functions and absorptive parts. The behavior of the scattering amplitude at high energies is discussed.

1. INTRODUCTION

IN a recent paper, Regge, [1] by introducing partial waves with complex orbital angular momenta, obtained an interesting result in nonrelativistic theory concerning the asymptotic behavior of scattering amplitudes and spectral functions in the nonphysical region of large momentum transfers.

In a previous paper of this author [2] the asymptotic behavior was studied under these same conditions in relativistic theory in order to examine the possible types of asymptotic behavior of the scattering amplitude at high energies. In this an important part was played by the Mandelstam equation, obtained by analytic continuation of the usual unitarity condition.

In the present paper we show that partial waves with complex orbital angular momenta l can also be introduced in a relativistic theory. Their analytic properties are to some extent similar to those of the corresponding nonrelativistic quantities. The introduction of complex values of l enables one to find the general solution of the Mandelstam equation and to obtain some information about the possible asymptotic behavior of amplitudes and spectral functions.

2. PARTIAL WAVES WITH COMPLEX ORBITAL ANGULAR MOMENTA

Let us consider the dispersion relation in the momentum transfer in the t channel (region III in Fig. 1):

$$A(s,t) = A(s_0,t) + \frac{1}{\pi}\int_{4\mu^2}^{\infty} ds' A_1(s',t)\left[\frac{1}{s'-s} - \frac{1}{s'-s_0}\right] + \frac{1}{\pi}\int_{4\mu^2}^{\infty} du' A_2(u',t)\left[\frac{1}{u'-u} - \frac{1}{u'-u_0}\right]. \quad (1)$$

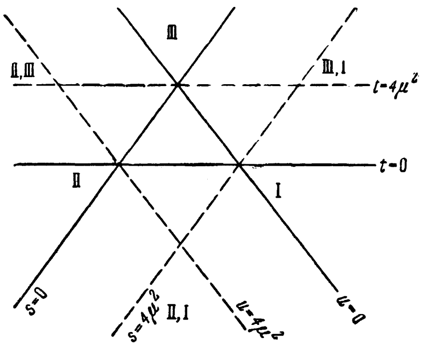

FIG. 1

To be specific, we have written the equation for one subtraction; as will be clear later, the number of subtractions is unimportant.

In addition we shall assume that the Mandelstam representation is valid. Then, assuming for simplicity that the particles all have the same mass μ and are the lightest particles (for example, π mesons), we find, following Mandelstam, that the spectral function $\rho(s,t)$ and the absorptive parts $A_1(s,t)$ and $A_2(u,t)$ must satisfy the equation

$$\rho(s,t) = \sqrt{\frac{t-4\mu^2}{t}} \int \frac{dz_1\, dz_2}{\sqrt{z^2+z_1^2+z_2^2-2zz_1z_2-1}} \times [A_1(s_1,t)A_1^*(s_2,t) + A_2(u_1,t)A_2^*(u_2,t)];$$

$$s = -\tfrac{1}{2}(t-4\mu^2)(1-z),$$

$$s_{1,2} = -\tfrac{1}{2}(t-4\mu^2)(1-z_{1,2}),$$

$$u_{1,2} = -\tfrac{1}{2}(t-4\mu^2)(1-z_{1,2}),$$

$$z > z_1z_2 + \sqrt{(z_1^2-1)(z_2^2-1)}, \quad (2)$$

for t below the threshold for inelastic processes. (For π mesons, $t < 16\mu^2$).

Using (1), we calculate the amplitude for the wave with angular momentum l:

$$f_l(t) = \frac{1}{2}\int_{-1}^{1} P_l(z) A(s,t)\, dz. \quad (3)$$

For $l \geq 1$

$$f_l(t) = \frac{1}{\pi}\int_{z_0}^{\infty} Q_l(z') A_1(s',t)\, dz' + (-1)^l \frac{1}{\pi}\int_{z_0}^{\infty} Q_l(z'') A_2(u'',t)\, dz'', \quad (4)$$

where

$$Q_l(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_l(x')}{x-x'}\, dx',$$

$$z' = 1 + \frac{2s'}{t-4\mu^2} \geqslant z_0 = 1 + \frac{8\mu^2}{t-4\mu^2} > 1,$$

$$z'' = 1 + \frac{2u''}{t-4\mu^2} \geqslant z_0. \quad (5)$$

If the asymptotic behavior of A for large s required not one but k subtractions, equation (4) would be valid for $l \geq k$. We write (4) in the form

$$f_l(t) = \varphi_l^{(1)}(t) + (-1)^l \varphi_l^{(2)}(t) \quad (6)$$

and study the properties of, say, $\varphi_l^{(1)}(t)$.

We consider the expression

$$\varphi_l^{(1)}(t) = \frac{1}{\pi}\int_{z_0}^{\infty} Q_l(z') A_1(z',t)\, dz' \quad (7)$$

as the definition of $\varphi_l^{(1)}$ for nonintegral l, where $Q_l(z)$ is the Legendre function of the second kind, and coincides with (5) for integer l. Then under the condition that $|A_1(z,t)| < Cz^a$, it follows from (7) that $\varphi_l^{(1)}$ is an analytic function of l in the half-plane Re $l > a$, since $Q_l(z')$ is an analytic function of l and behaves like $(z')^{-(l+1)}$ for $z' \to \infty$.

For $l \to \infty$,

$$\varphi_l^{(1)}(t) \sim e^{-\alpha l}, \qquad z_0 = \text{ch}\,\alpha. \quad (8)^*$$

Using the analytic properties and the asymptotic behavior (8), we can easily invert (7) and express $A_1(z',t)$ in terms of $\varphi_l^{(1)}(t)$. In order to do this, we use the following relation, which is proved in Appendix I:

$$\frac{i}{2\pi}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1) P_l(z_1) Q_l(z_2) = \delta(z_2 - z_1), \quad b > -1. \quad (9)$$

Integrating (7) over l for $b \geq a$, we get

$$A_1(z',t) = \frac{i}{2}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1)\varphi_l^{(1)}(t) P_l(z'). \quad (10)$$

We note that the right side of (10) is equal to zero for $z < z_0$, since then the asymptotic behavior of $\varphi_l^{(1)}$ and $P_l(z)$ for large l permits us to close the contour to the right for $l \to \infty$. The formula becomes meaningless for $z < 1$.

*ch = cosh.

Formulas (7) and (10) are analogous to the direct and inverse Mellin transformations, but are more convenient than the latter for studying the unitary condition (2).

We can write a similar relation for $A_2(z'',t)$:

$$A_2(z'',t) = \frac{i}{2}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1)\varphi_l^{(2)}(t) P_l(z''). \quad (11)$$

Since $\rho(s,t) = \text{Im}\,A_1(s,t)$,

$$\rho(s,t) = \frac{i}{2}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1) P_l(z) \frac{i}{2}[\varphi_l^{(1)} - (\varphi_{l^*}^{(1)})^*]. \quad (12)$$

Now we can substitute (10), (11) and (12) into (2). As shown in Appendix II, Eq. (2) will be satisfied if

$$\frac{i}{2}[\varphi_l^{(1)} - (\varphi_{l^*}^{(1)})^*] = \frac{q}{\omega}[\varphi_l^{(1)}(\varphi_{l^*}^{(1)})^* + \varphi_l^{(2)}(\varphi_{l^*}^{(2)})^*];$$

$$q = \tfrac{1}{2}\sqrt{t-4\mu^2}, \qquad \omega = \tfrac{1}{2}\sqrt{t}. \quad (13)$$

For real l, (13) goes over into

$$\text{Im}\,\varphi_l^{(1)} = (q/\omega)\,[|\varphi_l^{(1)}|^2 + |\varphi_l^{(2)}|^2]. \quad (14)$$

If together with Eq. (2), which is obtained by analytic continuation of the unitarity condition in the t channel into the region I, III of Fig. 1 $(s > 4\mu^2)$, we also considered the relation between $\rho(u,t)$ and A_1, A_2 which follows from the analytic continuation of the same unitarity condition into the region II, III of Fig. 1 $(u > 4\mu^2)$, then as shown by Mandelstam we would get the equation

$$\rho(u,t) = \text{Im}\,A_2(u,t) = \frac{q}{\omega}\int \frac{dz_1 dz_2}{\sqrt{z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1}} \times [A_1(z_1)A_2^*(z_2) + A_1^*(z_1)A_2(z_2)] \quad (15)$$

and, analogous to (12),

$$\rho(u,t) = \frac{i}{2}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1) P_l(z)\frac{i}{2}[\varphi_l^{(2)} - (\varphi_{l^*}^{(2)})^*]. \quad (16)$$

Substitution of (10), (11) and (16) in (15) gives

$$\frac{i}{2}[\varphi_l^{(2)} - (\varphi_{l^*}^{(2)})^*] = \frac{q}{\omega}[\varphi_l^{(1)}(\varphi_{l^*}^{(2)})^* + (\varphi_{l^*}^{(1)})^*\varphi_l^{(2)}], \quad (17)$$

where for real l,

$$\text{Im}\,\varphi_l^{(2)} = (q/\omega)\,[\varphi_l^{(1)}\varphi_l^{(2)*} + \varphi_l^{(1)*}\varphi_l^{(2)}]. \quad (18)$$

If for integer l we multiply (18) by $(-1)^l$ and add to (14), we get the usual unitarity condition:

$$\text{Im}\,f_l = (q/\omega)|f_l|^2. \quad (19)$$

Thus formulas (14) and (18) are the generalization of (19) to arbitrary real l, while (13) and (17) generalize it to arbitrary complex l.

For the case of interaction of identical particles, $\varphi_l^{(1)} = \varphi_l^{(2)} = \varphi_l$, but since the integration in the original unitarity condition must be taken only over half the sphere, the right sides of (1) and (15) contain an additional factor $1/2$, so that instead of (13) and (17) we get

$$\frac{i}{2}[\varphi_l-(\varphi_{l^*})^*]=\frac{q}{\omega}\varphi_l(\varphi_{l^*})^*,\qquad f_l=[1+(-1)^l]\varphi_l. \quad (20)$$

Thus we have shown that $A_1(s,t)$, $A_2(u,t)$, $\rho(s,t)$ and $\rho(u,t)$ have the form (10), (11), (12) and (16) and satisfy the unitarity condition if $\varphi_l^{(1)}$ and $\varphi_l^{(2)}$ satisfy (13) and (17). The functions $\varphi_l^{(1)}$, $\varphi_l^{(2)}$ are analytic in l, their behavior for large l is given by (8), and for integral $l > a$ they are related simply to the phases for the t channel (Eq. 6).

It is also easy to get the expression for the amplitude $A(z,t)$ in terms of $\varphi_l^{(1)}(t)$ and $\varphi_l^{(2)}(t)$. This can be done either by using the dispersion relation (1) or by analytic continuation of the series

$$A(z,t)=\sum_{l=0}^{\infty}(2l+1)f_l(t)P_l(z),$$

in a way similar to that for the nonrelativistic theory,[1] but including both cuts.

With some simple transformations we get

$$A(z,t)=\sum_{l=0}^{n}(2l+1)f_l(t)P_l(z)+\frac{i}{2}\int_{b-i\infty}^{b+i\infty}\frac{(2l+1)\,dl}{\sin l\pi}\times[\varphi_l^{(1)}(t)\,P_l(-z)+\varphi_l^{(2)}(t)\,P_l(z)], \quad (21)$$

where $k \le n \le b$.

If we resolve $A(z,t)$ into parts which are symmetric and antisymmetric with respect to the substitution $z \to -z$:

$$A(z,t)=A^+(z,t)+A^-(z,t),$$

then

$$A^{\pm}(z,t)=\frac{1}{2}\sum_{l=0}^{n}(2l+1)f_l(t)\,[1\pm(-1)^l]\,P_l(z)+\frac{i}{4}\int_{b-i\infty}^{b+i\infty}\frac{(2l+1)\,dl}{\sin l\pi}f_l^{\pm}\,[P_l(-z)\pm P_l(z)], \quad (22)$$

where

$$f_l^{\pm}=\varphi_l^{(1)}\pm\varphi_l^{(2)}$$

satisfy the conditions

$$\frac{i}{2}[f_l^{\pm}-(f_{l^*}^{\pm})^*]=\frac{q}{\omega}f_l^{\pm}(f_{l^*}^{\pm})^*. \quad (23)$$

Contrary to the situation for the nonrelativistic theory, we have not been able to show that $\varphi_l^{(1)}$ and $\varphi_l^{(2)}$ are a) meromorphic functions for $\mathrm{Re}\, l > -1/2$, and b) have singularities only in the upper half-plane. We know of no reasons why the first property (meromorphy) should be retained in the exact theory even when $t-4\mu^2 \ll 4\mu^2$, i.e., in the nonrelativistic region in the t channel. This is related to the fact that in analyzing the analytic properties of $f_l(t)$ it was necesary to assume that the potential at small distances is not too singular (which was necessary in the nonrelativistic theory, in order for a solution of the form r^l to exist for small r).

It is easy to give examples of potentials for which this property does not hold. For a potential of the type $-\alpha/r^2$, the partial wave has branch points at $l = -1/2 + \sqrt{\alpha}$, which for $\alpha > 1/4$ represent a collapse into the center. For a δ-function potential, $f_l = 0$ for all $l \ne 0$.

As for the second property (singularities only in the upper half-plane), in nonrelativistic theory it is a consequence of the hermiticity of the Hamiltonian and is closely related to the fact that the singularities of the amplitude as a function of the energy t, which correspond to unstable states, lie only on the second sheet of the t plane.

It may well be that this distinction of the upper half-plane is retained in the relativistic theory, but we have simply been unable to prove it.

In conclusion we consider the analytic properties of $\varphi_l^{(1)}(t)$ as a function of t for arbitrary l where $\mathrm{Re}\, l > \max_t a$. These properties are easily understood, starting from formula (7). Since $A_1(s,t)$ is an analytic function of t, Eq. (7) is conveniently written in the form

$$\varphi_l^{(1)}(t)=\frac{2}{\pi(t-4\mu^2)}\int_{4\mu^2}^{\infty}Q_l\Big(1+\frac{2s}{t-4\mu^2}\Big)A_1(s,t)\,ds. \quad (24)$$

For $t > 4\mu^2$, the singularities of $\varphi_l^{(1)}(t)$ correspond to singularities of $A_1(s,t)$. For $t \to 4\mu^2$,

$$Q_l\Big(1+\frac{2s}{t-4\mu^2}\Big)\sim\Big(\frac{t-4\mu^2}{2s}\Big)^{l+1},$$

consequently,

$$\varphi_l^{(1)}(t)\sim q^{2l}\ \text{for}\ q\to 0,$$

which coincides with the usual relation for integer l, with the one difference that $\varphi_l^{(1)}$ for nonintegral l remains complex when $t < 4\mu^2$.

As t decreases, singularities of $\varphi_l^{(1)}(t)$ appear for two reasons: 1) $Q_l(z)$ has a singularity at $z = -1$ and consequently $\varphi_l^{(1)}(t)$ has a singularity at $t = 0$; 2) $A_1(s,t)$ has a singularity for $4\mu^2 - s - t = u = u_0$.

It is also easy to write a dispersion relation for $\varphi_l^{(1)}(t)$. We note that in nonrelativistic theory $A_1(s,t)$ has no singularities for $t < 4\mu^2$, and the left-hand cut for $\varphi_l^{(1)}(t)$ comes only from the singularity of Q_l.

3. ASYMPTOTIC BEHAVIOR OF $A_1(s,t)$ AS $s \to \infty$

To investigate the asymptotic behavior of $A_1(s,t)$ for $s \to \infty$ $(z \to \infty)$ it is convenient to use formula (10). Since when $z \to \infty$, $P_l(z) \sim z^l$ (for $\mathrm{Re}\, l > -\frac{1}{2}$), the asymptotic behavior of A_1 as $z \to \infty$ is determined by the position of the nearest singularity on the right for the function $\varphi_l^{(1)}$.

The simplest asymptotic behavior, of the form $sf(t)$, which is discussed in [2], corresponds to the case where $\varphi_l^{(1)}$ has as its nearest singularity for arbitrary t a simple pole at $l = 1$. However, the unitarity condition (18) excludes the possibility that $\varphi_l^{(1)}$ goes to infinity for real l as we approach the first singular point on the right, since it follows from the unitarity condition that $(q/\omega)|\varphi_l^{(1)}| \le 1$.

In the nonrelativistic theory [1] the only singularities of $\varphi_l^{(1)}$ are poles in the upper half-plane and, consequently, the asymptotic behavior has the form

$$A_1 \sim f(t)\, s^{l_1(t)}, \tag{25}$$

where $l_1(t) = \alpha(t) + i\beta(t)$ is necessarily complex. The same asymptotic behavior was assumed in relativistic theory in the work of Chew and Frautschi.[3] Such behavior is possible, but it is important to understand that it leads to an essentially nondiffractive character of the scattering in the s channel for $s \to \infty$ (cf. [4]). In fact, if we continue (25) into the region $t < 4\mu^2$, then $\beta(t) = 0$ [the pole moves onto the real axis; for $t < 4\mu^2$ we do not have the condition (18)] and we will have

$$A_1 \sim f(t)\, s^{\alpha(t)},$$

where $\alpha(t) \ne \mathrm{const}$ (since the location of the pole of the function $\varphi_l^{(1)}(t)$, which is analytic in t, is a function of t). The assertion of Chew and Frautschi that α is approximately constant in the interval $-20m_\pi^2 < t < 0$ is, to say the least, not understandable.

For $t = 0$, as shown by Froissart,[5] $|A_1| \le Cs \ln^2 s$, and therefore $\alpha(0) \le 1$. In order to get a constant total cross section, it is necessary that $\alpha(0) = 1$. Then in the physical region of the s channel (region I in Fig. 1), for small t

$$A_1 \sim sf(t)\, e^{\gamma t \ln s} \tag{26}$$

falls off very rapidly with increasing $-t$ for large s, so that the diffraction cone (the region of values of t for which $d\sigma/dt$ does not tend to zero) is not independent of energy, but has a size $t \sim -1/\ln s$. In particular this has the consequence that the elastic scattering cross section tends to zero at high energies. If with such a behavior for $A_1(s,t)$ we calculated the partial wave amplitudes $a_l(s)$ in the channel s, then in contrast to the case of diffraction, in which $a_l(s) \sim 1$ for $l \lesssim p/\mu$ while the amplitudes drop rapidly for $l > p/\mu$, we would find $a_l(s) \sim 1/\ln s$ for $l \lesssim p\mu^{-1} \ln^{1/2} s$, while the amplitudes would fall off rapidly for $l > p\mu^{-1} \ln^{1/2} s$. This means that as the energy increases the particles "swell" and become more transparent.

The asymptotic behavior $A_1(s,t) = sB(\xi)f(t)$ (where $\xi = \ln s$), which was discussed in [2] and corresponds to a decreasing cross section, occurs if $\varphi_l^{(1)}(t)$ has a branch point for $l = 1$ (for arbitrary t). The fact that by virtue of the unitarity condition $\varphi_l^{(1)}(t)$ remains finite as we approach the branch point, has the consequence that $B(\xi)$ falls off faster than $1/\xi$. The appearance of such a branch point can be pictured classically, for example as follows. Suppose that the interaction in states with l values other than $l = 0$ has the character $-\alpha/r^2$ with $\alpha = \frac{9}{4}$. Then for $l = 1$ the function $\varphi_l^{(1)}(t)$ will have a branch point corresponding to collapse into the center, but $\varphi_l^{(1)}$ will still have a meaning. We note that in nonrelativistic theory among the interactions $1/r^n$ for small r only the interaction $1/r^2$ has a real effect on the analytic properties of $\varphi_l^{(1)}(t)$, since for $n < 2$ the function $\varphi_l^{(1)}$ is meromorphic for $\mathrm{Re}\, l > -\frac{1}{2}$, while for $n > 2$ it does not exist in general. If $\varphi_l^{(1)}$ has a branch point, then $B(\xi) \sim 1/\xi^{3/2}$ for $\xi \to \infty$.

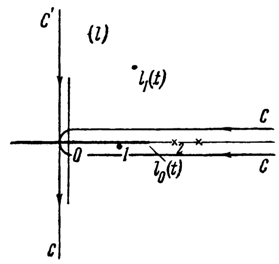

FIG. 2

Let us discuss briefly the possibility of such an asymptotic behavior of $A_1(s,t)$ that, no matter how complicated the asymptotic form for $t > 4\mu^2$ (region I, III in Fig. 1), for $t < 4\mu^2$ it is equal to $sf(t)$. It is clear that if $\varphi_l^{(1)}$ is a meromorphic function, then as we go to $t < 4\mu^2$ a pole cannot develop for $l = 1$ independently of t. But if there is a branch point $l = l_0(t)$ on the real axis (Fig. 2), then it may turn out that $\varphi_l^{(1)}$ has a pole for $l = 1$ which lies on only one side of the cut. The presence of such a pole does not contradict the unitarity condition. If, as t decreased, $l_0(t)$ moved toward the left, and for $t < 4\mu^2$ became less than unity, leaving a pole on the right, the asymptotic behavior when $t < 4\mu^2$ would be determined by the pole and would have the form sf(t). It is easy to see that in order for the pole to remain at the same place when $t < 4\mu^2$, it is necessary that it be located initially on the lower edge of the cut.

One may also consider a possible behavior like $l_0(t)$ in the region $t > 4\mu^2$. This problem contains several interesting points which we shall not discuss here. We note only that, even though one can make no rigorous assertions, from studies of these possibilities one becomes convinced that a behavior like sf(t) in the region $t > 4\mu^2$ is impossible to obtain if we restrict ourselves to only two-particle states in the unitarity condition for the t channel, and that even the inclusion of any finite number of states may not be enough.

In conclusion I wish to thank I. T. Dyatlov, L. D. Landau, I. Ya. Pomeranchuk and K. A. Ter-Martirosyan for valuable discussions, and Chew, Frautschi and Froissart for sending me their preprints.

APPENDIX I

To prove formula (9), we consider the well-known relation

$$\frac{1}{z_2 - z_1} = \frac{1}{2i}\sum_{l=0}^{\infty}(2l+1)\,P_l(z_1)\,Q_l(z_2), \qquad \text{(AI.1)}$$

which is valid for

$$|z_1 + \sqrt{z_1^2 - 1}| < |z_2 + \sqrt{z_2^2 - 1}|. \qquad \text{(AI.2)}$$

We shall assume that z_2 is real and larger than one, and write an expression for the right side of (AI.1) which is valid for arbitrary complex z_1. To do this we use the Watson transformation:

$$\frac{1}{z_2 - z_1} = -\frac{i}{2}\int_C dl\,(2l+1)\,\frac{P_l(-z_1)}{\sin l\pi}\,Q_l(z_2). \qquad \text{(AI.3)}$$

The contour C is shown in Fig. 2. The integral converges if (AI.2) is satisfied. For $z_1 > 1$, the expression (AI.3) appears at first glance to be undefined, since $P_l(x)$ for integer l has a branch point at $x = -1$. But since

$$\frac{i}{2}\,[P_l(-z - i\varepsilon) - P_l(-z + i\varepsilon)] = \sin l\pi \cdot P_l(z), \qquad \text{(AI.4)}$$

the integral on the right side of (AI.3) vanishes for the difference of the values on the two sides of the cut.

Since for large l and $|z| > 1$

$$P_l(z) \sim (2\pi l \,\mathrm{sh}\,\xi)^{-1/2} e^{(l+1/2)\xi}, \quad Q_l(z) \sim (\pi/2l\,\mathrm{sh}\,\xi)^{1/2} e^{-(l+1/2)\xi},$$
$$z = \mathrm{ch}\,\xi \qquad \text{(AI.5)*}$$

and are entire functions of l for $\mathrm{Re}\, l > -1$, for $z_1 < z_2$ the integration contour on the right side of (AI.3) can be deformed into the contour C′. The integral over the contour C′ will converge for all complex z and for real values satisfying $1 < z_1 < z_2$. We may therefore calculate the difference of the values with $z_1 \pm i\epsilon$ for the right and left sides. As a result we get, using (AI.4),

$$\delta(z_2 - z_1) = \frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty} dl\,(2l+1)\,P_l(z_1)\,Q_l(z_2), \qquad \text{(AI.6)}$$

QED.

This formula can also be proved directly. In particular, the fact that the right side is equal to zero for $z_1 < z_2$ follows immediately from the fact that in this case the contour of integration can be closed to the right.

In order to show that the right side of (AI.6) is equal to zero for $z_1 > z_2$, it is convenient to take $a = -1/2$: then using the relations

$$P_{-1/2-\gamma} = P_{-1/2+\gamma}, \quad Q_{-1/2-\gamma} = Q_{-1/2+\gamma} + \frac{\pi}{\mathrm{tg}(-1/2+\gamma)\pi}\,P_{1/2+\gamma}, \qquad \text{(AI.7)}$$

it is easy to show that

$$\int_{-i\infty}^{i\infty}\gamma d\gamma P_{-1/2+\gamma}(z_1)\,Q_{-1/2+\gamma}(z_2) = \int_{-i\infty}^{i\infty}\gamma d\gamma P_{-1/2+\gamma}(z_2)\,Q_{-1/2+\gamma}(z_1), \qquad \text{(AI.8)}$$

from which it follows that the right side of (AI.6) is a symmetric function of z_1 and z_2.

APPENDIX II

To prove that the Mandelstam equation (2) is equivalent to Eq. (13), we may use the relation

$$2\pi^2\frac{\vartheta\left(z - z_1 z_2 - \sqrt{(z_1^2-1)(z_2^2-1)}\right)}{\left[\sqrt{z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1}\right]^{1/2}} = \frac{i}{2}\int_{a-i\infty}^{a+i\infty}(2l+1)\,dl Q_l(z_1)\,Q_l(z_2)\,P_l(z),$$

$$a > -1, \quad \vartheta(x) = \begin{cases}1 & x > 0\\ 0 & x < 0\end{cases}, \qquad \text{(AII.1)}$$

which can, for example, be proved as follows.

Consider the expression for the absorptive part of the square diagram and carry out the angle integration using (AI.1); we then get

$$\int\frac{dz_1' dz_2'}{[1 + 2zz_1'z_2' - z^2 - z_1'^2 - z_2'^2]^{1/2}}\,\frac{1}{z_1 - z_1'}\,\frac{1}{z_2 - z_2'} = \sum_{l=0}^{\infty} Q_l(z_1)\,Q_l(z_2)\,P_l(z)\,(2l+1),$$

$$-1 < z_1',\ z_2' < 1, \quad z_1, z_2 > 1. \qquad \text{(AII.2)}$$

The left side of (AII.2) is easily calculated, and its imaginary part coincides with the left side of

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*sh = sinh.

(AII.1). The imaginary part of the right side of (AII.2) is calculated in exactly the same way as we used for obtaining formula (AI.6), and is equal to the right side of (AII.1). Substituting (AII.1) in Eq. (2) and using (10), (11) and (12), we easily obtain Eq. (13).

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