HYDRODYNAMICS OF A NONISOTHERMAL PLASMA

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The single-fluid magnetohydrodynamic equations with particle collisions taken into account are obtained for a nonisothermal plasma. The effect of particle collisions on the spectrum of magnetohydrodynamic and magnetic-sound plasma waves is investigated.

1. Klimontovich and Silin[1] (cf. also [2] and [3], Secs. 15 and 24) have carried out a single-fluid magnetohydrodynamic analysis of a nonisothermal plasma in which particle collisions were neglected. The possibility of carrying out this hydrodynamic analysis rests on the fact that weakly damped magnetohydrodynamic and magnetic-sound waves can propagate in a nonisothermal plasma in which $T_e \gg T_i$. These waves are damped by Cerenkov absorption and cyclotron absorption in the plasma. When particle collisions are taken into account, however, the dispersion relation for the weakly damped waves, which is obtained when particle collisions are neglected can be changed markedly, even when the collision integral in the kinetic equation is a relatively small term. Specifically, it will be shown below that under certain conditions the damping factor for the plasma waves is determined by the particle collisions while the wave frequencies are essentially unaffected by collisions.

In the high-frequency region, where the thermal motion of the plasma particles can be neglected, the important collisions are electron-ion collisions (cf. [3], Secs. 16 and 23). Here we consider a nonisothermal plasma in the frequency region ω (or the region of the characteristic time of the problem $1/\omega$) defined by the condition

$$kv_i \ll \omega \ll kv_e, \tag{A}$$

where v_e and v_i are the thermal velocities of the electrons and ions, while k is the wave vector (the characteristic dimension of the problem is $1/k$); the ion-ion collisions predominate under these conditions. Electron-electron and electron-ion collisions can be neglected. Below we derive the magnetohydrodynamic equations for a nonisothermal rarefied plasma with ion-ion collisions taken into account. We assume that (A) and the following conditions are satisfied:

$$\omega \ll \Omega_i \ll \omega_{Li}, \tag{B}$$

where $\Omega_i = e_i B_0/Mc$ is the ion Larmor frequency and $\omega_{Li} = \sqrt{4\pi e_i^2 N_i/M}$ is the ion Langmuir frequency.

2. If particle collisions are neglected the complete system of magnetohydrodynamic equations for a single-fluid nonisothermal plasma is

$$\partial \mathbf{B}/\partial t = \operatorname{rot}[\mathbf{vB}], \quad \operatorname{div}\mathbf{B} = 0, \tag{1*}$$

$$\partial\rho/\partial t + \operatorname{div}\rho\mathbf{v} = 0, \tag{2}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v} = -\frac{v_s^2}{\rho}\frac{\partial\rho}{\partial \mathbf{r}} + \frac{1}{4\pi\rho}[\operatorname{rot}\mathbf{B}, \mathbf{B}] + \frac{1}{\rho_0}\mathbf{F}_1^{\mathrm{dis}}, \tag{3}$$

where $v_s = \sqrt{|e_i/e|\varkappa T_e/M}$ is the velocity of sound while $\mathbf{F}_1^{\mathrm{dis}}$ represents the dissipative forces due to Cerenkov absorption and cyclotron absorption of the plasma waves. In contrast with the nondissipative terms, the dissipative term in the magnetohydrodynamic equations is obtained in an approximation linear in the function that describes the deviation of the particle distribution from the equilibrium (Maxwellian distribution); $\mathbf{F}_1^{\mathrm{dis}}$ is given by the expression[1]

$$F_{1i}^{\mathrm{dis}} = \frac{\rho_0 v_s^2}{B_0^2}\left\{\left[B_{0i}\left(\mathbf{B}_0\frac{\partial}{\partial \mathbf{r}}\right) + \left[\mathbf{B}_0\left[\mathbf{B}_0\frac{\partial}{\partial \mathbf{r}}\right]\right]_i\right]\frac{1}{B_0^2}\left(\mathbf{B}_0\frac{\partial}{\partial \mathbf{r}}\right)B_{0j} - \left[\mathbf{B}_0\left[\mathbf{B}_0\frac{\partial}{\partial \mathbf{r}}\right]\right]_i\frac{\partial}{\partial r_j}\right\}\int d\mathbf{r}'\, Q_1(\mathbf{r}-\mathbf{r}')\, v_j(\mathbf{r}', t), \tag{4}$$

where

$$Q_1(\mathbf{r}) = \frac{1}{(2\pi)^3}\sqrt{\frac{\pi}{2}\frac{m}{\varkappa T_e}}\int d\mathbf{k}\, e^{i\mathbf{kr}}\frac{B_0}{|\mathbf{kB}_0|}.$$

Account of plasma particle collisions in the equation of motion (3) and in the expression for $\mathbf{F}_1^{\mathrm{dis}}$, gives rise to a dissipative force $\mathbf{F}_2^{\mathrm{dis}}$ due to ion-ion collisions. To obtain an expression for $\mathbf{F}_2^{\mathrm{dis}}$ we recall the derivation of Eq. (3). Mul-

*rot = curl, $[\mathbf{vB}] = \mathbf{v}\times\mathbf{B}$

tiplying the ion kinetic equation by $\mathbf{p}$ and integrating over the momenta we have

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial r_j}\Pi_{ij}(\mathbf{r}, t) + \rho\frac{e_i}{M}\left\{E_i + \frac{1}{c}[\mathbf{vB}]_i\right\}, \quad (5)$$

where $\Pi_{ij}(\mathbf{r}, t)$ is the momentum flow tensor:

$$\Pi_{ij}(\mathbf{r}, t) = \int d\mathbf{p} p_i v_j f^{(i)}(\mathbf{p}, \mathbf{r}, t). \quad (6)$$

The thermal motion of the ions in a nonisothermal plasma can be neglected if (A) is satisfied; hence, as a first approximation, the momentum flow tensor can be written in the form

$$\rho v_i v_j. \quad (7)$$

Substituting this expression in (5) and eliminating the electric field $\mathbf{E}$ (this is done by taking account of the motion of the electrons in the plasma) we obtain the equation of motion (3).

When ion-ion collisions are introduced, Eq. (7), which represents the nondissipative part of the momentum flow tensor, must be supplemented by the dissipative part of $\Pi_{ij}(\mathbf{r}, t)$. We limit ourselves to the approximation linear in the deviation of the distribution function from the equilibrium (Maxwellian) function. Solving the ion kinetic equation by successive approximations in the collision integral and expressing the dissipative part of $\Pi_{ij}(\mathbf{r}, t)$ in terms of the hydrodynamic quantities we can write the equation of motion of the plasma in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v} = -\frac{v_s^2}{\rho}\frac{\partial \rho}{\partial \mathbf{r}} + \frac{1}{4\pi\rho}[\operatorname{rot}\mathbf{B}, \mathbf{B}] + \frac{1}{\rho_0}(\mathbf{F}_1^{\text{dis}} + \mathbf{F}_2^{\text{dis}}), \quad (8)$$

where the dissipative force $\mathbf{F}_2^{\text{dis}}$, due to ion-ion collisions, is given by

$$\frac{\partial^2 F_{2i}^{\text{dis}}}{\partial t^2} = -\frac{\sqrt{2}}{5}\rho_0 \nu_{ii} v_i^2\left[3\frac{B_{0i}(\mathbf{B}_0\partial/\partial\mathbf{r})}{B_0^2} - \frac{\partial}{\partial r_i}\right] \times \left[3\frac{B_{0j}(\mathbf{B}_0\partial/\partial\mathbf{r})}{B_0^2} - \frac{\partial}{\partial r_j}\right] v_j(\mathbf{r}, t). \quad (9)$$

Here,

$$\nu_{ii} = \frac{4}{3}\sqrt{2\pi/M} e_i^4 N_i L\ (\varkappa T_i)^{-3/2}$$

is the effective ion-ion collision frequency (L is the Coulomb logarithm), $v_i^2 = \kappa T_i/M$, while $v_A^2 = B_0^2/4\pi\rho_0$ is the Alfvén velocity.

Equations (1), (2), and (8) form the complete system of magnetohydrodynamic equations for a single-fluid nonisothermal plasma with ion-ion collisions taken into account.

When the gradients are parallel to the fixed magnetic field $\mathbf{B}_0$ (i.e., when $\mathbf{k}\cdot\mathbf{B}_0/B_0 = k$) Eqs. (1), (2), and (8) become the ordinary hydrodynamics equations for a nonisothermal plasma (i.e., with no fixed magnetic field). Obviously the continuity equation (2) retains its form under these conditions; using the equation of motion (8) we then find

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v} = -\frac{v_s^2}{\rho}\frac{\partial\rho}{\partial \mathbf{r}} + \mathbf{F}_2^{\text{dis}} + \frac{v_s^2}{(2\pi)^2}\sqrt{\frac{\pi}{2}\frac{m}{\varkappa T_e}}\frac{\partial}{\partial \mathbf{r}}\int d\mathbf{r}'\frac{1}{(\mathbf{r}-\mathbf{r}')^2}\operatorname{div}\mathbf{v}(\mathbf{r}', t), \quad (10)$$

where

$$\frac{\partial^2 \mathbf{F}_2^{\text{dis}}}{\partial t^2} = -\frac{4\sqrt{2}}{5}\rho_0\nu_{ii}v_i^2 \operatorname{grad}\operatorname{div}\mathbf{v}(\mathbf{r}, t). \quad (11)$$

We note that the dissipative term in (8) and (10), $\mathbf{F}_2^{\text{dis}}$, which is due to ion-ion collisions, is a spatially localized quantity and takes account of ion inertia; this is in contrast with the other dissipative term $\mathbf{F}_1^{\text{dis}}$, due to Cerenkov and cyclotron absorption. This last feature is a consequence of the time dependence of $\mathbf{F}_2^{\text{dis}}$ [cf. (9) and (11)].

3. We now consider the effect of ion-ion collisions on the spectrum of magnetohydrodynamic and magnetic-sound plasma waves. In the approximation used here the magnetohydrodynamic waves for which $\mathbf{v}$ is perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$ are not damped. This result follows because we have neglected terms of order ω/Ω_i in deriving the expressions for the dissipative forces $\mathbf{F}_1^{\text{dis}}$ and $\mathbf{F}_2^{\text{dis}}$. The magnetohydrodynamic dispersion relation is [2,3]

$$\omega^2 = k^2 v_A^2 \cos^2\vartheta, \quad (12)$$

where ϑ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$.

The dispersion relation for the magnetic-sound waves for which $\mathbf{v}$ lies in the plane of $\mathbf{k}$ and $\mathbf{B}_0$ is

$$\omega_\pm^2 = \frac{1}{2}k^2\{v_A^2 + v_s^2 \pm [(v_A^2 + v_s^2)^2 - 4v_A^2 v_s^2\cos^2\vartheta]^{1/2}\},$$

$$\gamma_\pm = \frac{k v_s}{2|\cos\vartheta|}\sqrt{\frac{\pi}{8}\left|\frac{e_i}{e}\right|\frac{m}{M}} \times\left\{1 \pm \frac{\cos 2\vartheta\,[(v_s/v_A)^2\cos 2\vartheta - 1]}{[1 + (v_s/v_A)^4 - 2(v_s/v_A)^2\cos 2\vartheta]^{1/2}}\right\} + \frac{2\sqrt{2}}{5}\nu_{ii}\left(\frac{v_i}{v_s}\right)^2$$
$$\times\left\{\frac{1}{2} + \frac{9}{8}\left(\frac{v_s}{v_A}\right)^2\sin^2\vartheta \mp \frac{1}{8}\left[1 + \left(\frac{v_s}{v_A}\right)^4 - 2\left(\frac{v_s}{v_A}\right)^2\cos 2\vartheta\right]^{-1/2} \times\left[\left(1 - \left(\frac{v_s}{v_A}\right)^2\cos 2\vartheta\right)\left(4 + 3\left(\frac{v_s}{v_A}\right)^2\sin^2\vartheta\right) + 2\sin^2\vartheta\left(2 + 3\left(\frac{v_s}{v_A}\right)^2(1 + \cos^2\vartheta)\right)\right]\right\}. \quad (13)$$

For waves that propagate along the fixed magnetic field ($\vartheta = 0$) we have

$$\omega_+^2 = k^2 v_A^2, \quad \gamma_+ = 0, \quad \omega_-^2 = k^2 v_s^2,$$

$$\gamma_- = \sqrt{\frac{\pi}{8}\left|\frac{e_i}{e}\right|\frac{m}{M}}\, kv_s + \frac{2\sqrt{2}}{5}\nu_{ii}\left(\frac{v_i}{v_s}\right)^2. \tag{14}$$

We note that ω_- and γ_- give the frequency and damping factor for the ordinary hydrodynamic waves, i.e., waves that propagate in the absence of an external magnetic field.

It follows from (13) and (14) that the collision contribution to the damping factor is independent of the magnitude of the wave vector. Thus, in the approximation used here particle collisions do not act to disperse a wave packet in any direction in space (do not affect the shape of the packet) and only cause an exponential damping in time. Hence the spreading of a wave packet is due completely to the Cerenkov absorption and is a linear function of time, as shown in [1]. The spreading rate is

$$\sqrt{(\pi/8)\,|e_i/e|\,(m/M)}\; v_s.$$

Equations (13) and (14) show that the collision absorption becomes greater than the Cerenkov absorption when

$$\frac{1}{k} = \lambda > l_i \frac{5\sqrt{\pi}}{8}\sqrt{\left|\frac{e_i}{e}\right|\frac{m}{M}}\left(\left|\frac{e_i}{e}\right|\frac{T_e}{T_i}\right)^{3/2},$$

where $l_i = v_i/\nu_{ii}$ is the ion mean free path. In the other limiting case, where

$$\lambda \ll l_i\sqrt{\left|\frac{e_i}{e}\right|\frac{m}{M}}\left(\left|\frac{e_i}{e}\right|\frac{T_e}{T_i}\right)^{3/2}, \tag{15}$$

the particle collisions in the plasma can be neglected. Thus, Eq. (15) determines the limits of applicability of the hydrodynamic analysis of Klimontovich and Silin. [1]

The appearance of a parameter with the dimensions of length l_i in the equations of motion (8) and (11) gives us some basis for postulating the existence of stationary shock waves of finite width in the hydrodynamics of a nonisothermal plasma with ion-ion collisions taken into account. However, a simple analysis of Eqs. (2) and (11) for the one-dimensional case shows that there can be no stationary shock wave with a finite front width in the approximation used here.

4. The dispersion relations (12) and (13) can also be obtained by solving the electromagnetic dispersion equation:

$$|\,k^2\delta_{ij} - k_i k_j - \omega^2 c^{-2}\,\varepsilon_{ij}\,(\omega, \mathbf{k})\,| = 0. \tag{16}$$

The dielectric tensor is computed in the usual way, by solving the kinetic equations for the electrons and ions with ion-ion collisions taken into account and introducing the conditions in (A) and (B). As a result we obtain

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}, \tag{17}$$

where*

$$\varepsilon_{11} = \omega_{Li}^2 / \Omega_i^2,$$

$$\varepsilon_{22} = \frac{\omega_{Li}^2}{\Omega_i^2} + i\sqrt{2\pi}\,\frac{\omega_{Li}^2 v_s^2 k \sin^2\vartheta}{\omega\Omega_i^2 v_s\,|\cos\vartheta\,|}\sqrt{\left|\frac{e_i}{e}\right|\frac{m}{M}} + i\frac{\sqrt{2}}{5}\frac{\omega_{Li}^2\nu_{ii}v_i^2k^2\sin^2\vartheta}{\omega^3\Omega_i^2},$$

$$\varepsilon_{33} = -\frac{\omega_{Li}^2}{\omega^2} + \frac{\omega_{Li}^2}{v_s^2k^2\cos^2\vartheta}\left(1 + i\sqrt{\frac{\pi}{2}\left|\frac{e_i}{e}\right|\frac{m}{M}}\,\frac{\omega}{kv_s\,|\cos\vartheta\,|}\right) + i\frac{4\sqrt{2}}{5}\frac{\omega_{Li}^2\nu_{ii}v_i^2k^2\cos^2\vartheta}{\omega^5},$$

$$\varepsilon_{23} = -\varepsilon_{32} = -i\frac{\omega_{Li}^2}{\omega\Omega_i}\,\mathrm{tg}\,\vartheta\left(1 + i\sqrt{\frac{\pi}{2}\left|\frac{e_i}{e}\right|\frac{M}{m}}\,\frac{\omega}{kv_s\,|\cos\vartheta\,|}\right) - \frac{2\sqrt{2}}{5}\frac{\omega_{Li}^2\nu_{ii}v_i^2k^2\sin\vartheta\cos\vartheta}{\omega^4\Omega_i}.$$

If there is no fixed magnetic field the acoustic wave spectrum (14) corresponds to the longitudinal electromagnetic plasma wave spectrum

$$\varepsilon^l(\omega, k) = 0, \tag{18}$$

where

$$\varepsilon^l(\omega, k) = \frac{\omega_{Li}^2}{k^2v_s^2} - \frac{\omega_{Li}^2}{\omega^2} + i\left(\sqrt{\frac{\pi}{2}\left|\frac{e_i}{e}\right|\frac{m}{M}}\,\frac{\omega\omega_{Li}^2}{k^3v_s^3} + \frac{4\sqrt{2}}{5}\frac{\omega_{Li}^2\nu_{ii}k^2v_i^2}{\omega^5}\right). \tag{19}$$

5. In conclusion we point out the limits of applicability of the magnetohydrodynamic equations obtained above for a nonisothermal plasma with ion-ion collisions taken into account. As we have indicated, in solving the ion kinetic equation one usually makes use of an expansion in powers of the collision integral. An estimate of the successive terms in the expansion shows that the expansion parameter is the small quantity

$$\frac{\lambda}{l_i}\left(\left|\frac{e}{e_i}\right|\frac{T_i}{T_e}\right)^{3/2} \ll 1. \tag{20}$$

It is evident that (20) is more general than (15).

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313

*tg = tan.