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DEGREE OF GROWTH OF MATRIX ELEMENTS IN THE AXIOMATIC APPROACH

B. V. MEDVEDEV and M. K. POLIVANOV

Mathematics Institute, Academy of Sciences, U.S.S.R. and Joint Institute for Nuclear Research

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It is shown that within the "axiomatic" approach for the construction of the scattering matrix, supplemented by the requirement that the theory be "renormalizable," some very strong restrictions arise on the possible degree of growth of the matrix elements.

1. INTRODUCTION

In the last five years a lot of attention has been devoted to the study of the general structure of local quantum field theory.[1-4] A central question in these investigations is the problem to what extent is the theory determined by only general requirements—relativistic invariance, unitarity and completeness of the system of positive energy states, locality—and without specific dynamic assumptions, that are made when the theory is constructed on the basis of the Hamiltonian approach.

The basic system of physical assumptions may be formulated in various ways. It seems convenient to us to start from the scattering matrix S, as was first proposed by Heisenberg,[5] and to formulate these physical assumptions as requirements that must be satisfied by the matrix elements of S. In addition to the S matrix it is necessary to introduce into the theory some local operators, since without them different points in space-time cannot be distinguished and the causality requirement cannot be formulated. This can be done (cf.[4]) by writing the S matrix as an expansion in normal products of asymptotic fields:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \ldots \; dx_n \, \Phi^n(x_1, \ldots, x_n) : \varphi(x_1) \ldots \varphi(x_n) : \tag{1}$$

and then extending it off the energy shell by removing the condition

$$(\Box - m^2)\,\varphi(x) = 0. \tag{2}$$

Then Heisenberg local operators can be constructed by variational differentiation with respect to the fields $\varphi(x)$.

A system of basic assumptions of this type has been formulated by Bogolyubov[6] for a theory with adiabatic switching on and off of the interaction; this system, as was shown by Bogolyubov and Shirkov,[7] gave within the framework of perturbation theory the same results as the conventional Lagrangian formulation and renormalization theory. Later this set of assumptions was reformulated and made more precise by Bogolyubov and the authors[4]* especially for derivation of dispersion relations and spectral representations of the Källen-Lehmann type. We shall refer to this approach for the construction of quantum field theory, based on the set of fundamental assumptions of BMP, Sec. 2, and resting on the methods of dispersion theory, as the dispersion approach. The significance of the dispersion approach to quantum field theory is not restricted to the limited number of exact results, that have been obtained with its help, but determines a new method for the construction of the entire theory.

In particular, if one attempts to satisfy the fundamental conditions of the dispersion approach by a formal series expansion in powers of some small parameter, then one will, as always in perturbation theory, be able to obtain consistently one term in the expansion after another. The advantage over the conventional theory will lie in the fact that it will now no longer be necessary to resort to the physically unsatisfactory adiabatic switching on and off procedure, and one will be able to work with only renormalized quantities, thus avoiding the meaningless in the modern theory question of the relation between "renormalized" and "unrenormalized" quantities. As was recently shown,[8] the consecutive terms in the expansion are determined in this way accurately to within a finite number of constants, whose sig-

*To be referred to in the following as BMP.

nificance is that of the counter terms in the conventional Hamiltonian approach.*

The number of such constants is determined by the degree of growth of the matrix elements. Given the interaction Lagrangian, these degrees of growth are determined in a well known manner. It has been repeatedly suggested[1,4] that specifying the degree of growth replaces in the dispersion approach the specification of the Lagrangian. It was obvious at the same time,[1,8] that the degree of growth could not be specified completely arbitrarily.

This work is devoted to the clarification of the extent of the arbitrariness with which the degree of growth of various matrix elements can be specified. We find, somewhat unexpectedly, that this arbitrariness is quite limited, and that for the important class of "properly renormalizable" theories the degree of growth need not be specified as a separate postulate, since it is almost uniquely determined by the set of basic assumptions of BMP and the transformation properties of the fields.

It turns out that the investigation for the simplest case of a self-interacting spin-zero field can be carried out without explicit use of perturbation theory.

2. EQUATIONS FOR THE MATRIX ELEMENTS

From the set of basic physical assumptions formulated in BMP one can deduce in a variety of ways[9] a system of equations that couple with each other generalized vertices with various numbers of legs, i.e., matrix elements corresponding to various numbers of particles in the initial and final states. Since in the derivation of this system of equations use is made of the causality condition it is necessary, as has been already remarked, to consider in addition to the S matrix some kind of local Heisenberg operators. A minimum of two such operators must be introduced, their significance being that of first and second variational derivatives of the scattering matrix or, more precisely, of "radiation operators" (see BMP) of first and second order. We then can formulate the theory in such a way that all other "legs," except for one and two respectively, can be real and lie on the energy shell.*

We thus investigate the matrix elements of two Hermitian operators J and J(x) taken between states on the energy shell:

$$J(p_1, \ldots, p_l; q_1, \ldots, q_s) = \langle \mathbf{p}_1, \ldots, \mathbf{p}_l | J | \mathbf{q}_1, \ldots, \mathbf{q}_s \rangle, \tag{3}$$

$$J(x | p_1, \ldots, p_l; q_1, \ldots, q_s) = \langle \mathbf{p}_1, \ldots, \mathbf{p}_l | J(x) | \mathbf{q}_1, \ldots, \mathbf{q}_s \rangle \tag{4}$$

The operator J is simply the Heisenberg current operator evaluated at the origin of the coordinate system in order to exclude its trivial x-dependence

$$J = j(0), \qquad j(x) = i \frac{\delta S}{\delta \varphi(x)} S^+. \tag{5}$$

As a consequence of translational invariance its matrix elements are related to the matrix elements $j(x | \ldots p \ldots; \ldots q \ldots)$ of the operator j(x) by the formula

$$J(p_1, \ldots, p_l; q_1, \ldots, q_s) = j(x | p_1, \ldots, p_l; q_1, \ldots, q_s) \exp\{-i(\Sigma p_i - \Sigma q_j)x\}. \tag{6}$$

The second operator, J(x), is the retarded radiation operator from which again the trivial coordinate dependence has been removed:

$$J(x) = -\delta j\left(-\frac{x}{2}\right) \Big/ \delta\varphi\left(\frac{x}{2}\right), \qquad J^+(x) = J(x). \tag{7}$$

Its matrix elements (4) coincide with the functions $F^{ret}_{\omega\omega'}$ introduced in BMP. Let us note that in fact the matrix elements (3) contain one momentum that does not lie on the energy shell

$$P - Q \neq 0, \qquad P = \sum_1^l p_i, \qquad Q = \sum_1^s q_j, \tag{8}$$

and the matrix elements (4) contain two such momenta: (8) and the momentum corresponding to the explicitly appearing coordinate x.

It is easy to see that the matrix elements (3) and (4) are connected, independently of the causality condition, by the relations†

$$J(p, p_1, \ldots, p_l; q_1, \ldots, q_s) = P\left(\frac{q_1}{q_2, \ldots, q_s}\right) \times \delta(\mathbf{p} - \mathbf{q}_1) J(p_1, \ldots, p_l; q_2, \ldots, q_s) - \frac{1}{(2\pi)^{3/2}\sqrt{2p^0}} \times \int dx\, J(x | p_1, \ldots, p_l; q_1, \ldots, q_s) \times \exp\left\{i\left(p + \frac{P-Q}{2}\right)x\right\}, \tag{9'}$$

*Let us note, by the way, that in the dispersion approach—and this represents another of its advantages—the origin of the counter terms, as well as the reason for the appearance of divergences in the conventional theory when ϑ-functions are unjustifiably multiplied by insufficiently regular functions, become particularly clear. Such an operation reduces in momentum representation to an application of the integral Cauchy formula to a function that does not vanish at infinity without taking into account the integral along the large circle (cf. discussion in BMP, Secs. 1 & 4).

*The authors are grateful to N. N. Bogolyubov who called their attention to the usefulness of such an approach.

†The operator P in Eq. (9) is the symmetrization operator in the appropriate arguments, as defined by Bogolyubov (see[7], §18).

$$J(p_1, \ldots, p_l; q, q_1, \ldots, q_s) = P\left(\frac{p_1}{p_2, \ldots, p_l}\right)\delta(\mathbf{p}_1 - \mathbf{q})\, J(p_2, \ldots, p_l; q_1, \ldots, q_s) - \frac{1}{(2\pi)^{3/2}\sqrt{2q^0}}\int dx\, J(x \mid p_1, \ldots, p_l; q_1, \ldots, q_s) \times \exp\left\{i\left(-q + \frac{P-Q}{2}\right)x\right\}, \quad (9'')$$

$$J(x) - J(-x) = i\{j(x/2)\, j(-x/2) - j(-x/2)\, j(x/2)\}, \quad (10)$$

and the causality condition imposes on the operator $J(x)$ the additional restriction

$$J(x) = 0 \text{ for } x \lesssim 0. \quad (11)$$

And so we have obtained a system of equations (9) – (11) for determining the matrix elements of the operators J and $J(x)$. It may be that this system is sufficient to determine the operators in question; in any case it can be shown, by methods analogous to those used previously,[8] that this is indeed so within the framework of perturbation theory.

In order to exclude from the above system the operator $J(x)$ we first rewrite Eq. (10) in terms of matrix elements:

$$J(x \mid p_1, \ldots, p_l; q_1, \ldots, q_s) - J(-x \mid p_1, \ldots, p_l; q_1, \ldots, q_s) = i\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int d\mathbf{k}_1 \ldots d\mathbf{k}_\nu\, J(p_1, \ldots, p_l; k_1, \ldots, k_\nu) \times J(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \times \left[\exp\left\{i\left(\frac{P+Q}{2} - K\right)x\right\} - \exp\left\{-i\left(\frac{P+Q}{2} - K\right)x\right\}\right], \quad K = \sum_{\kappa=1}^{\nu} k_\kappa. \quad (12)$$

Now one must further take into account the restriction imposed on $J(x)$ by the causality condition (11), as a consequence of which $J(-x \mid \ldots p \ldots; \ldots q \ldots)$ vanishes when $x^0 < 0$. Formally this can be accomplished by introducing into Eq. (12) a factor $\theta(x^0)$:

$$J(x \mid p_1, \ldots, p_l; q_1, \ldots, q_s) = i\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int d\mathbf{k}_1 \ldots d\mathbf{k}_\nu\, J \times (p_1, \ldots, p_l; k_1, \ldots, k_\nu)\, J(k_1, \ldots, k_\nu; q_1, \ldots, q_s)\, \theta(x^0) \times \left[\exp\left\{i\left(\frac{P+Q}{2} - K\right)x\right\} - \exp\left\{-i\left(\frac{P+Q}{2} - K\right)x\right\}\right]. \quad (13)$$

Naturally this operation, as is well known, may turn out to be devoid of precise meaning if the integrands do not fall off sufficiently rapidly and would then result, when applied literally, in the appearance of divergences. In such cases, as is known from dispersion relations, it is necessary to perform a subtraction under the integral sign, as a consequence of which a certain polynomial will appear in momentum representation on the right hand side of Eq. (13). It is in this sense precisely that we will understand Eq. (13) and will not, in what follows, write out the polynomial explicitly.

Keeping these reservations in mind we can substitute Eq. (13) into, for example, Eq. (9′). We then arrive at an infinite system of coupled equations:

$$J(p, p_1, \ldots, p_l; q_1, \ldots, q_s) = P\left(\frac{q_1}{q_2, \ldots, q_s}\right)\delta(\mathbf{p} - \mathbf{q}_1)\, J(p_1, \ldots, p_l; q_2, \ldots, q_s) - \frac{(2\pi)^{3/2}}{\sqrt{2p^0}}\sum_{\nu}^{\infty}\frac{1}{\nu!}\int d\mathbf{k}_1 \ldots d\mathbf{k}_\nu\, J(p_1, \ldots, p_l; k_1, \ldots, k_\nu) \times J(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \times \left\{\frac{\delta(\mathbf{p} + \mathbf{P} - \mathbf{K})}{K^0 - P^0 - p^0 - i\varepsilon} - \frac{\delta(\mathbf{p} - \mathbf{Q} + \mathbf{K})}{-K^0 + Q^0 - p^0 - i\varepsilon}\right\} \quad (14)$$

and an analogous system, arising from Eq. (9″), from which to determine the matrix elements of the operator J.

It will be more convenient to deal not directly with the matrix elements of the operator J but rather with the relativistically invariant matrix elements

$$I(p_1, \ldots, p_l; q_1, \ldots, q_s) = \sqrt{2p_1^0 \ldots 2p_l^0\, 2q_1^0 \ldots 2q_s^0}\, J(p_1, \ldots, p_l; q_1, \ldots, q_s), \quad (15)$$

which are normalized in the conventional manner. (In order to establish the connection with the usual results let us recall that since we have already taken translational invariance into account, the number of arguments explicitly appearing in the matrix element is by one less than the number of legs of the corresponding generalized diagram.) For the matrix elements $I(\ldots p \ldots; \ldots q \ldots)$ the basic set can be rewritten in the form

$$I(p, p_1, \ldots, p_l; q_1 \ldots, q_s) = P\left(\frac{q_1}{q_2, \ldots, q_s}\right)\sqrt{2p^0 2q_1^0}\,\delta(\mathbf{p} - \mathbf{q}_1) \times I(p_1, \ldots, p_l; q_2, \ldots, q_s) - (2\pi)^{3/2}\sum_{\nu}^{\infty}\frac{1}{\nu!}\int \frac{d\mathbf{k}_1 \ldots d\mathbf{k}_\nu}{2k_1^0 \ldots 2k_\nu^0} \times I(p_1, \ldots, p_l; k_1, \ldots, k_\nu)\, I(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \times \left\{\frac{\delta(\mathbf{p} + \mathbf{P} - \mathbf{K})}{K^0 - P^0 - p^0 - i\varepsilon} - \frac{\delta(\mathbf{p} - \mathbf{Q} + \mathbf{K})}{-K^0 + Q^0 - p^0 - i\varepsilon}\right\}. \quad (16')$$

The lower limit in the summation over ν is deduced from the following considerations. First of all, as is easy to show, only connected diagrams contribute to the current J; therefore the summation over ν cannot include the value $\nu = 0$. Next, we recall the conditions for the stability of the

vacuum and the one-particle states, BMP I, Eq. (6), as a result of which it is necessary to set

$$I(-;-) = I(p;-) = I(-;q) = 0. \qquad (17)$$

Therefore, if even one of the numbers l or s is equal to zero, then the summation over ν starts at two.

The second half of the system of coupled equations is obtained in a fully analogous manner from Eq. (9″)

$$I(p_1, \ldots, p_l; q, q_1, \ldots, q_s) = P\left(\frac{p_1}{p_2, \ldots, p_l}\right)\delta(\mathbf{p}_1 - \mathbf{q})\sqrt{2p_1^0 2q^0} \times I(p_2, \ldots, p_l; q_1, \ldots, q_s) - (2\pi)^{3/2}\sum_{\nu}^{\infty}\frac{1}{\nu!}\int\frac{dk_1 \ldots dk_\nu}{2k_1^0 \ldots 2k_\nu^0} \times I(p_1, \ldots, p_l; k_1, \ldots, k_\nu)\, I(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \times \left\{\frac{\delta(-\mathbf{q}+\mathbf{P}-\mathbf{K})}{K^0 - P^0 + q^0 - i\varepsilon} - \frac{\delta(-\mathbf{q}-\mathbf{Q}+\mathbf{K})}{-K_0 + Q^0 + q^0 - i\varepsilon}\right\}. \qquad (16'')$$

We recall once more the polynomials discussed in connection with Eq. (13)—although not explicitly written out they should not be forgotten on the right sides of Eqs. (16).

In accordance with Eq. (17), matrix elements with two legs on the energy shell are absent from the system (16). Therefore it is necessary to go off the energy shell in order to be able to include proper energy parts in the considerations. It is convenient to do so by defining the four-dimensional Fourier transform of the matrix elements (4) by the formula

$$\tilde{I}(k \mid p_1, \ldots, p_l; q_1, \ldots, q_s) = [2p_q^0 \ldots 2p_l^0 2q_1^0 \ldots 2q_s^0]^{-1/2} \times \int e^{ikx} J(x \mid p_1, \ldots, p_l; q_1, \ldots, q_s)\, dx. \qquad (18)$$

According to Eq. (12) we obtain for these Fourier transforms the following expressions

$$\tilde{I}(k \mid p_1, \ldots, p_l; q_1, \ldots, q_s) = (2\pi)^3\sum_{\nu}^{\infty}\frac{1}{\nu!}\int\frac{dk_1 \ldots dk_\nu}{2k_1^0 \ldots 2k_\nu^0} I(p_1, \ldots, p_l; k_1, \ldots, k_\nu) \times I(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \times \left\{\frac{\delta[\mathbf{k} + (\mathbf{P}+\mathbf{Q})/2 - \mathbf{K}]}{-k^0 - (P^0+Q^0)/2 + K^0 - i\varepsilon} - \frac{\delta[\mathbf{k} - (\mathbf{P}+\mathbf{Q})/2 + \mathbf{K}]}{-k^0 + (P^0+Q^0)/2 - K^0 - i\varepsilon}\right\}, \qquad (19)$$

which, in particular, make it possible to find the proper energy parts when the remaining matrix elements on the energy shell are known. It can be said that Eqs. (19) represent the formulas for "leaving the energy shell," since the vector $\mathbf{k}$ in them is not restricted by the condition $\mathbf{k}^2 = m^2$.

One can also write the inverse relations:

$$I(p, p_1, \ldots, p_l; q_2, \ldots, q_s) = P\left(\frac{q_1}{q_2, \ldots, q_s}\right)\sqrt{2p^0 2q_1^0}\, I(p_1, \ldots, p_l; q_2, \ldots, q_s) - (2\pi)^{-3/2}\tilde{I}\left(p + \frac{P-Q}{2} \,\middle|\, p_1, \ldots, p_l; q_1, \ldots, q_s\right) \qquad (20)$$

under the condition $\mathbf{p}^2 = m^2$.

Without going into the ways of solving the system (16),* which represents in essence the relativistic analogue of the Low equation, we shall make use of it to make estimates of possible degrees of growth of the matrix elements of the operator J.

3. DEGREE OF GROWTH OF MATRIX ELEMENTS

The dependence of the matrix elements on the many momenta may be quite complex and we do not pretend here to discover what it is in detail. We address ourselves to a much simpler problem. Namely, in analogy to the procedure followed in perturbation theory (cf., for example, [7], Sec. 26) when it is desired to establish the degree of growth of some diagram, we will be interested only in the total degree of growth when all momenta are uniformly stretched.

We require for every matrix element l and s the existence for every kind of momentum of a finite growth exponent—that is a minimal integer $\omega(l, s)$ such that upon stretching of all momenta

$$p_1 = \xi_1 P, \ldots, \quad p_l = \xi_l P, \quad q_1 = \eta_1 P, \ldots, \quad q_s = \eta_s P, \quad P \to \infty \qquad (21)$$

the matrix element $I(\mathbf{p}_1, \ldots, \mathbf{p}_l; \mathbf{q}_1, \ldots, \mathbf{q}_s)$ grows slower than $P^{\omega(l,s)+\alpha}$ for an arbitrary $\alpha > 0$. We will refer to theories for which this condition is satisfied as renormalizable.† In what follows we consider only renormalizable theories.

*Difficulties of two kinds stand in the way of getting a solution for a system of this type. Such systems express lower matrix elements (i.e. matrix elements with fewer arguments) in terms of higher ones; it is therefore altogether not clear how to go about getting an exact solution for such a system. If, on the other hand, an approximation of some sort is being considered then one is faced with the problems arising from the overdeterminacy of the system—from the causality condition follows not only the system (16), but also a large variety [9] of other infinite systems of similar character. If we were to find an exact solution of the system (16), then that solution would automatically satisfy all the other possible systems; an approximation, on the other hand, good for one system may turn out to be very poor for another system.

†Such a definition of renormalizable theories is somewhat broader than the conventional definition, wherein it is also required that the number of matrix elements with nonnegative growth exponents be finite. The discussion that follows tends to support the view that the difference of the classes of theories defined in these two ways is empty.

We now wish to see whether the equations in (16) impose some kind of restrictions on the possible choice of the numbers $\omega(l, s)$. The right side of (16) consists, except for the uninteresting contribution from nonconnected diagrams, of an (infinite) sum of nonlinear terms of identical structure:

$$\int \frac{d\mathbf{k}_1 \ldots d\mathbf{k}_\nu}{2k_1^0 \ldots 2k_\nu^0} I(p_1, \ldots, p_l; k_1, \ldots, k_\nu) \times I(k_1, \ldots, k_\nu; q_1, \ldots, q_s) \times \left\{ \frac{\delta(\mathbf{p}+\mathbf{P}-\mathbf{K})}{K^0 - P^0 - p^0 - i\varepsilon} - \frac{\delta(\mathbf{p}-\mathbf{Q}+\mathbf{K})}{-K^0 + Q^0 - p^0 - i\varepsilon} \right\}.$$

Each of them contains 3ν integrations over the components of the momenta $\mathbf{k}$, ν factors k^0 in the denominator, and the three-dimensional δ-function divided by a one-dimensional energy denominator. In addition there appears under the integral sign the product of the matrix elements I corresponding to the numbers l, ν and ν, s, whose precise dependence on the momenta is unknown. Clearly, therefore, we cannot determine the momentum dependence of the entire integral. If, however, we make the plausible assumption that in the integration over $\mathbf{k}_1, \ldots, \mathbf{k}_\nu$ the important contributions come from large values of the momenta, then it will be sufficient to know the asymptotic behavior of the matrix elements in the integrand when all momenta are increased, which is governed by the growth exponent $\omega(l, \nu)$ or $\omega(\nu, s)$. But then the determination of the behavior of the integral when all momenta are increased reduces, just as in perturbation theory, to the counting of the powers of momenta. In view of what has been said such a count gives

$$3\nu - \nu - 3 - 1 + \omega(l, \nu) + \omega(\nu, s) = \omega(l, \nu) + \omega(\nu, s) + 2\nu - 4. \quad (22)$$

In order to make now an estimate of the entire right side degree of growth we remark that it would be most unnatural if the higher momentum powers in various terms in the summation over ν on the right side of Eq. (16) were to cancel each other without some physical reason. Such a cancellation could only be due to the existence of some sort of a group, as is for example the case for the well known compensation of high powers in electrodynamics as a consequence of the group of gauge transformations.

Therefore, if the theory does not admit any groups (which is assumed to be the case in what follows for the sake of simplicity*) it follows from Eq. (16) that the exponent of growth of the matrix element on the left side can in no case be less than the degree of growth (22) of each term of the right side. In this way we arrive at the two sets of inequalities:

$$\omega(l+1, s) \geqslant \omega(l, \nu) + \omega(\nu, s) + 2\nu - 4, \quad (23')$$

$$\omega(l, s+1) \geqslant \omega(l, \nu) + \omega(\nu, s) + 2\nu - 4, \quad (23'')$$

which must be satisfied for all l, s, and ν which fulfill the conditions

$$\nu \geqslant 1, \quad l + s \geqslant 1, \quad \nu + l \geqslant 2, \quad \nu + s \geqslant 2. \quad (24)$$

In a fully analogous manner we obtain from Eq. (19) an estimate for the degree of growth of the off the energy shell matrix element $\tilde{I}$:

$$\tilde{\omega}(l, s) \geqslant \omega(l, \nu) + \omega(\nu, s) + 2\nu - 4. \quad (25)$$

It is easy to imagine that the degree of growth should not depend on the numbers l and s separately, but only on their sum (that is the total number of legs in the diagram), i.e., that

$$\omega(l, s) = \Omega(l+s). \quad (26)$$

In that case the two sets (23) become one:

$$\Omega(l + s + 1) \geqslant \Omega(l + \nu) + \Omega(\nu + s) + 2\nu - 4, \quad (27)$$

with the same conditions (24) on the numbers l, s, and ν. It is easy to see that if all the inequalities are replaced by equalities then the system (27) has as a particular solution

$$\Omega_0(n) = 3 - n. \quad (28)$$

Therefore the general solution can be conveniently looked for in the form of a sum of the particular solution and a certain addition N(n):

$$\Omega(n) = 3 - n + N(n). \quad (29)$$

After this substitution the basic system takes on the form

$$N(l + s + 1) \geqslant N(l + \nu) + N(s + \nu). \quad (30)$$

The considerations that have led us to Eq. (30) cannot, of course, be taken as yet as a rigorous mathematical proof. A mathematician would call them heuristic and would try to produce counter examples. Without attempting to give here a rigorous proof let us present one more argument. The matrix elements in the integrands on the right side of Eq. (16) consist, in part, of the counterterms. Since the momentum dependence of the counterterms is in the form of polynomials the integrand is explicitly known and so the part of the integral involved in the counter term can be exhaustively studied by elementary methods. Actually such a study is unnecessary since exactly the same prob-

*The case when such a group exists must be investigated separately.

lem has been studied in detail in the theory of R-operation (see [7], Sec. 26).

Let us go on to an investigation of the system of inequalities (30). We establish first of all that all N(n) are nonpositive. To prove this it is sufficient for odd n to set in Eq. (30) $\nu = s + 1 \geq 2$, l —arbitrary. Then the left side cancels the first term on the right and we get

$$0 \geqslant N(2s+1) \text{ for } s \geqslant 1. \tag{31}$$

For even n we set $s = l \geq 1$ and $\nu \geq 1$, which again is allowed by the conditions (24). We then get

$$2N(l+\nu) \leqslant N(2l+1) \leqslant 0, \tag{32}$$

where we have used Eq. (31) for the last inequality. Thus

$$\Omega(n) \leqslant \Omega_0(n) = 3 - n \text{ for all } n \geqslant 2. \tag{33}$$

In this way we verify that the particular solution (28) gives the maximum possible growth exponent for the matrix elements $I(p_1, \ldots p_l; q_1, \ldots q_s)$. From this result it follows in particular that the number of matrix elements with positive growth exponents is finite and that consequently our definition of renormalizable theories coincides with the conventional one.

We show next that the possible growth exponents are bounded not only from above but also from below. To this end we set $\nu = 1$ —its minimum possible value, and denote the argument on the left side of Eq. (30) by the single letter n, so that

$$N(n) \geqslant N(n-s) + N(s+1), \tag{34}$$

where n and s are restricted according to Eq. (24) by the conditions

$$n \geqslant s+2, \quad s \geqslant 1. \tag{35}$$

The same formula (with a new value for n) can be applied again to the first term on the right side of Eq. (34). Repeating this procedure k times we obtain for N(n) the lower bound

$$N(n) \geqslant N(n-ks) + kN(s+1)N. \tag{36}$$

At that, in view of Eq. (35), in choosing k one must satisfy the condition

$$k \geqslant (n-2)/s. \tag{37}$$

Setting in Eq. (36) s = 1 and k equal to its maximum possible value k = n − 2, we find that for all $n \geq 2$ we must have

$$N(n) \geqslant N(2) + (n-2)N(2) = (n-1)N(2). \tag{38}$$

The result (38) is not an underestimate, by which we mean that the lower bound for the values of N(n) indicated by it can actually be attained. Indeed, if we let the N(n) in the basic system (30) take on the minimum values allowed by Eq. (38), then that system reduces to the condition $\nu \geq 1$, which is satisfied according to Eq. (24).

The limitations on the possible values of N(n) are not exhausted by the above conditions (32) and (38). Namely, should N(n) for some $n = n_0 > 2$ exceed the minimum value (38) then new restrictions are imposed on N(n) for $n > n_0$, which can be derived from Eq. (36) by setting in it s equal to some number larger than one. We will not derive these conditions here.

Finally, relations (25) provide us with a lower bound for the growth exponent for diagrams with two legs off the energy shell:

$$\tilde{N}(1) = \tilde{\Omega}(1) - 2 = \tilde{\omega}(1 \mid -; -) - 2 \geqslant 2N(\nu), \quad \nu \geqslant 2. \tag{38'}$$

In this case there is no upper bound.

4. PROPERLY RENORMALIZABLE THEORIES

In the preceding section we have derived from the basic equations (16) of the theory the system of inequalities (23) by assuming the theory to be renormalizable. To that end we made use of the requirement that the growth exponent on the left side of Eq. (16) could not be exceeded by the growth exponent of any of the terms on the right side, i.e., it had to be either larger than or equal to the largest growth exponent of the terms on the right side. It would be desirable to make this condition stronger and replace the inequality by an equality. Since, as we have repeatedly emphasized, in addition to the explicitly written out integrals it is understood that there appear on the right side of Eq. (16) certain counter terms, such a strengthening of the conditions requires an additional assumption.

We shall call a theory properly renormalizable if the power of the polynomials added to the T products does not exceed the growth exponent of the corresponding T products. Let us now suppose that we are dealing with a properly renormalizable theory. Then the conditions (23) can indeed be rewritten in the stronger form:

$$\omega(l+1, s) = \max_{\nu} \{\omega(l, \nu) + \omega(\nu, s) + 2\nu - 4\},$$
$$\omega(l, s+1) = \max_{\nu} \{\omega(l, \nu) + \omega(\nu, s) + 2\nu - 4\}. \tag{39}$$

For the quantities N(n) the stronger conditions become

$$N(l+s+1) = \max \{N(l+\nu) + N(s+\nu)\}, \tag{40}$$

where the maximum is to be taken over all values of the arguments allowed by the conditions (24).

Table I

2							
5=3+2	*4*						
7=4+3	5+2	*6*					
9=5+4	6+3	7+2	*8*				
11=6+5	7+4	8+3	9+2	*10*			
13=7+6	8+5	9+4	10+3	11+2	*12*		
15=8+7 17=9+8	9+6 10+7 ...	10+5 11+6 ...	11+4 12+5 ...	12+3 13+4 ...	13+2 14+3 ...	*14* 15+2 ...	...

We attempt to solve this system. Let us write it in the form:

$$N(n_1) = \max\{N(n_2) + N(n_3)\}, \tag{41}$$

where now the maximum is sought over values of n_2 and n_3 satisfying the set of conditions

$$n_2 + n_3 = n_1 - 1 + 2\nu, \quad n_2 > \nu \geqslant 1, \; n_3 > \nu \geqslant 1. \tag{42}$$

To obtain a solution of the system (41) we write the totality of pairs N(n), over which the maximum on the right side of Eq. (41) is taken, in the form of two tables (I and II, for even and odd values separately) with two entries, in which the rows are labeled by the sums of n_2 and n_3 and the columns by the differences. The sums in the cells in the tables represent all possible decompositions of $n_1 - 1 + 2\nu$ into the sum of n_2 and n_3, satisfying Eq. (42); it is obvious that they represent simultaneously the sums standing under the maximum sign in Eq. (41). If any of these combinations of n_2 and n_3 satisfy the inequalities (42) then so will obviously, for fixed n_1, all the combinations appearing below in the same column since displacement downwards in a column corresponds simply to an increase in ν. By giving ν the smallest possible value $\nu = 1$ we convince ourselves that, according to the first equality (42), only the rows with $n_2 + n_3 \geq n_1 + 1$ enter into the region over which the maximum in Eq. (41) is sought for a given n_1. As regards the number of columns that enter this region, this is determined from the condition $(n_2 - n_3) \leq (n_2 + n_3) - 4$ which follows from Eq. (42). The resultant regions are shown in Tables I and II and marked by the corresponding values of n_1, which are denoted in italics. It follows right away from a study of Tables I and II that

$$N(2) \leqslant N(4), \tag{43}$$

since it is seen from Table I that the region over which the maximum for N(2) is sought, is fully contained within the region over which the maximum for N(4) is sought.

Table II

4=2+2	*3*				
6=3+3	4+2	*5*			
8=4+4	5+3	6+2	*7*		
10=5+5	6+4	7+3	8+2	*9*	
12=6+6 14=7+7	7+5 8+6 ...	8+4 9+5 ...	9+3 10+4 ...	10+2 11+3 ...	*11* 12+2 ... ...

Next the situation becomes more complicated since in passing from N(3) to N(5) not only a new (the third in Table II) column is added, but also the element 2 + 2 gets eliminated from the first row. However as a consequence of Eq. (43) N(2) + N(2) ≤ N(4) + N(2), and the combination 4 + 2 also enters both allowable regions. Consequently we may write a new inequality:

$$N(3) \leqslant N(5). \tag{44}$$

It is easy to see that this state of affairs will be preserved at each successive step. That is, each time as we go from n to n+2 the shrinkage of the region over which the maximum is sought will turn out to be immaterial as soon as chains of inequalities of the type Eqs. (43), (44) are established for all $k \leq n+1$. In this manner a complete induction turns out to be possible and we arrive at the infinite chains

$$N(2) \leqslant N(4) \leqslant N(6) \leqslant \ldots \leqslant N(2k) \leqslant \ldots, \tag{45'}$$

and

$$N(3) \leqslant N(5) \leqslant N(7) \leqslant \ldots \leqslant N(2k+1) \leqslant \ldots. \tag{45''}$$

On the other hand, however, all N(n) are bounded from above by the condition (33). Thus the two nondecreasing sequences of integers (34) should reach their upper bounds, i.e., there should exist

$$\max N(2k) = -a \leqslant 0 \quad \text{and}$$

$$\max N(2k+1) = -b \leqslant 0 \quad (k \geqslant 1). \tag{46}$$

Observing now that the region (42), over which the maximum in Eq. (41) is sought, includes for any n_1 arbitrarily large n_2 and n_3 we conclude that instead of Eq. (41) we could write

$$N(2k) = \max\{-a-b\} = -a-b,$$
$$N(2k+1) = \max\{(-a-a),(-b-b)\}. \tag{47}$$

Consequently N(2k) and N(2k+1) are independent of k. But then

$$N(2k) = -a = -a-b, \text{ i. e. } b = 0,$$

and, further,

$$N(2k+1) = -b = \max\{-2a, -2b\},$$

i.e., $-a \leq 0$.

And so the solution of the system (40) is of the form

$$N(2k) = -a, \quad N(2k+1) = 0, \tag{48}$$

where a is an arbitrary nonnegative integer. According to Eq. (29) this means that the general expression for the possible growth exponents of matrix elements I in a properly renormalizable theory is given by

$$\Omega(2k) = 3 - 2k - a,$$
$$\Omega(2k+1) = 2 - 2k, \quad a \geqslant 0, \; a \in N. \tag{49}$$

There remains to write the condition for the growth exponent for the diagram with two legs:

$$\tilde{N}(1) = 2\max_{\nu \geqslant 2} N(\nu) = 0, \quad \tilde{\Omega}(1) = \tilde{N}(1) + 2 = 2. \tag{50}$$

5. DISCUSSION

The most important of the results obtained in the two preceding sections is the upper bound (33) for the possible growth exponents. Only matrix elements with three or four legs (n = 2 or 3) (beside the proper energy parts, that do not enter the system (16)) can have nonnegative exponents. But this means that only these matrix elements may possess counterterms (being polynomials in the momenta they could not have negative growth exponents!). We thus see that within the dispersion approach framework the "dynamical principle" is almost superfluous—the specification of the transformation properties of the fields alone specializes the character of admissible interactions accurate to within a small number of constants. For the case of the spinless field here considered only two such constants appear—the constant quadrilinear counterterm (n = 3) as well as the trilinear one (the formally admissible linear counterterm is forbidden by relativistic invariance considerations).

Particularly strong limitations arise in properly renormalizable theories. We note that a condition, that appears to be identical to the proper renormalizability condition, is always imposed in the conventional construction of perturbation theory; namely the choice for counterterms in momentum representation of polynomials of the minimum possible degree (cf., for example, [7], Sec. 26) (without such a condition it would be impossible to construct the R-operation). A significant difference arises, however, in the following point. Conventionally one deals in fact with counterterms (in a broad sense) of two kinds. Along with proper counter terms (renormalization constants), arising in the determination of products of singular functions, one also considers "charges," which come from the original Lagrangian. The degrees of the corresponding polynomials are not determined from a minimality condition but are prescribed ad hoc when the theory is formulated. In our approach all counterterms are treated uniformly, the "charges" appearing on the same footing as the "renormalization constants." Both kinds play the role of inhomogeneities or a kind of boundary conditions for the basic system (16); there are reasons to believe that in their absence only the trivial zero solution would be admissible. This is certainly so if we accept an expansion in some small parameter in the spirit of [8].

In that connection the requirement of minimum growth adds to the logical scheme of the theory a particular beauty: in the absence of counterterms the system (16) would possess only the trivial solution; nontrivial solutions are obtained only if counter terms are introduced; introducing them, however, does not add extraneous elements to the system (16) but rather use is made of the nonuniqueness present in the system, owing to its singular nature. Were the system (16) regular there would be no internal reason for adding counterterms and we would arrive at a unique—zero—solution (cf. [10]).

We thus impose the condition of "minimal growth" also on the counterterms, normally included in the "bare Lagrangian." This, naturally, may lead to stronger restrictions on the acceptable class of theories—the class of properly renormalizable theories already is the class of theories renormalizable in the conventional sense. In particular this eliminates the theory with one trilinear interaction of scalar particles—the Hurst-Thirring field. In this theory the growth exponent of the simplest diagram with three legs is equal to minus two, and is even smaller for more complicated three-legged diagrams. Therefore from the point of view of minimal growth such diagrams should not be accompanied by counterterms and the charge should be equal to zero.*

It is interesting that a properly renormalizable theory with two types of scalar interactions—trilinear and quadrilinear simultaneously—is pos-

*It is interesting to compare these considerations with the indications that have appeared in the literature[11] that this theory is internally inconsistent.

sible. Indeed, the growth exponent of the trilinear vertex $\Omega(2)$ may be chosen in Eq. (49) either equal to zero or negative on account of a, but the one corresponding to the quadrilinear vertex $\Omega(3)$ is necessarily equal to zero, i.e., in the theory of interacting spin zero particles there necessarily appears the quadrilinear interaction.

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