ANALYTIC PROPERTIES OF A SQUARE DIAGRAM WITH NON-DECAYING MASSES

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An explicit form of the dispersion representations in energy and momentum transfer for the simplest square diagram with non-decaying masses is derived.

THE analytic properties of the simplest square diagram (four-branch vertex) have been investigated by many authors.[1] The aim of the present paper is to determine the explicit form of the dispersion representations in the energy s and the square of the momentum transfer t in the center-of-mass system for the simplest square diagram with arbitrary external mass values consistent with the stability condition.

If the external masses are sufficiently small, one can write the amplitude under consideration as a dispersion integral of the imaginary part, which can be determined with the help of the unitarity condition. As the masses of the external particles are increased, anomalous terms appear in the dispersion representation; these are due to the anomalous thresholds or to the presence of complex singularities in the amplitude. These terms can be determined by analytic continuation with respect to the masses appearing in the dispersion relations. The method of analytic continuation with respect to the masses, as proposed by Mandelstam,[2] has been used earlier in the investigation of the anomalies of the square diagram with different masses.[3]

In the first section of this paper we shall consider the singularities of the imaginary part of the amplitude, which must be known for the analytic continuation with respect to the masses in the dispersion relations. In Sec. 2 we shall carry out the analytic continuation of the dispersion integral over s with respect to one of the masses with the condition $t < 0$. In Sec. 3 we shall obtain dispersion representations which are valid for arbitrary values of t by analytic continuation with respect to t. We shall see that the amplitude has a complex singularity in s for certain nonphysical values of t. It is interesting to note that this singularity lies in the lower half-plane of s. Thus the amplitude is an analytic function of s in the upper half-plane for arbitrary mass values and for arbitrary real

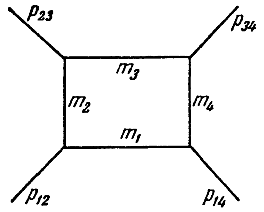

FIG. 1

t. This result remains valid when the external masses do not satisfy the stability condition, as will be shown in a subsequent paper.

In Sec. 4 we carry out the analytic continuation with respect to the masses of the other particles. In this way we obtain a dispersion representation in s for arbitrary external mass values satisfying the stability condition. It follows from the formulas of the present paper that the anomalous term can in all cases be written as an integral of a function which is formally identical with the Mandelstam function $A_{12}(s, t)$.[4] The dispersion representation in t can obviously be obtained from the representation in s by making the replacement $s \rightleftarrows t$ and relabeling the masses.

1. FORMULATION OF THE PROBLEM. POSITION OF THE SINGULARITIES OF THE IMAGINARY PART OF THE AMPLITUDE

Let us consider the anomalous singularities of the diagram shown in Fig. 1. Let $p_{12} = p_{21}$, $p_{23} = p_{32}$, $p_{34} = p_{43}$, $p_{14} = p_{41}$ be the four-momenta of the particles involved in the reaction, and let m_1^2, m_2^2, m_3^2, and m_4^2 be the squares of the masses of the virtual particles. The amplitude corresponding to the diagram of Fig. 1 can be written in the form

$$A = \frac{ig^4}{32\,(2\pi)^{19}\,(2p_{12}^0\cdot 2p_{23}^0\cdot 2p_{34}^0\cdot 2p_{14}^0)^{1/2} m_1 m_2 m_3 m_4}\,A(\mu_{ik}), \qquad (1)$$

where g is the coupling constant and the μ_{ik} are given by*

*The notation is the same as in [5].

$$\mu_{ik} = \mu_{ki} = (m_i^2 + m_k^2 - p_{ik}^2)/2m_i m_k,$$

with

$$p_{13}^2 = p_{31}^2 = (p_{12} + p_{23})^2 = s, \quad p_{24}^2 = p_{12}^2 = (p_{12} + p_{14})^2 = t.$$

It follows from the definition of μ_{ik} that in the physical region of the reaction $(p_{12}, p_{23}) \to (p_{34}, p_{14})$

$$\mu_{13} < -1, \qquad \mu_{24} > 1.$$

The function $A(\mu_{ik})$ is an analytic function of each of its arguments if the other arguments are fixed. It is known that for sufficiently small internal masses, the function $A(\mu_{ik})$ (with $\mu_{24} > 1$) can be written as a dispersion integral over μ_{13}:

$$A(\mu_{ik}) = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{A_1}{\mu'_{13} - \mu_{13}} d\mu'_{13}, \tag{2}$$

where $A_1 = \mathrm{Im}\, A(\mu_{ik})$ is determined from the unitarity condition. After rather involved calculations one finds that

$$A_1 = K'^{-1/2} \ln(\xi_1/\xi_2), \quad \xi_1 = V - \sqrt{K'}, \quad \xi_2 = V + \sqrt{K'}, \tag{3}$$

where $K' = (\mu_{13}^2 - 1)K$, and V and K have the form

$$V = \mu_{24}(\mu_{13}^2 - 1) - \mu_{13}(\mu_{12}\mu_{34} + \mu_{23}\mu_{14}) + \mu_{23}\mu_{34} + \mu_{12}\mu_{14}, \tag{4}$$

$$K = \begin{vmatrix} 1 & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{12} & 1 & \mu_{23} & \mu_{24} \\ \mu_{13} & \mu_{23} & 1 & \mu_{34} \\ \mu_{14} & \mu_{24} & \mu_{34} & 1 \end{vmatrix}. \tag{5}$$

We assume in formula (3) that $\arg \xi_1 = \arg \xi_2 = 0$ and $\sqrt{K} > 0$ in the physical region.

The function $A(\mu_{ik})$ in formula (1) is the limit of the analytic function (2) obtained by letting μ_{13} approach the real axis from the lower half-plane.

In order to obtain for $A(\mu_{ik})$ a dispersion relation which is valid for large external masses, we must continue formula (2) analytically with respect to the masses. For this purpose we must investigate the positions of the singularities of A_1 in the variable μ_{13} as functions of the external masses. The singular points of A_1 are evidently determined by the equations

$$\xi_1 = 0, \tag{6}$$

$$\xi_2 = 0, \tag{7}$$

$$K = 0. \tag{8}$$

It is easily verified that (6) and (7) imply

$$\lambda(\mu_{12}, \mu_{23}, \mu_{13}) = 0, \tag{9}$$

$$\lambda(\mu_{34}, \mu_{14}, \mu_{13}) = 0, \tag{10}$$

where

$$\lambda(\alpha, \beta, \gamma) = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma - 1. \tag{11}$$

Equations (8) — (10) are the same as those obtained by applying the Landau method [6] for the determination of the singular points to the diagram under consideration.

The solution of (9) and (10) can be written in the form

$$\Delta^{\pm}(\mu_{12}, \mu_{23}) = \mu_{12}\mu_{23} \pm \sqrt{(1 - \mu_{12}^2)(1 - \mu_{23}^2)}, \tag{12}$$

$$\Delta^{\pm}(\mu_{34}, \mu_{14}) = \mu_{14}\mu_{34} \pm \sqrt{(1 - \mu_{14}^2)(1 - \mu_{34}^2)}. \tag{13}$$

If the absolute values of μ_{12}, μ_{23}, μ_{14}, and μ_{34} are smaller than unity, then

$$\Delta^{\pm}(\mu_{12}, \mu_{23}) = \cos(\theta_{12} \pm \theta_{23}), \tag{14}$$

$$\Delta^{\pm}(\mu_{14}, \mu_{34}) = \cos(\theta_{14} \pm \theta_{34}), \tag{15}$$

where

$$\theta_{ik} = \arccos \mu_{ik}, \qquad 0 \leqslant \theta_{ik} \leqslant \pi. \tag{16}$$

We shall denote the four roots of (9) and (10) also by the symbols $\Delta_i(\mu_{13})$ (i = 1, 2, 3, 4), numbering them in the order of increasing values:

$$\Delta_1(\mu_{13}) < \Delta_2(\mu_{13}) < \Delta_3(\mu_{13}) < \Delta_4(\mu_{13}).$$

The solution of (8) has the form

$$\mu_{13}^{\pm} = (\mu_{24}^2 - 1)^{-1}\{\omega(\mu_{24}, \mu_{14}, \mu_{23}, \mu_{12}, \mu_{34}) \pm [\lambda(\mu_{12}, \mu_{14}, \mu_{24})\,\lambda(\mu_{23}, \mu_{34}, \mu_{24})]^{1/2}\}. \tag{17}$$

Solving (8) with respect to μ_{24}, we have

$$\mu_{24}^{\pm} = (\mu_{13}^2 - 1)^{-1}\{\omega(\mu_{13}, \mu_{23}, \mu_{14}, \mu_{12}, \mu_{34}) \pm [\lambda(\mu_{12}, \mu_{23}, \mu_{13})\,\lambda(\mu_{34}, \mu_{14}, \mu_{13})]^{1/2}\}. \tag{18}$$

Here

$$\omega(\alpha, \beta, \gamma, \delta, \nu) = \alpha(\beta\gamma + \delta\nu) - (\beta\nu + \gamma\delta). \tag{19}$$

The functions $\mu_{13}^{\pm}$ and $\mu_{24}^{\pm}$ will also be denoted by $\square_i(\mu_{24})$ and $\square_i(\mu_{13})$, where i = 1, 2 and

$$\square_1(\mu_{24}) < \square_2(\mu_{24}), \qquad \square_1(\mu_{13}) < \square_2(\mu_{13}).$$

It is seen from (17) and (18) that the curves (8) lie in the dashed regions of the (μ_{13}, μ_{24}) plane (see Fig. 2). For brevity, curves lying in the regions I, . . . , V will in the following be called curves I, . . . , V, respectively.

The curves I — IV have as asymptotes the straight lines $\mu_{13} = \pm 1$, $\mu_{24} = \pm 1$. These curves either lie entirely inside the region between the asymptotes or touch the straight lines $\Delta_1(\mu_{13})$, $\Delta_4(\mu_{13})$, $\Delta_1(\mu_{24})$, $\Delta_4(\mu_{24})$, where $\Delta_i(\mu_{24})$ are the roots of the equations

$$\lambda(\mu_{12}, \mu_{14}, \mu_{24}) = 0, \qquad \lambda(\mu_{23}, \mu_{34}, \mu_{24}) = 0, \tag{20}$$

i.e.,

$$\mu_{24} = \Delta^{\pm}(\mu_{12}, \mu_{14}), \qquad \mu_{24} = \Delta^{\pm}(\mu_{14}, \mu_{34}),$$

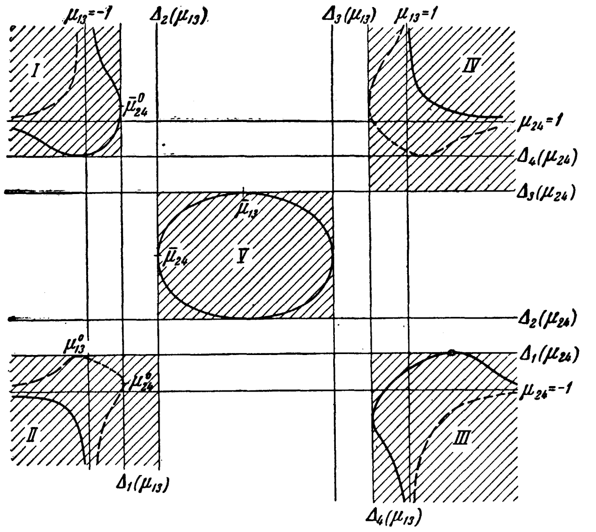

FIG. 2

numbered in the order of increasing values.

It can be shown that curve I touches the straight line $\Delta_1(\mu_{13})$ if the condition

$$(\mu_{12}+\mu_{23})(\mu_{14}+\mu_{34})>0 \tag{21}$$

is fulfilled. If the inequality sign is reversed, curve II touches the straight line $\Delta_1(\mu_{13})$.

If

$$(\mu_{12}+\mu_{14})(\mu_{23}+\mu_{34})>0, \tag{22}$$

curve III touches the straight line $\Delta_1(\mu_{24})$, whereas curve II touches the straight line $\Delta_1(\mu_{24})$ if the inequality sign in (22) is reversed. The position of the curves I — III under the conditions (21) and (22) is shown in Fig. 2 (solid lines). Figure 2 also shows the position of these curves when the conditions (21) and (22) are reversed (dotted lines).

It is easily seen that part of curve I can lie in the physical region for certain values of the masses. But this part of the curve is actually not singular for A_1.

As will become clear in the following, formula (2) is valid if

$$\mu_{12}+\mu_{23}>0, \qquad \mu_{14}+\mu_{34}>0, \tag{23}$$

i.e., if

$$\theta_{12}+\theta_{23}<\pi, \qquad \theta_{14}+\theta_{34}<\pi.$$

Condition (23) implies the inequality (21). The positions of the curves I and II for this case are indicated by solid lines in Fig. 2.

2. ANALYTIC CONTINUATION WITH RESPECT TO μ_{12} AND μ_{23} FOR $\mu_{24} > 1$

Let us now turn to the analytic continuation of formula (2) with respect to μ_{12} in the region $\mu_{24} > 1$. A dispersion relation for $A(\mu_{ik})$ for other values of μ_{24} can be obtained by analytic continuation with respect to μ_{24}.

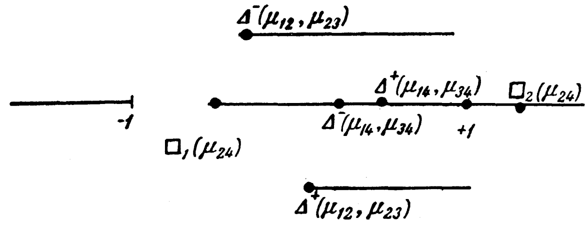

FIG. 3

In order to carry out the analytic continuation with respect to μ_{12}, we assume that μ_{12} has an infinitesimal imaginary part $-i\delta$. To be definite, we assume that $\delta > 0$. The case $\delta < 0$ is treated analogously. It can be shown that, for $\delta > 0$, the point $\Delta^-(\mu_{12}, \mu_{23})$ moves into the upper half-plane of μ_{13}, whereas $\Delta^+(\mu_{12}, \mu_{23})$ moves into the lower half-plane. If the mass p_{12}^2 is sufficiently small, the point $\square_1(\mu_{24})$ lies in the lower half-plane, and the point $\square_2(\mu_{24})$, in the upper half-plane. The positions of the singular points in the complex μ_{13} plane and the cuts leading away from them are shown in Fig. 3 for the case where the masses are sufficiently small, i.e., condition (23) is fulfilled.

The cuts coming from the singular points must be chosen in such a way that $\sqrt{K}$ is positive in the region $\mu_{13} < (-1, \square_1(\mu_{24}))$ and $\arg \xi_1 = \arg \xi_2 = 0$.

As p_{12}^2 increases, i.e., as μ_{12} decreases, the point $\Delta^-(\mu_{12}, \mu_{23})$ remains in the upper half-plane until μ_{12} reaches the value

$$\mu_{12}=-\mu_{23}, \tag{24}$$

i.e., $\theta_{12} + \theta_{23} = \pi$.

For small $\mu_{12} < -\mu_{23}$ the sign of the inequality (21) is reversed. The positions of the curves I and II for this case are indicated in Fig. 2 by dotted lines.

If $\mu_{12} = -\mu_{23}$, $\Delta^-(\mu_{12}, \mu_{23})$ hits the half-axis $\mu_{13} < -1$ in the point

$$\Delta^-(-\mu_{23}, \mu_{23}) = -1 - a\delta^2,$$

where a is some positive number. As μ_{12} decreases further, the point $\Delta^-(\mu_{12}, \mu_{23})$ moves into the lower half-plane. The point $\square_1(\mu_{24})$ moves into the upper half-plane for a certain $\mu_{12} > -\mu_{23}$, without touching the half-axis $\mu_{13} < -1$. The points $\Delta^+(\mu_{12}, \mu_{23})$ and $\square_2(\mu_{24})$ are of no interest, since they are always far removed from the half-axis $\mu_{13} < -1$.

In order to prevent the point $\Delta^-(\mu_{12}, \mu_{23})$ from touching the contour of integration during these operations, we must deform this contour in the

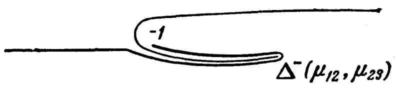

FIG. 4

way shown in Fig. 4. The integral along this contour will be an analytic function of μ_{12} and is identical with the integral (2) for $\mu_{12} < -\mu_{23}$. Thus the integral along the contour shown in Fig. 4 represents an analytic continuation of formula (2) into the region $\mu_{12} < -\mu_{23}$.

The integral along the contour C shown in Fig. 4 gives an anomalous addition to formula (2). This term can be reduced to an integral of the discontinuity of A_1 along the segment $(-1, \Delta^-(\mu_{12}, \mu_{23}))$ of the cut coming from the point $\Delta^-(\mu_{12}, \mu_{23})$. Let us consider, for example, the case where

$$\Delta^-(\mu_{12}, \mu_{23}) = \Delta_1(\mu_{13}).$$

In the region $-1 < \mu_{13} < \Delta_1(\mu_{13})$ we have

$$V^2 > K', \tag{25}$$

$$\sqrt{K'} < 0, \tag{26}$$

$$V < 0; \tag{27}$$

the inequality (25) follows from (6) and (7), and (26) is obvious from Fig. 4. In order to prove (27), we note that the quantity V does not vanish for $\mu_{12} + \mu_{23} < 0$ in the region $\mu_{24} > 1$, $-1 < \mu_{13} < \Delta_1(\mu_{13})$, and $V \to -\infty$ for $\mu_{24} \to \infty$, as is seen from formula (4).

It follows from (25) — (27) that $\Delta^-(\mu_{12}, \mu_{23})$ is a singular point for $\ln \xi_1$, and $\xi_1 < 0$ on the cut. On the other hand, $d\xi_1/d\mu_{13} > 0$ in the point $\mu_{13} = \Delta_1(\mu_{13})$, since

$$\frac{d\xi_1}{d\mu_{13}}\Big|_{\mu_{13}=\Delta_1} = \frac{1}{\xi_2}\frac{d(\xi_1\xi_2)}{d\mu_{13}}\Big|_{\mu_{13}=\Delta_1}$$

$$= \frac{1}{\xi_2}\frac{d}{d\mu_{13}}[\lambda(\mu_{12}, \mu_{23}, \mu_{13})\lambda(\mu_{14}, \mu_{34}, \mu_{13})]\big|_{\mu_{13}=\Delta_1}$$

$$= \frac{\lambda(\mu_{14}, \mu_{34}, \mu_{13})}{\xi_2}\frac{d}{d\mu_{13}}\lambda(\mu_{12}, \mu_{23}, \mu_{13})\big|_{\mu_{13}=\Delta_1} > 0. \tag{28}$$

Hence $\text{Im}\,\xi_1 > 0$ near the point $\Delta_1(\mu_{13})$ if μ_{13} is in the upper half-plane, and $\text{Im}\,\xi_1 < 0$ if μ_{13} is in the lower half-plane. In other words, $\arg \xi_1 = \pi$ on the upper branch of the branch cut and $\arg \xi_2 = -\pi$ on the lower branch. The anomalous term $A_{an}(\mu_{ik})$ is therefore equal to

$$A_{an}(\mu_{ik}) = -\frac{2\pi i}{\pi}\int\limits_{-1}^{\Delta^-(\mu_{12}, \mu_{23})}\frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13}-\mu_{13})},$$

$$\sqrt{K} = i\,|\sqrt{K}|. \tag{29}$$

Formula (29) is evidently also valid in the case when the point $\Delta^-(\mu_{12}, \mu_{23})$ is not the one with the lowest value of all the $\Delta_i(\mu_{13})$, since the function in (29) is analytic in μ_{12}. For the same reason formula (29) maintains its form also when one continues with respect to μ_{23}. Hence formula (29) describes the anomalous term in the region $\mu_{24} > 1$ for

$$\theta_{12}+\theta_{23} > \pi, \qquad \theta_{14}+\theta_{34} < \pi \tag{30}$$

It is seen from (29) that the anomalous addition to formula (2) is expressed in the form of an integral of a function which is formally identical with the Mandelstam function A_{12}, which in our case is equal to

$$A_{12}(\mu_{ik}) = 2\pi/|\sqrt{K}| \tag{31}$$

(see the following section).

3. ANALYTIC CONTINUATION INTO THE REGION $\mu_{24} < 1$

Let us now establish a dispersion representation for $A(\mu_{ik})$ in the region $\mu_{24} < 1$. Here we must distinguish between two cases:

1) $$\theta_{12}+\theta_{23}+\theta_{14}+\theta_{34} < 2\pi, \tag{32}$$

2) $$\theta_{12}+\theta_{23}+\theta_{14}+\theta_{34} > 2\pi. \tag{33}$$

In the first case Mandelstam's double representation is valid, as will be shown below. We shall call this case the normal case. In the second (anomalous) case there does not exist a double representation, since the function $A(\mu_{ik})$ has complex singularities.

1) Let us consider the analytic continuation in the normal case. If condition (32) is satisfied, we have, according to (14) and (15),

$$\Delta^-(\mu_{12}, \mu_{23}) = \Delta_1(\mu_{13}).$$

The expression under the integral sign in formula (29) for $A_{an}(\mu_{ik})$, regarded as a function of μ_{24}, has the singular points $\square_1(\mu_{13})$ and $\square_2(\mu_{13})$, which lie on the curve II. We must draw the cut between these points in such a way that $\sqrt{K} = i\,|\sqrt{K}|$ for $\mu_{24} > \square_2(\mu_{13})$. Then formula (29) can be written as a double dispersion integral

$$\frac{2\pi}{\pi^2}\int d\mu'_{13}\,d\mu'_{24}\,/\,[\sqrt{K}(\mu'_{13}-\mu_{13})(\mu'_{24}-\mu_{24})] \tag{34}$$

in the region bounded by the curve II and the straight line $\mu_{13} = -1$.

The analytic continuation of the dispersion integral (2) in its general form has been carried out by Mandelstam. He obtains the double integral (34) in the region $\mu_{24} < \square_1(\mu_{13})$, $\mu_{13} < -1$. The sum of this integral and the double integral for A_{an} gives the double integral (34) in the region bounded by

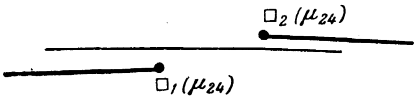

FIG. 5

the curve II. The double Mandelstam representation is therefore valid if condition (32) is satisfied. This result has been obtained earlier.[1]

For real μ_{ik} in the region $\mu_{24} < 1$ we shall interpret the function $A(\mu_{ik})$ as the limit

$$A(\mu_{13} - i\delta,\ \mu_{24} - i\delta), \qquad \delta \to +0. \tag{35}$$

With this definition, the function $A(\mu_{ik})$ is the scattering amplitude for the process

$$(p_{12},\ p_{14}) \to (p_{23},\ p_{34})$$

in the region $\mu_{13} > 1$, $\mu_{24} < -1$.

We shall show now that under condition (32) those points of the curve II for which $\mu_{13} < \mu_{13}^0$, $\mu_{24} < \mu_{24}^0$, where μ_{13}^0 and μ_{24}^0 are the tangent points of the curve II and the straight lines $\Delta_1(\mu_{24})$ and $\Delta_1(\mu_{13})$, are not singular points of the function $A(\mu_{ik})$ as defined by (35). For this purpose we write (2) in the form

$$A(\mu_{ik}) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{1}{\sqrt{K}} \ln\frac{-\xi_1}{-\xi_2}\, \frac{d\mu_{13}'}{\mu_{13}' - \mu_{13}} - \frac{2\pi i}{\pi}\int_{-\infty}^{-1} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})}, \tag{36}$$

where

$$\sqrt{K} > 0 \text{ for } \mu_{24} < \Box_1(\mu_{13}),$$

$$\sqrt{K} = i\,|\sqrt{K}| \text{ for } \mu_{24} > \Box_1(\mu_{13});$$

$$\arg(-\xi_1) = \arg(-\xi_2) = 0, \text{ if } \mu_{24} < \Box_1(\mu_{13}).$$

Curve II is not a singular curve for the first term on the right-hand side of (36). The second term is an integral along the upper branch of the cut coming from the point $\Box_1(\mu_{24})$ and along the lower branch of the cut coming from the point $\Box_2(\mu_{24})$ (see Fig. 5).

If the function $A(\mu_{ik})$ is defined according to (35) for real μ_{ik}, we can move the contour of integration into the upper half-plane in such a way that it does not go through the point $\Box_1(\mu_{24})$, i.e., this point is actually not singular. It will become clear in the following that this result holds true also in the presence of anomalies.

The point $\Box_2(\mu_{24})$ is a singular point of the second term on the right-hand side of (2) if $\Box_2(\mu_{24}) < -1$, and of $A_{an}(\mu_{ik})$ if $\mu_{24} > \mu_{24}^0$, $\Box_2(\mu_{24}) > -1$. Therefore the point $\Box_2(\mu_{24})$ is a singular point of $A(\mu_{ik})$ for $\mu_{24} > \mu_{24}^0$. For $\mu_{24} < \mu_{24}^0$ the point $\Box_2(\mu_{24}) > -1$ is not a singular point of $A(\mu_{ik})$ if $A(\mu_{ik})$ is defined according to (35), as can easily be verified.

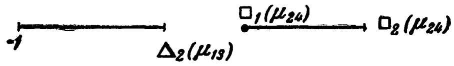

FIG. 6

Thus only those points of curve II are singular points of $A(\mu_{ik})$ which satisfy the conditions $\mu_{13} > \mu_{13}^0$, $\mu_{24} > \mu_{24}^0$.

2) Let us now consider the anomalous case. The analytic continuation of formula (29) for $A_{an}(\mu_{ik})$ will be carried out in three steps: first into the region $\Delta_1(\mu_{24}) < \mu_{24} < 1$, then into the region $-1 < \mu_{24} < \Delta_1(\mu_{24})$, and finally into the region $\mu_{24} < -1$.

We first consider the analytic continuation into the region

$$\Delta_1(\mu_{24}) < \mu_{24} < 1.$$

Here two cases can occur:

a) $$\theta_{12} + \theta_{23} < 2\pi - |\theta_{14} - \theta_{34}|, \tag{37}$$

b) $$\theta_{12} + \theta_{23} > 2\pi - |\theta_{14} - \theta_{34}|. \tag{38}$$

It follows from (14) and (15) that in the first case

$$\Delta^-(\mu_{12},\ \mu_{23}) = \Delta_2(\mu_{13}), \tag{39}$$

whereas in the second case

$$\Delta^-(\mu_{12},\ \mu_{23}) = \Delta_3(\mu_{13}). \tag{40}$$

a) Let us consider the first case. As long as $\mu_{24} > \bar{\mu}_{24}$, where $\bar{\mu}_{24}$ is the tangent point of the straight line $\Delta_2(\mu_{13})$ and the curve V (see Fig. 2), formula (29) evidently represents a function which is analytic in μ_{12} with the condition that $\sqrt{K} = i\,|\sqrt{K}|$. The singular points of $\sqrt{K}$ in the μ_{13} plane and the cuts leading away from these points are shown in Fig. 6.

For the analytic continuation into the region $\mu_{24} < \bar{\mu}_{24}$ we shall assume that

$$\mu_{24} = \tilde{\mu}_{24} - i\delta. \tag{41}$$

If $\delta > 0$ in (41), it can be easily shown that for $\mu_{24} > \bar{\mu}_{24}$ the point $\Box_1(\mu_{24})$ lies in the lower and the point $\Box_2(\mu_{24})$ in the upper half-plane of μ_{13}. For $\mu_{24} = \bar{\mu}_{24}$ the point $\Box_1(\mu_{24})$ crosses the segment $(-1, \Delta_2(\mu_{13}))$ and moves then into the upper half-plane, taking the contour along (Fig. 7). For μ_{24}

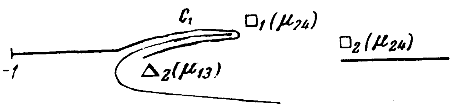

FIG. 7

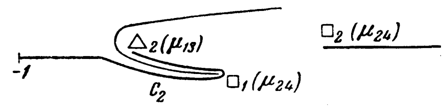

FIG. 8

$< \bar{\mu}_{24}$, the integration must therefore be carried out along the contour shown in Fig. 7.

The same thing happens if $\delta < 0$ in (41), only that now the point $\square_1(\mu_{24})$ moves from the upper into the lower half-plane (Fig. 8).

As long as $\mu_{24} > \Delta_2(\mu_{24})$, the contours C_1 and C_2 shown in Figs. 7 and 8 lie on the real axis and the integrals along them coincide. The point $\bar{\mu}_{24}$ is therefore not a singular point of $A(\mu_{ik})$.

The anomalous term in the region $\Delta_2(\mu_{24}) < \mu_{24} < 1$ has the form

$$A_{\text{an}} = -\frac{2\pi i}{\pi} \int\limits_{-1}^{\Delta^-(\mu_{12},\mu_{23})} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})} - \frac{4\pi i}{\pi}\theta(\bar{\mu}_{24} - \mu_{24}) \int\limits_{\Delta^-(\mu_{12},\mu_{23})}^{\square_1(\mu_{24})} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})}, \tag{42}$$

where $\sqrt{K} = i|\sqrt{K}|$, and $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

For $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$ the contours C_1 and C_2 move into the complex plane and the integrals along them will no longer coincide. The point $\Delta_2(\mu_{24})$ is therefore a singular point for $A(\mu_{ik})$. It follows from (41), (14), and (15) that

$$\Delta_2(\mu_{24}) = \max[\Delta^-(\mu_{12}, \mu_{14}),\ \Delta^-(\mu_{23}, \mu_{34})].$$

We note that the singular point with respect to μ_{13} for $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$ is situated in the upper half-plane of μ_{13}, if $A(\mu_{ik})$ is defined according to (35). Thus the amplitude is an analytic function of s in the upper half-plane.

b) Let us now assume that condition (40) is satisfied. In this case the singular points $\square_1(\mu_{24})$ and $\square_2(\mu_{24})$ fall on the segment $(-1, \Delta^-(\mu_{12}, \mu_{23}))$ immediately after passing the point

$$\Delta_3(\mu_{24}) = \max[\Delta^-(\mu_{12}, \mu_{14}),\ \Delta^-(\mu_{23}, \mu_{34})].$$

Considerations completely analogous to those in case a) show that the point $\Delta_3(\mu_{24})$ is a singular point for $A(\mu_{ik})$, and after passing the point $\bar{\mu}_{24}$, where $\bar{\mu}_{24}$ is the tangent point of the curve V and the straight line $\Delta_2(\mu_{13})$, the contour of integration is deformed, just as in case a). The anomalous term in the region $\Delta_2(\mu_{24}) < \mu_{24} < 1$ has the form

$$A_{\text{an}} = -\frac{2\pi i}{\pi} \int\limits_{-1}^{\Delta^-(\mu_{12},\mu_{23})} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})}, \tag{43}$$

where $\sqrt{K}$ is defined in the following way:

For $\mu_{24} > \bar{\mu}_{24}$

$$\sqrt{K} = i|\sqrt{K}|, \text{if } \mu_{13} \text{ is outside the interval } (\square_1(\mu_{24}), \square_2(\mu_{24})); \tag{44}$$

$$\sqrt{K} > 0, \text{ if } \square_2(\mu_{24}) < \mu_{13} < \square_1(\mu_{24}) \text{ and in (41) } \delta > 0; \tag{45}$$

$$\sqrt{K} < 0, \text{ if } \mu_{13} \text{ is inside the interval } (\square_1(\mu_{24}), \square_2(\mu_{24})) \text{ and in (41) } \delta < 0. \tag{46}$$

When $\mu_{24} < \bar{\mu}_{24}$ we have

$$\sqrt{K} = i|\sqrt{K}| \quad \text{for } \mu_{13} < \square_1(\mu_{24}), \tag{47}$$

$$\sqrt{K} = -i|\sqrt{K}| \quad \text{for } \mu_{13} > \square_1(\mu_{24}) \tag{48}$$

and when

$$\square_1(\mu_{24}) < \mu_{13} < \square_2(\mu_{24})$$

$\sqrt{K}$ is defined by (45) and (46). Here it turns out that only those points of the curve V are singularities of the function $A(\mu_{ik})$, defined according to (35), which satisfy the condition $\mu_{13} > \bar{\mu}_{13}$, $\mu_{24} > \bar{\mu}_{24}'$, where $\bar{\mu}_{13}$ and $\bar{\mu}_{24}'$ are the tangent points of the curve V and the straight lines $\Delta_3(\mu_{24})$ and $\Delta_2(\mu_{13})$.

As in the case considered above, $A(\mu_{ik})$ has a complex singularity in the upper half-plane as a function of μ_{13}, if $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$. However, the point

$$\Delta_2(\mu_{24}) = \min[\Delta^+(\mu_{12}, \mu_{14}),\ \Delta^+(\mu_{23}, \mu_{34})]$$

is not a singular point of $A(\mu_{ik})$ in this case. The anomalous term is in this region equal to the sum of the integral (29) and an integral along the contours C_1 or C_2, which are shown in Figs. 7 and 8.

Let us now consider the analytic continuation into the region $-1 < \mu_{24} < \Delta_1(\mu_{24})$.

a) Let us assume that condition (22) is fulfilled. In this case the curve III touches the straight line $\Delta_1(\mu_{24})$. After passing the point $\Delta_1(\mu_{24})$ the contours of integration therefore assume the position shown in Fig. 9 for the case $\delta > 0$ in (41) and in Fig. 10 for the case $\delta < 0$. The anomalous term has the form

$$A_{\text{an}}(\mu_{ik}) = -\frac{2\pi i}{\pi} \int\limits_{-1}^{\Delta^-(\mu_{12},\mu_{23})} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})} - \frac{4\pi i}{\pi} \int\limits_{\Delta^-(\mu_{12},\mu_{23})}^{\square_2(\mu_{24})} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})}, \tag{49}$$

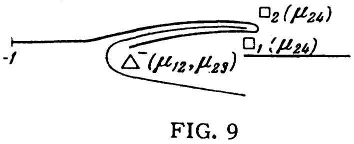

FIG. 9

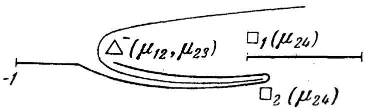

FIG. 10

where

$$\sqrt{K} = i\,|\sqrt{K}\,|, \quad \text{if} \quad \mu_{13} < \Box_1(\mu_{24});$$
$$\sqrt{K} > 0 \qquad \text{for } \mu_{13} > \Box_1(\mu_{24}),\ \delta > 0; \tag{50}$$

$$\sqrt{K} < 0 \qquad \text{for } \mu_{13} > \Box_1(\mu_{24}),\ \delta < 0. \tag{51}$$

Considerations analogous to those above show that the point $\Box_1(\mu_{24})$ is not a singularity for $A(\mu_{ik})$, defined according to (35). The point $\Delta_1(\mu_{24})$ is a singular point for $A(\mu_{ik})$ if

$$\theta_{12} + \theta_{14} > \pi, \qquad \theta_{23} + \theta_{34} > \pi, \tag{52}$$

but is not a singular point in the opposite case. This can be seen by analytic continuation of the dispersion representation in μ_{24} with respect to the masses.

b) Let us now turn to the case

$$(\mu_{12} + \mu_{14})(\mu_{23} + \mu_{34}) < 0. \tag{53}$$

In this case the straight line $\Delta_1(\mu_{24})$ touches the curve II. Considerations analogous to those above show that the anomalous term will be

$$A_{\mathrm{an}}(\mu_{ik}) = -\frac{4\pi i}{\pi}\,\theta(-1 - \Box_2(\mu_{24})) \int_{\Box_2(\mu_{24})}^{-1} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{-1}^{\Delta^-(\mu_{12},\,\mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})}, \tag{54}$$

where $\sqrt{K} = -i\,|\sqrt{K}\,|$.

As in the above-mentioned cases, it can be shown that the point $\Box_2(\mu_{24})$ is not a singular point of $A(\mu_{ik})$. The point $\Delta_1(\mu_{24})$ will evidently be a singular point, since it is in this point that the cut for $A(\mu_{ik})$ as a function of μ_{24}, coming from the points $\Delta_2(\mu_{24})$ or $\Delta_3(\mu_{24})$, ends.

Let us consider, finally, the analytic continuation into the region $\mu_{24} < -1$. It can be shown by the above-mentioned method that the anomalous term in this region is given by the formula

$$A_{\mathrm{an}}(\mu_{ik}) = -\frac{4\pi i}{\pi}\,\theta(-1 - \Box_1(\mu_{24})) \int_{\Box_1(\mu_{24})}^{-1} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{-1}^{\Delta^-(\mu_{12},\,\mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})}, \tag{55}$$

where

$$\sqrt{K} = -i\,|\sqrt{K}\,|, \quad \text{if} \quad \delta > 0 \text{ in (41)},$$
$$\sqrt{K} = i\,|\sqrt{K}\,| \qquad \text{for } \delta < 0.$$

The point $\Box_1(\mu_{24})$ is not a singular point of $A(\mu_{ik})$, as defined according to (35).

4. ANALYTIC CONTINUATION WITH RESPECT TO μ_{14} AND μ_{34}

The formulas for the anomalous term obtained in the preceding section are valid if condition (30) is satisfied. In order to determine the anomalous term for the case when the masses of all particles are large, i.e., when

$$\theta_{12} + \theta_{23} > \pi, \qquad \theta_{14} + \theta_{34} > \pi, \tag{56}$$

we must carry out an analytic continuation of the dispersion representation for $A(\mu_{ik})$ with respect to μ_{14} and μ_{34}. Condition (56) implies again the inequality (21), so that the curve I touches the straight line $\Delta_1(\mu_{13})$.

The analytic continuation with respect to μ_{14} and μ_{34} into the region $1 < \mu_{24} < \bar{\mu}^0_{24}$, where $\bar{\mu}^0_{24}$ is the tangent point of the curve I and the straight line $\Delta_1(\mu_{13})$, is carried out in the same way as the analytic continuation with respect to μ_{12} and μ_{23}. The formula for the anomalous term in this region has the form

$$A_{\mathrm{an}} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})}, \tag{57}$$

where

$$\beta = \max\,[\Delta^-(\mu_{12}, \mu_{23}),\ \Delta^-(\mu_{14}, \mu_{34})], \qquad \sqrt{K} = i\,|\sqrt{K}\,|.$$

The anomalous terms for other values of μ_{24} can be obtained by analytic continuation with respect to μ_{24}.

In the region $\mu_{24} > \bar{\mu}^0_{24}$ the anomalous term is

$$A_{\mathrm{an}} = -\frac{4\pi i}{\pi} \int_{\Box_1(\mu_{24})}^{\Delta_1(\mu_{13})} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})}, \tag{58}$$

where $\sqrt{K} = i\,|\sqrt{K}\,|$.

Therefore, only those points of the curve I are singular points of $A(\mu_{ik})$ which satisfy the condition $\mu_{24} > \bar{\mu}^0_{24}$.

In the region $\Delta_2(\mu_{24}) < \mu_{24} < 1$ the formula for the anomalous term with condition (37) has the form

$$A_{\mathrm{an}} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})} - \frac{4\pi i}{\pi}\,\theta(\bar{\mu}_{24} - \mu_{24}) \int_{\beta}^{\Box_1(\mu_{24})} \frac{d\mu'_{13}}{\sqrt{K}\,(\mu'_{13} - \mu_{13})}, \tag{59}$$

where $\sqrt{K} = i\,|\sqrt{K}\,|$, and $\bar{\mu}_{24}$ is the tangent point of the curve V and the straight line $\Delta_2(\mu_{13})$.

If the condition (38) is satisfied, the anomalous term is equal to

$$A_{an} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad \sqrt{K} = i|\sqrt{K}|. \tag{60}$$

In the region $\Delta_2(\mu_{24}) < \mu_{24} < \Delta_1(\mu_{24})$ the anomalous term A_{an} is equal to the sum of expression (60) and an integral along the contours C_1 or C_2, which are shown in Figs. 8 and 9. All considerations of the preceding section concerning the singular points $\Delta_2(\mu_{24})$ and $\Box_2(\mu_{24})$ are also valid in this case.

In the region $-1 < \mu_{24} < \Delta_1(\mu_{24})$ we have, with condition (22),

$$A_{an} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{4\pi i}{\pi} \int_{\beta}^{\Box_2(\mu_{24})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \tag{61}$$

where $\sqrt{K}$ is defined according to (50) and (51).

If condition (53) is fulfilled,

$$A_{an} = -\frac{4\pi i}{\pi} \theta(\Delta_1(\mu_{13}) - \Box_2(\mu_{24})) \int_{\Box_2(\mu_{24})}^{\Delta_1(\mu_{13})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \tag{62}$$

where $\sqrt{K} = -i|\sqrt{K}|$.

If $\mu_{24} < -1$, then

$$A_{an} = -\frac{4\pi i}{\pi} \int_{\Box_1(\mu_{24})}^{\Delta_1(\mu_{13})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \tag{63}$$

where $\sqrt{K}$ is defined according to (58).

In the case (56), as in the case (30), the curve II is not a singular curve for $A(\mu_{ik})$.

Thus, if (33) is satisfied, the anomalous additions to the dispersion representation (2) are given for $\mu_{24} > \Delta_4(\mu_{24})$ by formulas (29), (57), and (58), for $\Delta_2(\mu_{24}) < \mu_{24} < \Delta_4(\mu_{24})$ by formulas (42), (43), (59), and (60), for $-1 < \mu_{24} < \Delta_1(\mu_{24})$ by formulas (49), (54), (61), and (62), and for $\mu_{24} < -1$ by formulas (55) and (63).

In the region $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$ the function $A_{an}(\mu_{ik})$ has a complex singularity in μ_{13}, which is situated in the upper half-plane. In this region the anomalous term is given by the sum of the integrals (43) or (60) and integrals along the contours C_1 or C_2, shown in Figs. 7 and 8. In all cases the anomalous term is given by some integral of a function which is formally identical with the Mandelstam function A_{12}.

If condition (32) is satisfied, there exists a double Mandelstam representation in the form of the integral (34) over the region bounded by the curve II.

All the formulas above have been derived under the assumption that the squares of the masses of the external particles have an infinitesimal positive imaginary part. It turns out, however, that one obtains the same results if this small addition is taken to be negative. Therefore the values of the masses for which the anomalous additional terms appear are not singularities of $A(\mu_{ik})$.

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160