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THE ANALYTIC PROPERTIES OF THE TOTAL πN INTERACTION CROSS SECTION AS A FUNCTION OF VIRTUALITY

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The analytic properties of the imaginary part of the forward amplitude for elastic scattering of π mesons on nucleons as a function of virtuality (i.e., the square of the four-momentum) of the π meson are considered in perturbation theory as well as on the basis of the Jost-Lehmann-Dyson spectral representation. In virtue of the optical theorem such an analysis yields information on the analytic properties of the total πN interaction cross section as a function of virtuality. The role of the "anomalous" thresholds is discussed. The possibilities for obtaining the total cross section as a multiplicative function of the total energy in the center-of-mass system and of the virtuality are discussed.

1. A number of recent papers have been devoted to the investigation of inelastic processes at high energies in the pole approximation (see, for example,[1-7]). The assumptions restricting the pole approximation have been enumerated earlier.[4] Clearly it is highly desirable to go beyond the limitations of the pole approximation. For this purpose it is necessary, in particular, to obtain some information on the behavior of the total cross section as a function of virtuality.* A first step in this direction is the investigation of the analytic properties of the total cross section as a function of virtuality for a fixed value of the energy in the center-of-mass system. In virtue of the optical theorem, which relates the imaginary part of the forward scattering amplitude to the total cross section, the analytic properties of the total cross section are given by the analytic properties of the imaginary part of the forward scattering amplitude. With this in mind, we now turn to the consideration of the analytic properties of the imaginary part of the forward amplitude for elastic πN scattering as a function of the virtuality of the π meson.

2. Let us first determine the analytic properties of the imaginary part of the forward elastic scattering amplitude in the k^2 plane, using fourth order perturbation theory. The corresponding diagram is shown in Fig. 1. The squares of the masses of each of the particles and their four-

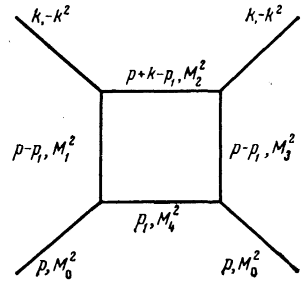

FIG. 1. Fourth order diagram in perturbation theory.

momenta are indicated. To keep the considerations general, we shall not as yet assume that $M_0 = M_1 = M_2 = M_3 = m$ and $M_4 = \mu$ (m is the nucleon mass and μ is the pion mass). The imaginary part of the amplitude for this process is written in the form

$$\operatorname{Im} f_4(0^\circ) = \frac{g^4}{8\pi^2} \tag{1}$$

$$\times \int \frac{d^4 p_1 \delta\left((p+k-p_1)^2 + M_2^2\right) \delta\left(p_1^2 + M_4^2\right) \theta\left(k_0 + p_0 - p_{10}\right) \theta\left(p_{10}\right)}{\left[(p-p_1)^2 + M_1^2\right]\left[(p-p_1)^2 + M_3^2\right]}.$$

Evaluating the integral on the right hand side of (1) and introducing the notation

$$s = -(p+k)^2, \quad \tau = -k^2/s, \quad \beta_i^2 = M_i^2/s, \tag{2}$$

we find

$$\operatorname{Im} f_4(0^\circ) = \frac{g^4}{32\pi s^2} \frac{\left[(1+\beta_4^2-\beta_2^2)^2 - 4\beta_4^2\right]^{1/2}}{\beta_1^2 - \beta_3^2}$$

$$\times \ln \frac{(a_1 + Q^{1/2})(a_3 - Q^{1/2})}{(a_1 - Q^{1/2})(a_3 + Q^{1/2})}, \tag{3}$$

where

$$a_i = 2(\beta_0^2 + \beta_4^2 - \beta_i^2) - (1 + \beta_0^2 - \tau)(1 + \beta_4^2 - \beta_2^2)$$

$$(i = 1, 3),$$

$$Q = \left[(1 + \beta_0^2 - \tau)^2 - 4\beta_0^2\right]\left[(1 + \beta_4^2 - \beta_2^2)^2 - 4\beta_4^2\right].$$

*The term "virtuality" is used to describe the square of the four-momentum of one of the incoming particles. We remark that one is most interested in the behavior of the total cross section as a function of virtuality on the real half-axis Re $k^2 > 0$ (the metric is chosen such that $k^2 = \mathbf{k}^2 - k_0^2$).[4]

The analysis of this expression is considerably simplified if we assume that $M_1 = M_3$; this assumption does not diminish the generality of our discussion in an essential manner. Then formula (3) takes the form

$$\operatorname{Im} f_4(0^\circ) = \frac{g^4}{16\pi s^2}\frac{[(1+\beta_4^2-\beta_2^2)^2-4\beta_4^2]^{1/2}}{(\tau-\tau_3)(\tau-\tau_4)},$$

$$\tau_{3,4} = [\beta_4^2+\beta_1^2-\beta_0^2+\beta_0^2(\beta_2^2+\beta_4^2)$$
$$-(\beta_1^2-\beta_4^2)(\beta_2^2-\beta_4^2)\pm iP^{1/2}]/2\beta_4^2,$$
$$P = ((1+\beta_4^2-\beta_2^2)^2-4\beta_4^2)((\beta_0+\beta_4)^2$$
$$-\beta_1^2)(\beta_1^2-(\beta_0-\beta_4)^2). \quad (4)$$

It is seen from expression (4) that the imaginary part of the forward elastic scattering amplitude has two poles in the complex τ plane. The position of these poles in the k^2 plane (s is assumed fixed and real) can be easily obtained with the help of (2) and (4).

We see that the poles are situated symmetrically with respect to the real k^2 axis. At the reaction threshold [i.e., for $s = (M_2 + M_4)^2$] they coalesce into a single point on the real axis, and with increasing s they move away from the real axis along a hyperbola in the k^2 plane with the asymptotes

$$\operatorname{Im} k^2 = \pm\frac{[(M_0+M_4)^2-M_1^2]^{1/2}[M_1^2-(M_0-M_4)^2]^{1/2}}{M_1^2+M_4^2-M_0^2}$$
$$\times[\operatorname{Re} k^2+M_1^2+M_2^2].$$

It is interesting to note that for $M_1^2 + M_4^2 - M_0^2 > 0$ (this case corresponds to the so-called "normal" thresholds[8]) these poles will always move away from the region of interest $\operatorname{Re} k^2 > 0$ as the energy s increases.[4] On the other hand, if $M_1^2 + M_4^2 - M_0^2 < 0$ (this corresponds to the "anomalous" thresholds[8]), the poles will move into the region $\operatorname{Re} k^2 > 0$ as the energy increases beyond a certain value. The distance from these poles to the region of interest can be of the same order of magnitude as the distance from this region to the point $k^2 = -\mu^2$ corresponding to the pole of the propagation function of the π meson.

We shall not discuss this problem in any detail, since the presence of poles in the complex k^2 plane in perturbation theory and their role in the extrapolation procedure proposed by Chew and Low[1] have already been discussed by Ascoli.[9] We emphasize only that the above discussion shows that these poles correspond to poles of the total cross section as a function of the virtuality. For an illustration we give the location of the poles of the πN scattering cross section as a function of virtuality in fourth order perturbation theory (i.e., for $M_0 = M_1 = M_2 = M_3 = m$, $M_4 = \mu$). These poles are situated at the following points of the k^2 plane:

$$k_{3,4}^2 = -\tfrac{1}{2}\{s+3m^2-\mu^2$$
$$\pm i(4m^2/\mu^2-1)^{1/2}[(s-m^2+\mu^2)^2-4s\mu^2]^{1/2}\}.$$

At the reaction threshold $s = (m+\mu)^2$ we have

$$k_3^2 = k_4^2 = -m(2m+\mu) \leqslant -\mu^2.$$

3. Let us now consider the analytic properties of the total cross section in the k^2 plane on the basis of the spectral representation of Jost, Lehmann, and Dyson.[10] It was shown in the paper of Lehmann[10] [formula (23)] that the imaginary part of the forward elastic scattering amplitude as a function of s and k^2 can be written in the form [with $s \geq (m+\mu)^2$ in the case of πN scattering]

$$\operatorname{Im} T(0^\circ) = \frac{1}{16\pi K_0^2}\int du_{0i}\,u_i\,du_i\,d\varkappa_{0i}^2$$
$$\times\int_0^{2\pi} d\alpha\int_0^{2\pi} d\chi\int_{-1}^{1} d(\cos\varphi_1)\int_{-1}^{1} d(\cos\varphi_2)$$
$$\times\frac{\Phi(u_{0i}, u_i^2, \varkappa_{0i}^2, \cos\alpha\sin\varphi_1\sin\varphi_2+\cos\varphi_1\cos\varphi_2, s)}{[X_1-\sin\varphi_1\cos\chi][X_2-\sin\varphi_2\cos(\chi-\alpha)]}, \quad (5)$$

where

$$X_i = \{K_0^2+u_i^2+\varkappa_{0i}^2-[(m^2+k^2)/2s^{1/2}+u_{0i}]^2\}/2K_0u_i$$
$$(i = 1, 2),$$
$$K_0^2 = [(s+m^2+k^2)^2-4m^2s]/4s,$$

and the function Φ is arbitrary in the region

$$0\leqslant u_i\leqslant s^{1/2}/2, \quad -s^{1/2}/2+u_i\leqslant u_{0i}\leqslant s^{1/2}/2-u_i,$$
$$\varkappa_{0i}\geqslant\max\{0;\ m_1-\sqrt{(s^{1/2}/2+u_{0i})^2-u_i^2},$$
$$m_2-\sqrt{(s^{1/2}/2-u_{0i})^2-u_i^2}\} \quad (6)$$

($m_1 = 3\mu$ and $m_2 = m+\mu$ in the case of πN scattering); outside this region the function Φ vanishes. From (5) we can derive the analytic properties with respect to the virtuality k^2. Indeed, singularities can occur only at points k^2 for which the denominator in (5) vanishes, since the function Φ is independent of k^2. If the denominator is to vanish, at least one of the conditions

$$-1\leqslant X_1\leqslant 1, \quad -1\leqslant X_2\leqslant 1. \quad (7)$$

must be satisfied. These conditions are identical in the sense that they define the same region of analyticity in the k^2 plane owing to the symmetry of the conditions (6) with respect to the index i.

We shall therefore leave out this index in the following.

Using formula (6), we can express the virtuality k^2 as a function of X, u, u_0, κ_0, and s. In addition to the variables (2) it is convenient to introduce also the variables

$$x=\left[\left(\frac{1}{2}+\frac{u_0}{s^{1/2}}\right)^2-\frac{u^2}{s}\right]^{1/2},$$

$$y=\left[\left(\frac{1}{2}-\frac{u_0}{s^{1/2}}\right)^2-\frac{u^2}{s}\right]^{1/2},$$

$$\varkappa^2=\varkappa_0^2/s,\qquad z=1-X^2,\quad \beta^2=m^2/s. \tag{8}$$

Then

$$\tau=1+\beta^2-\frac{2(y^2+\beta^2-\varkappa^2)(1-x^2+y^2)}{4y^2+(1-(x+y)^2)(1-(x-y)^2)z}$$
$$\pm\frac{2\sqrt{1-z}\,[(1-(x+y)^2)(1-(x-y)^2)]^{1/2}}{4y^2+[1-(x+y)^2][1-(x-y)^2]z}$$
$$\times\{(y^2+\beta^2-\varkappa^2)^2-4\beta^2y^2$$
$$-[1-(x+y)^2][1-(x-y)^2]\beta^2z\}^{1/2}. \tag{9}$$

As we have shown above [formulas (6) and (7)], the imaginary part of the forward scattering amplitude can have singularities only in the following region of the variables:

$$x\geqslant 0,\qquad y\geqslant 0,\qquad x+y\leqslant 1,$$
$$\varkappa\geqslant\max\{0,\ \beta_1-x;\ \beta_2-y\},\quad 0\leqslant z\leqslant 1. \tag{10}$$

The expression under the root sign in (9) can be positive or negative. If it is negative, the singularities will appear in the complex k^2 plane. Then

$$\tau=\tau_1\pm i\tau_2, \tag{11}$$

where

$$\tau_1=1+\beta^2-\frac{2(y^2+\beta^2-\varkappa^2)(1-x^2+y^2)}{4y^2+[1-(x+y)^2][1-(x-y)^2]z}$$
$$\equiv 1+\beta^2-2b,$$
$$\tau_2^2=4(1-z)[1-(x+y)^2][1-(x-y)^2]$$
$$\times\{\beta^2(4y^2+[1-(x+y)^2]$$
$$\times[1-(x-y)^2]z)-(y^2+\beta^2-\varkappa^2)^2\}$$
$$\times\{4y^2+[1-(x+y)^2][1-(x-y)^2]z\}^{-2}.$$

It is seen from (11) that the singularities are situated symmetrically with respect to the real τ axis.

In order to determine the boundaries of the region of analyticity, we require first of all the minimal value of τ_1 for which the variables x, y, κ, and z are in the region (10). An elementary calculation yields

$$\tau_{1\,min}=1-\sqrt{3}\,\beta_2/y_0$$
$$+9(\beta_2^2-\beta^2)/8y_0^2+(\beta_2^2+\beta^2)/2, \tag{12}$$

where

$$y_0=\cos\frac{\pi+\varphi}{3},\qquad \cos\varphi=\frac{3\sqrt{3}}{2}\frac{\beta_2^2-\beta^2}{\beta_2}.$$

Hence the imaginary part of the forward elastic scattering amplitude cannot have singularities for $\tau_1<\tau_{1\,min}$ if $\tau_2^2>0$.

Expression (12) is simplified considerably in the case of πN scattering with $s\geq(m+\mu)^2$. Noting that in this case $(\beta_2^2-\beta^2)/\beta_2\ll 1$, we obtain

$$\tau_{1\,min}\approx-(2\beta^2-\beta_2^2)/2(\beta_2^2-\beta^2)+(3\beta^2-\beta_2^2)/2s. \tag{13}$$

From this we find

$$(\mathrm{Re}\,k^2)_{max}$$
$$\approx s(2m^2-m_2^2)/2(m_2^2-m^2)-\tfrac{1}{2}(3m^2-m_2^2), \tag{14}$$

i.e., asymptotically (for $s\to\infty$) the boundary of the region of analyticity in the k^2 plane moves toward the region of positive values of Re k^2, the shift being proportional to s.

Let us now determine the maximal and minimal values of τ_2^2 for a given fixed τ_1. We notice, first of all, that

$$\tau_2^2=4(b-r)(\beta^2/r-b),$$
$$r=(y^2+\beta^2-\varkappa^2)/(1-x^2+y^2). \tag{15}$$

Thus, for a given τ_1 (i.e., for a given b), τ_2^2 is a function of the single variable r. The signs of b and r are identical and are determined by the sign of the expression $y^2+\beta^2-\kappa^2$.

It is seen from (15) that it is sufficient for finding the maximal value of τ_2^2 for given b to determine the minimal value of r for the same b [taking account of the restrictions on the region of the variables (10)] and substitute it in (15). In this way we obtain for $b\geq 0$

$$\frac{\tau_{2max}^2}{(1-q^2)^2}+\frac{(1+\beta^2-\tau_1)^2}{(1+q^2)^2}=\frac{\beta^2}{q^2}, \tag{16}$$

where $q=(\beta_2^2-\beta^2)/2\beta_2$. For πN scattering and $s\geq(m+\mu)^2$, we have $q\ll 1$, and (16) takes the form

$$\tau_{2max}^2+(1+\beta^2-\tau_1)^2=4\beta_2^2\beta^2/(\beta_2^2-\beta^2)^2. \tag{17}$$

Thus, for $\tau_{1\,min}\leq\tau_1\leq 1+\beta^2$, singular points can only be found inside the ellipse described by Eq. (16).

To find the minimal value of τ_2^2 for given τ_1, we must determine the maximal value of r for given b. The resulting expressions are very complicated and are, therefore, given in the Appendix. Here we quote only the expression for the minimal value of τ_2^2 as a function of τ_1, with $\tau_{1\,min} \le \tau_1 \le \beta_1^2$ and the following restrictions on s:

$$s \gg (m_2 - m_1)^2, \quad s \geqslant (m + m_1)^2. \tag{18}$$

In this case we have

$$\tau_{2min}^2 = (1 - \tau_1)^2 \left(\frac{2\beta^2}{\beta_2^2 - (1 - \tau_1)(\beta_2^2 - \beta^2) + \beta_2 [\beta_2^2 - 2(1 - \tau_1)(\beta_2^2 - \beta^2)]^{1/2}} - 1 \right). \tag{19}$$

Equation (19) defines the boundary of the region of analyticity closest to the real axis. It is easily seen from (17) and (19) that the boundaries of the region of analyticity in the k^2 plane move asymptotically (for $s \to \infty$) away from the real axis, the shift being proportional to s.

The shape of the region of analyticity in the k^2 plane for πN scattering at the reaction threshold [$s = (m + \mu)^2$] is shown in Fig. 2 [condition (18) is not satisfied here, and we must therefore use the formulas given in the Appendix]. It is interesting to note that at the reaction threshold the cross section has singularities on the real axis close to the pole of the π meson propagation function at $k^2 = \mu^2$. Indeed, the closest singularity (point A in Figs. 2 and 3) for $s = (m + \mu)^2$ is situated at the point $k^2 = -3\mu^2(2m + \mu)/(2m - \mu)$.

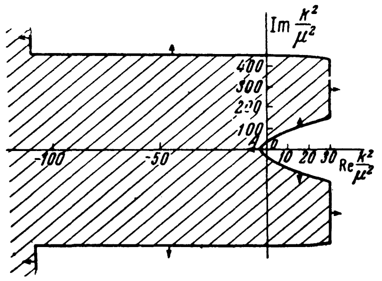

FIG. 2. Region of analyticity of the total πN interaction cross section in the k^2 plane with $s = (m + \mu)^2$ (not shaded). The scale of the imaginary axis is chosen ten times smaller than that of the real axis.

The directions in which the boundaries of the region of analyticity and the point A move as s is increased are indicated in Fig. 2 by arrows. Figure 3 shows in a larger scale the shape of the boundary of the region of analyticity near the point $k^2 = -\mu^2$ for $s = (m + \mu)^2$. For all s satisfying condition (18) [for πN scattering this means $s \ge (m + 3\mu)^2$] the shape of the boundary near the point $k^2 = -\mu^2$ changes (see Fig. 4), while the other boundaries keep their previous shape, as shown in Fig. 2. The boundary point Re $k^2 = -(3\mu)^2$ is not shifted any more as the energy changes. The other boundary points move along the directions indicated by the arrows as the energy s is increased. As already noted, this shift is asymptotically proportional to s.

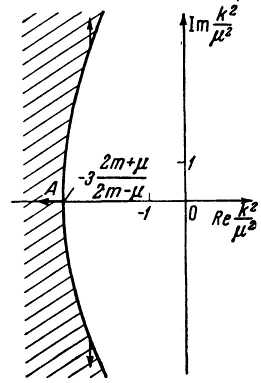

FIG. 3. Shape of the boundaries of the region of analyticity near the point $k^2 = -\mu^2$ for $s = (m + \mu)^2$.

4. From the results obtained above we can draw a number of conclusions. First of all, we note the following circumstance. We found in fourth order perturbation theory that the singularities of the πN interaction cross section as functions of the virtuality k^2 can occur only in the region Re $k^2 < 0$ and are considerably farther removed from the region of interest Re $k^2 > 0$ (see the footnote above) than the pole at $k^2 = -\mu^2$. The spectral representation then indicates the possibility that singularities may also appear in the region Re $k^2 > 0$ of the complex k^2 plane; near the reaction threshold these singularities may be situated at a distance from the region of interest which is comparable with the distance from the pole of the π meson propagation function to that region.

The occurrence of these singularities may have a great effect on the possibility of carrying out the extrapolation procedure proposed by Chew and Low[4] not only in the "anomalous" cases but also in the case of an inelastic NN interaction with formation of a single π meson. Since, however,

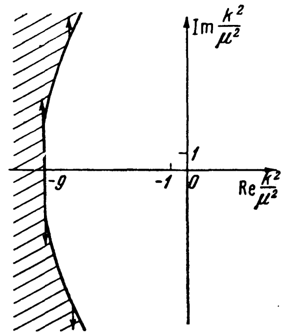

FIG. 4. Shape of the boundaries of the region of analyticity near the point $k^2 = -\mu^2$ for $s > (m + 3\mu)^2$.

the singularities in the complex plane move away from the real axis as s is increased, it appears to be probable that their effect will become smaller as the energy increases.

Furthermore, it follows immediately from the analytic properties of the total cross section as a function of the virtuality, as obtained above, that the total cross section cannot be a multiplicative function of s and k^2 for arbitrary values of these variables. Indeed, if the function $\sigma(s, k^2)$ were multiplicative, $\sigma(s, k^2) = f_1(s) f_2(k^2)$, this would imply that the position of the poles in the k^2 plane is independent of the value of s. We have shown above that this is not the case either in perturbation theory or on the basis of the spectral representation.

However, sometimes one is interested in a multiplicative total cross section on the real k^2 axis for $0 < k^2 \lesssim \mu^2$ (this problem has been discussed earlier[5]). The requirement that the total cross section be multiplicative on a small segment of the real axis and at the same time have the analytic properties obtained above imposes certain restrictions on the form of the total cross section as a function of s and k^2. This problem will be investigated separately.

Ascoli[11] has recently obtained the analytic properties of the amplitude for processes with an arbitrary number of outgoing lines with respect to two transferred momenta. If, in particular, we consider such amplitudes in the one-meson approximation, we expect that the region of their analyticity with respect to one of the transferred momenta, the other one remaining fixed, is at most as large as the region of analyticity of the cross section as a function of the virtuality (the virtuality of the π meson coincides here with the transferred momentum). The region of analyticity that we have obtained is considerably larger than the region of analyticity in the work of Ascoli,[11] since we are interested only in the one-meson term in the processes considered by Ascoli.

In conclusion I take this opportunity to thank D. S. Chernavskii for his steadfast interest in this work and for a discussion of the results.

APPENDIX

We give here the formulas for the minimal value of τ_2^2 for given τ_1.

1) For $b \geq b_0 = [1 + (\beta_2 - \beta_1)^2][\beta_2^2 + \beta^2 - 2\beta_2\beta_1] \times 4^{-1}(\beta - \beta_1)^{-2}$:

$$\tau^2_{2min} = 4\,[b - R_0]\left[\frac{\beta^2}{R_0} - b\right];$$

$$R_0 = \frac{2\beta_2(y_1 - q)}{1 + y_1^2}, \qquad y_1 = \frac{p}{3} - \frac{2}{3}\,|p^2 - 3|^{1/2}\cos\frac{\pi + \varphi}{3},$$

$$\cos\varphi = \frac{|2p^3 - 9p + 27q|}{2\,|p^2 - 3|^{3/2}}, \qquad p = \frac{2b}{\beta_2} + q. \qquad \text{(A.1)}$$

2) For $b \leq b_0$:

$$\tau^2_{2min} = 4\,[b - R_1]\,[\beta^2/R_1 - b];$$

$$R_1 = 2\beta_2(y_2 - q)/[1 - (\beta_2 - \beta_1)^2 + 2(\beta_2 - \beta_1)y_2],$$

$$y_2 = \frac{1}{4}\{\beta_2[1 - (\beta_2 - \beta_1)^2] - (\beta_2 - \beta_1)(\beta_2^2 - \beta^2)$$
$$+ ([\beta_2(1 - (\beta_2 - \beta_1)^2) + (\beta_2 - \beta_1)(\beta_2^2 - \beta^2)]^2 - 4b(\beta_2^2 - \beta^2)$$
$$\times [1 - (\beta_2 - \beta_1)^2])^{1/2}\}\,[b - \beta_2(\beta_2 - \beta_1)]^{-1}. \qquad \text{(A.2)}$$

If conditions (18) are satisfied, (A.1) and (A.2) go over into formula (19).

3) If s does not satisfy condition (18), the coordinate of the point A (see Fig. 2) is determined by the expression

$$k_A^2 = -[s\sqrt{s} - (m_2 + m_1)s$$
$$- (m^2 - 2m_1m_2)\sqrt{s} + m^2(m_2 - m_1)]$$
$$\times (\sqrt{s} + m_2 - m_1)^{-1}. \qquad \text{(A.3)}$$

Formula (A.3) agrees with formula (4.5) of the paper of Todorov,[12] in which the boundaries of the region of analyticity on the real k^2 axis were determined.

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