# *A MECHANISM FOR ABSORPTION OF ENERGY BY ANISOTROPIC BODIES*

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A body located in a varying external field can execute periodic motion by absorbing the field energy and dissipating it in the viscous medium, if the magnetic susceptibility or electric polarizability of the body are anisotropic. The absorption coefficient has a resonant dependence on the frequency of the external field. In particular, only periodic oscillations of nuclei of a certain size are possible for the liquid—solid transition.

THE purpose of this research was to ascertain whether electromagnetic energy at audio frequencies (classical effect) can be absorbed, as a result of anisotropy of the magnetic susceptibility (or electric polarizability), by macroscopic bodies immersed in a liquid. Such bodies acquire in an external magnetic field a magnetic (or electric) moment of direction, in general, different than the field; under certain conditions this should lead to periodic motion of the body and to a corresponding absorption of energy. To be specific, we shall refer henceforth to the magnetic moment of an anisotropic nucleus of solid phase floating in a liquid. Let the nucleus be spherical with principal moments of inertia $I_x = I_y = I_z = I$, let the components of the magnetic susceptibility tensor be $\chi_x = \chi_y \neq \chi_z$, and let the external magnetic field be $F$.

We introduce two rectangular coordinate frames, one moving with axes xyz aligned with the principal inertia axes of the nucleus, and one stationary $\xi\eta\zeta$, with the $\zeta$ axis in the direction of the external field. The cosines of the angles between the direction of the external field and the axes of the moving frame will be designated $\gamma_1$, $\gamma_2$, and $\gamma_3$. They can be expressed in terms of the Euler angles $\theta$, $\psi$, and $\varphi$:

$$\gamma_1 = \sin\varphi\sin\theta;\quad \gamma_2 = \cos\varphi\sin\theta;\ \gamma_3 = \cos\theta. \tag{1}$$

The equation describing the rotary motion of the nucleus has the form

$$d\mathbf{L}/dt = [\mathbf{MF}] - \beta\boldsymbol{\omega}, \tag{2}^*$$

where $\mathbf{L}$ is the angular momentum of the nucleus, $\boldsymbol{\omega}$ is the angular velocity of rotation, $M_i = \chi_i F_i$ is the magnetic moment induced in the field $\mathbf{F}$, and $\beta$ is the moment of the force of resistance to rotation in a viscous liquid.

*$[\mathbf{MF}] = \mathbf{M}\times\mathbf{F}$.

We project (2) on the axes of the moving coordinate system:

$$Id\omega_x/dt = -\Delta\chi F^2\gamma_2\gamma_3 - \beta\omega_x,\quad Id\omega_y/dt = \Delta\chi F^2\gamma_1\gamma_3 - \beta\omega_y,$$
$$Id\omega_z/dt = -\beta\omega_z. \tag{3}$$

It is quite obvious that undamped motion of the nucleus is possible in the presence of an anisotropy $\Delta\chi = \chi_z - \chi_x \neq 0$.

Euler's dynamic equations (3) must be solved simultaneously with Euler's kinematic equations

$$\omega_x = \dot\psi\gamma_1 + \dot\theta\cos\varphi,\quad \omega_y = \dot\psi\gamma_2 - \dot\theta\sin\varphi,$$
$$\omega_z = \dot\psi\gamma_3 + \dot\varphi. \tag{4}$$

It is also useful to employ the fact that the axes $\xi$, $\eta$, and $\zeta$ are stationary:

$$d\gamma_1/dt = \gamma_2\omega_z - \gamma_3\omega_y;\quad d\gamma_2/dt = \gamma_3\omega_x - \gamma_1\omega_z;$$
$$d\gamma_3/dt = \gamma_1\omega_y - \gamma_2\omega_x. \tag{5}$$

To ascertain the energy absorption, we are interested in the undamped solutions of the system (3)—(5). The last equation of (3) has only damped solutions $\omega_z = C\exp\{-\beta t/I\}$. Therefore, for sufficiently large $t$, we can assume

$$\omega_z = 0. \tag{6}$$

Multiplying the first two equations of (3) by $\gamma_1$ and $\gamma_2$ respectively, adding them, and using (5) we obtain $\gamma_1\omega_x + \gamma_2\omega_y = C_1\exp\{-\beta t/I\}$. For the reasons indicated above, we put

$$\gamma_1\omega_x + \gamma_2\omega_y = 0. \tag{7}$$

The physically obvious conditions (6) and (7) signify that the external field cannot cause the nucleus to rotate in either the plane perpendicular to the external field or in the plane xy where the susceptibility is symmetrical.

Substitution of (4) in (7) leads to the following solution of the problem of periodic motion of the anisotropic nucleus:

$$\psi = \text{const}, \quad \varphi = \text{const}, \tag{8}$$

and the nutation angle $\theta$ is a solution of the differential equation

$$I\ddot{\theta} + \beta\dot{\theta} + \tfrac{1}{2}\Delta\chi F^2 \sin 2\theta = 0. \tag{9}$$

Assuming that the external field is a monochromatic wave $F = F_0 \cos pt$ and expanding $\sin 2\theta$ we obtain

$$\ddot{\theta} + 2\varkappa\dot{\theta} + \omega_0^2(1 + \cos 2pt)\left(1 - \tfrac{2}{3}\theta^2\right)\theta = 0,$$
$$\varkappa = \beta/2I, \ \omega_0^2 = F_0^2\Delta\chi/2I. \tag{10}$$

All the coefficients of (10) can be expressed in terms of the parameters of the substance and the radius R of the nucleus. The moment of the forces of resistance to the rotation of the sphere is [1] $\beta = 8\pi R^3\lambda$, where $\lambda$ is the coefficient of viscosity of the liquid; the anisotropy of the susceptibility of the nucleus is $\Delta\chi = (\Delta\chi)_m 4\pi R^3\rho/3$, where $(\Delta\chi)_m$ is the anisotropy of the specific susceptibility (polarizability) of the solid phase, $\rho$ is the density, and the principal moment of inertia of the sphere is $I = 8\pi\rho R^5/15$. Then

$$\varkappa = 7{,}5\lambda/\rho R^2, \ \omega_0^2 = 1{,}25\,(\Delta\chi)_m F_0^2 R^{-2}. \tag{11}$$

Equation (10) takes account also of the nonlinear terms. The solution of the linear differential equations contains constant factors determined by the boundary conditions. Account of the nonlinearity enables us to find the characteristics of the steady-state motion independently of the initial conditions, which is precisely the purpose of our investigation.

Assuming the nonlinearity small, we rewrite (10) in the form

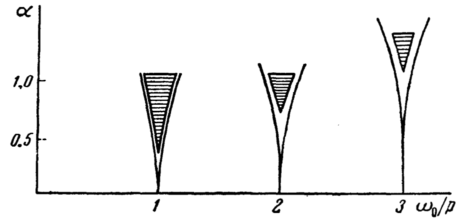
$$\ddot{\theta} + 2\varkappa\dot{\theta} + \omega_0^2(1 + \alpha\cos 2pt)\theta + V(t, \theta) = 0,$$
$$V(t, \theta) = -\tfrac{2}{3}\omega_0^2(1 + \alpha\cos 2pt)\theta^3. \tag{12}$$

For convenience we have introduced in (12) the parameter $\alpha = 1$. In the linear approximation we have

$$\ddot{\theta} + 2\varkappa\dot{\theta} + \omega_0^2(1 + \alpha\cos 2pt)\theta = 0. \tag{13}$$

Equations similar to (12) and (13) are encountered in various problems where the parameters of the system have periodically variable parameters, as in the case of parametric resonance; these have been treated in an extensive literature.[2-6]

We consider the existence of periodic solutions of (13) first, after which we take account of the nonlinearity [Eq. (12)]. Periodic solutions of (13) are not obtained for all values of the coefficients




$\kappa$ and $\omega_0$, i.e., according to (11), not for all values of R. Let us investigate the dimensions R of nuclei capable of executing periodic motions in a field of a given frequency p. The figure, plotted in coordinates $\alpha$ and $\omega_0/p$, shows the instability and stability regions of the solutions of the Mathieu equation, to which Eq. (13) reduces when $\kappa = 0$. When friction is taken into account, the instability regions shrink and shift upward, the shift increasing with increasing $\omega_0$ (for given p). The instability regions for $\kappa \neq 0$ are shown shaded in the figure.

In our problem the periodic solutions of (13) are determined by the intersection of the line $\alpha = 1$ with the boundaries of the shaded regions. The number of intersections is finite, since the boundaries of the instability regions of the solutions shift upward with increasing $\omega_0/p$. The fact that the abscissas of these points have a maximum abscissa denotes the existence of a minimum dimension $R_{min}$ of the nucleus capable of executing periodic oscillations in the external field. It is likewise obvious that there exists a value $R_{max}$, determined by the abscissa of $(\omega_0/p)_{min}$ in the figure. Thus, periodic oscillations and absorption of energy from a specified external field $F = F_0 \cos pt$ by the nuclei are possible only if the nuclei have certain resonant dimensions $R_i$ ($i = 1, 2, \ldots, n$), with $R_{max} \geq R_i \geq R_{min}$.

Mathieu equations [when $\alpha = 1$ in (13)], cannot be solved analytically, but the determination of the numerical values of the coefficients $\omega_0$ and $\kappa$, for which (13) has periodic solutions, is in principle a simple although cumbersome problem.* After compiling such a table, it is necessary to calculate the resonant nuclear radii $R_i$ for the specific substance [$(\Delta\chi)_m$, $\lambda$ in (11)] and for the specific experimental conditions (amplitude $F_0$ and frequencies $p_1$, $p_2$, . . . , of the external field).

Turning now to Eq. (12), let us investigate the dependence of the amplitude of the nucleus oscillations at the frequency of the external field. Let the coefficients (11) of Eq. (12) be such that the periodic oscillations of the nucleus correspond to

*This effort is justified if suitable experiments are set up.

the first instability region in the figure. Since small nonlinearity is assumed, we seek an approximate solution of (12) in the form

$$\theta = A_1 \sin pt + B_1 \cos pt. \tag{14}$$

Substituting (14) in (13) and equating the coefficients of sin pt and cos pt to zero we obtain

$$A_1(\omega_0^2 - p^2 - \omega_0^2/2) - \tfrac{1}{2}\omega_0^2 A_1 A^2 - 2\varkappa p B_1 + \tfrac{1}{3}\omega_0^2 A_1^3 = 0,$$

$$B_1(\omega_0^2 - p^2 + \omega_0^2/2) - \tfrac{1}{2}\omega_0^2 B_1 A^2 + 2\varkappa p A_1 - \tfrac{1}{3}\omega_0^2 B_1^3 = 0. \tag{15}$$

In (15) we have introduced the oscillation amplitude $A = (A_1^2 + B_1^2)^{1/2}$.

To make the results clear, let us assume that the cubic terms in (15) are small and let us neglect the last terms in the left halves of (15).* Then the vanishing of the determinant of (15) leads to the following frequency dependence of the square of the oscillation amplitude:

$$A^2 = 2\omega_0^{-2}[\omega_0^2 - p^2 \pm (\omega_0^4/4 - 4\varkappa^2 p^2)^{1/2}]. \tag{16}$$

The nucleus will execute periodic oscillations only if

$$\omega_0^2 > 4\varkappa p, \tag{17}$$

i.e., the attenuation due to viscosity must not be too large.† Periodic oscillations are possible at external-field frequencies for which $A^2 > 0$. The condition $A^2 = 0$ determines the boundaries of the resonant frequency band.

Thus, the solution of (13) assumes the form

$$\theta = A \sin(pt - \delta), \quad \operatorname{tg} \delta = 2\varkappa p / [\omega_0^2/2 \pm (\omega_0^4/4 - 4\varkappa^2 p^2)^{1/2}],$$
$$A = \{2\omega_0^{-2}[\omega_0^2 - p^2 \pm (\omega_0^4/4 - 4\varkappa^2 p^2)^{1/2}]\}^{1/2}. \tag{18‡}$$

The absorption coefficient $\gamma$ is expressed in terms of the density of the dissipative function $F/V = \beta\dot{\theta}^2/2V$ as

$$\gamma = 8\pi |F| / pF_0^2 V.$$

For a comparison with experimental data, it is necessary to average over the period of variation of the angle of revolution $\theta$. As a result we obtain

$$\gamma = \frac{4\pi\beta p}{F_0^2\omega_0^2 V}\left[\omega_0^2 - p^2 \pm \left(\frac{\omega_0^4}{4} - 4\varkappa^2 p^2\right)^{1/2}\right]. \tag{19}$$

As can be seen from (19), for nuclei of given dimension, the dependence of the coefficient of absorption on the frequency of the external field is of resonant character.

At each instant of time t of a real liquid-solid transition the nuclei have a certain distribution over the dimensions R. The distribution function N(R, t) can evidently be determined with the aid of the kinetic equation in the particle dimension space. We can then obtain (with account of the remarks made in the last two footnotes) the frequency dependence of the absorption coefficient and of the susceptibility (polarizability), suitable for direct comparison with experiments.

The energy-absorption mechanism considered above can be used to investigate the kinetics of the liquid-solid transition of magnetically-isotropic or electrically-isotropic bodies, the number of which is quite large. D. N. Astrov and A. V. Voronel' of the Laboratory of the Institute of Physico-Technical and Radio Measurements have observed electromagnetic absorption at acoustic frequencies in the melting and crystallization of benzene. This absorption was observed only during the phase transition process and not in the solid and liquid phases.

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*A more exact solution of (15) and an account of the higher harmonics of the nucleus [in addition to (14)] can be obtained by successive approximations.

†The criterion (17) agrees poorly with the condition that the Reynolds number be small, i.e., at such high frequencies the moment of the friction forces is proportional, generally speaking, to the angular velocity raised to a power higher than the first. The subsequent calculation is therefore only qualitative. The disparity indicated occurs only for resonances at the overtones of the external field (parametric resonance) and obviously not at undertones of the field (resonance of the n-th kind).

‡tg = tan.

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