THEORY OF SCATTERING OF HIGH-ENERGY PHOTONS BY PHOTONS

S. S. SANNIKOV

Physico-Technical Institute, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor February 17, 1961, resubmitted April 24, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 467-477 (August, 1961)

Scattering of high-energy photons by photons is investigated by the dispersion-relation technique. The cross sections for zero and small angle scattering and the total scattering cross section are determined.

THE dispersion relation technique [1-3] is being used more widely for the study of various processes involving the scattering of particles. In the present article, this technique is applied to the investigation of photon-photon scattering in the high energy region.* The dispersion relation technique considerably simplifies the calculation of the scattering amplitude and cross section and enables one to obtain the integral cross section, which has not been calculated until now.

First, from the unitary condition we shall find an expression for the imaginary part of the scattering amplitude. Then, after investigating the symmetry properties and analytic properties of the total amplitude, we shall find the real part with the aid of the dispersion relations. The differential cross section for photon-photon scattering is calculated for scattering angles $\theta \gg m/\omega$ (m is the electron mass, ω is the c.m.s. photon frequency) and for zero angle. In addition to the main terms containing the fourth degree of the logarithm, we shall calculate terms containing the third degree of the logarithm.

1. The dispersion relations can be used to investigate the analytic properties of the photon-photon invariant scattering amplitude A which is related to the matrix element M in the following way:

$$M = \tfrac{1}{4}(2\pi)^4(\omega_1\omega_2\omega_3\omega_4)^{-1/2} A\delta(k_1 + k_2 - k_3 - k_4),$$

where ω_i and k_i are the frequencies and the wave vectors of the photons.

The amplitude A can be represented in the form

$$A = A_s + A_c + A_a + A_{s,e} + A_{c,e} + A_{a,e},$$

where the partial amplitudes A_s, A_c, and A_a correspond to the scattering processes

$$(s)\quad (k_1, e_1) + (k_2, e_2) \to (k_3, e_3) + (k_4, e_4),$$
$$(c)\quad (k_1, e_1) + (-k_4, e_4) \to (k_3, e_3) + (-k_2, e_2),$$
$$(a)\quad (k_1, e_1) + (-k_3, e_3) \to (-k_2, e_2) + (k_4, e_4).$$

The partial amplitudes $A_{s,e}$, $A_{c,e}$, and $A_{a,e}$ correspond to exchange scattering processes with the photons in the final states (k_3, e_3), (k_4, e_4) and are obtained from A_s, A_c, and A_a by means of the interchange

$$(k_3, e_3) \leftrightarrow (k_4, e_4). \tag{1}$$

The amplitudes A_c and A_a can be obtained from A_s by means of the interchange

$$A_s \to A_c \text{ for } (k_2, e_2) \leftrightarrow (-k_4, e_4),$$
$$A_s \to A_a \text{ for } (k_2, e_2) \leftrightarrow (-k_3, e_3), \tag{2}$$

and it is therefore sufficient to consider only the amplitude A_s.

In order to determine the imaginary part of this amplitude, we shall start from the unitary condition for the scattering matrix S:

$$S^+S = SS^+ = 1.$$

Setting S = 1 + iT, we obtain

$$T - T^+ = iT^+T. \tag{3}$$

The amplitude A_s is related to the matrix element of the operator T in the following way:

$$i\langle k_3e_3, k_4e_4 | T | k_1e_1, k_2e_2\rangle = \tfrac{1}{4}(2\pi)^4(\omega_1\omega_2\omega_3\omega_4)^{-1/2}A_s\delta(k_1 + k_2 - k_3 - k_4).$$

It follows from (3) that

$$\mathrm{Im}\,\langle k_3e_3, k_4e_4 | T | k_1e_1, k_2e_2\rangle = \tfrac{1}{2}\sum_n \langle k_3e_3, k_4e_4 | T^+ | n\rangle\langle n | T | k_1e_1, k_2e_2\rangle. \tag{4}$$

If from all the intermediate states in (4) we retain only the states of a free electron-positron

*Photon-photon scattering was first investigated by Euler[4] in the low-energy region and by Akhiezer[5] in the high-energy region. Karplus and Neuman investigated photon-photon scattering with the aid of invariant quantum electrodynamics.[6]

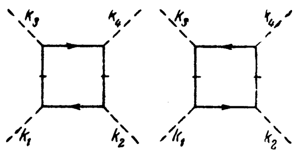

FIG. 1

pair, we obtain the following expression for the imaginary part of the amplitude A_s in the first nonvanishing approximation of perturbation theory (in the order e^4):

$$\operatorname{Im} A_s = \frac{(2\pi)^4}{2} \sum_{\substack{p_1 s_1 \\ -p_2 s_2 \\ (p_1+p_2=k_1+k_2)}} \frac{m^2}{p_1^0 p_2^0} A^+ (k_3 e_3, k_4 e_4, p_1 s_1, -p_2 s_2)$$

$$\times A (p_1 s_1, -p_2 s_2, k_1 e_1, k_2 e_2). \qquad (5)$$

In the first-order approximation of perturbation theory (in the order e^2), the expressions occurring in (5) for the amplitudes of the creation and annihilation of a pair in which two free photons take part are well known:

$$A (p_1 s_1, -p_2 s_2, k_1 e_1, k_2 e_2) = ie^2 \bar{u} (p_1) \left[\frac{\hat{e}_1 (i (\hat{p}_1 - \hat{k}_1) - m) \hat{e}_2}{-2 p_1 k_1} + \frac{\hat{e}_2 (i (\hat{p}_1 - \hat{k}_2) - m) \hat{e}_1}{-2 p_1 k_2} \right] v (-p_2),$$

$$A^+ (k_3 e_3, k_4 e_4, p_1 s_1, -p_2 s_2) = -ie^2 \bar{v} (-p_2) \left[\frac{\hat{e}_4 (i (\hat{p}_1 - \hat{k}_3) - m) \hat{e}_3}{-2 p_1 k_3} + \frac{\hat{e}_3 (i (\hat{p}_1 - \hat{k}_4) - m) \hat{e}_4}{-2 p_1 k_4} \right] u (p_1)$$

(Invariant normalization factors were chosen here, for the spinors u and v.)

It is seen directly from (5) and from similar expressions for Im A_c and Im A_a that the following equalities hold:

$$A_{s,e} = A_s, \qquad A_{c,e} = A_c, \qquad A_{a,e} = A_a.$$

Introducing the notation

$$A_s = \tfrac{1}{2} (A_1 + A_{1,e}), \qquad A_c = \tfrac{1}{2} (A_2 + A_3),$$
$$A_a = \tfrac{1}{2} (A_{2,e} + A_{3,e}),$$

we write the total amplitude A in the form

$$A = A_1 + A_2 + A_3 + A_{1,e} + A_{2,e} + A_{3,e}.$$

In the first nonvanishing approximation of perturbation theory, it follows from (5) that the imaginary part of the amplitude A_1 is depicted by the two Feynman diagrams shown in Fig. 1 (the electron lines with the horizontal stroke denote free particles).

The amplitudes A_i are functions of the scalar products $k_i e_j$ and of two independent scalar invariants associated with the total energy and momentum transfer. We introduce the following notation for these invariants:

$$s = -(k_1 + k_2)^2 = -(k_3 + k_4)^2,$$
$$t = (k_1 - k_3)^2 = (k_2 - k_4)^2,$$
$$u = (k_1 - k_4)^2 = (k_2 - k_3)^2.$$

By the law of conservation, s, t and u satisfy the equality $-s + t + u = 0$. Using the properties of crossing symmetry (1) and (2), we have

$$A_1 \to A_2 \quad \text{for} \quad (k_2, e_2) \leftrightarrow (-k_4, e_4), \qquad s \to -u, \quad t \to t;$$
$$A_1 \to A_3 \quad \text{for} \quad (k_2, e_2) \leftrightarrow (-k_3, e_3), \qquad s \to -t, \quad u \to u;$$
$$A_1 \to A_{1,e} \quad \text{for} \quad (k_3, e_3) \leftrightarrow (k_4, e_4), \qquad s \to s, \quad t \to u;$$
$$A_1 \to A_{2,e} \quad \text{for} \quad (k_4, e_4) \to (-k_2, e_2),$$
$$(k_3, e_3) \to (k_4, e_4), \ (k_2, e_2) \to (-k_3, e_3);$$
$$s \to -t, \quad t \to u, \quad u \to -s;$$
$$A_1 \to A_{3,e} \quad \text{for} \quad (k_2, e_2) \to (-k_4, e_4),$$
$$(k_4, e_4) \to (k_3, e_3), \ (k_3, e_3) \to (-k_2, e_2).$$
$$s \to -u, \quad u \to t, \quad t \to -s. \qquad (6)$$

The kinematics of all scattering process with four external free-photon lines is shown in Fig. 2. The energetically allowed region for the processes described by the amplitudes A_1 and $A_{1,e}$ is the half-plane $s \geq 0$, and the processes described by the amplitudes A_2, $A_{3,e}$ and A_3, $A_{2,e}$ are energetically possible in the half-planes $s - t \leq 0$ and $t \leq 0$, respectively. The imaginary parts of the amplitudes A_i and $A_{i,e}$, as will be seen below, are nonzero in the regions

$$\operatorname{Im} A_1, \ \operatorname{Im} A_{1,e} \neq 0 \ \text{for} \ s \geqslant 4m^2,$$
$$\operatorname{Im} A_2, \ \operatorname{Im} A_{3,e} \neq 0 \ \text{for} \ u \leqslant -4m^2,$$
$$\operatorname{Im} A_3, \ \operatorname{Im} A_{2,e} \neq 0 \ \text{for} \ t \leqslant -4m^2.$$

The physical regions (regions in which the scattering angles are real) are shown crosshatched in Fig. 2.

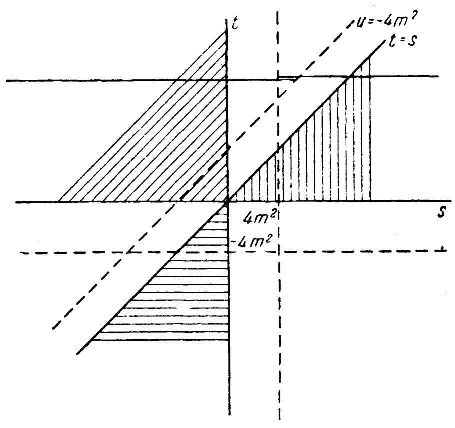

FIG. 2

2. We introduce the notation

$$p_1 - p_2 = v, \quad k_1 + k_2 = k_3 + k_4 = q, \quad k_1 - k_2 = p,$$
$$k_3 - k_4 = p' \quad (-q^2 = p^2 = p'^2 = s, \quad qp = qp' = 0),$$

and represent Im A_1 in the form

$$\operatorname{Im} A_1 = \frac{e^4}{8\pi^2} \int d^4 v \delta(qv)\, \delta(v^2 - s + 4m^2) \frac{S_1 + S_2}{(vp - s)(vp' - s)}; \tag{7}$$

$$S_1 = \operatorname{Sp}\Big(\frac{i}{2}(\hat{q} + \hat{v}) - m\Big)\gamma_\mu\Big(\frac{i}{2}(\hat{v} - \hat{p}) - m\Big)\gamma_\nu \times\Big(\frac{i}{2}(\hat{q} - \hat{v}) + m\Big)\gamma_\rho\Big(\frac{i}{2}(\hat{v} - \hat{p}') - m\Big)\gamma_\sigma e^1_\mu e^2_\nu e^3_\sigma e^4_\rho,$$
$$S_2 = \operatorname{Sp}\Big(\frac{i}{2}(\hat{q} - \hat{v}) - m\Big)\gamma_\nu\Big(\frac{i}{2}(\hat{v} - \hat{p}) + m\Big)\gamma_\mu \times\Big(\frac{i}{2}(\hat{q} + \hat{v}) + m\Big)\gamma_\sigma\Big(\frac{i}{2}(\hat{v} - \hat{p}') + m\Big)\gamma_\rho e^1_\mu e^2_\nu e^3_\sigma e^4_\rho$$
$$(\hat{q} = \gamma_\mu q_\mu). \tag{8}$$

The calculation of the spurs (8) leads to the following general expression for Im A_1:

$$\operatorname{Im} A_1 = \frac{e^4}{8\pi^2} \operatorname{Im} T^{(1)}_{\mu\nu\rho\sigma} e^1_\mu e^2_\nu e^3_\sigma e^4_\rho,$$

where the components of the tensor $T^{(1)}_{\mu\nu\rho\sigma}$ have the form

$$\begin{aligned} T^{(1)}_{\mu\nu\rho\sigma} = {} & (\delta_{\mu\nu}\delta_{\rho\sigma}A + \delta_{\mu\rho}\delta_{\nu\sigma}B + \delta_{\mu\sigma}\delta_{\nu\rho}C) \\ & + (\delta_{\mu\nu}q_\rho q_\sigma A^{(qq)} + \delta_{\mu\rho}q_\nu q_\sigma B^{(qq)} + \delta_{\mu\sigma}q_\nu q_\rho C^{(qq)} + \delta_{\nu\rho}q_\mu q_\sigma D^{(qq)} \\ & + \delta_{\nu\sigma}q_\mu q_\rho E^{(qq)} + \delta_{\rho\sigma}q_\mu q_\nu F^{(qq)} + \ldots) + (q_\mu q_\nu p_\rho p_\sigma A^{(qqpp)} \\ & + q_\mu q_\rho p_\nu p_\sigma B^{(qqpp)} + q_\mu q_\sigma p_\nu p_\rho C^{(qqpp)} + q_\nu q_\rho p_\mu p_\sigma^{(qqpp)} \\ & + q_\nu q_\sigma p_\mu p_\rho E^{(qqpp)} + q_\rho q_\sigma p_\mu p_\nu F^{(qqpp)} + \ldots) \\ & + (p_\mu p_\nu p_\rho p'_\sigma A^{(pppp')} + p_\mu p_\nu p_\sigma p'_\rho B^{(pppp')} + p_\mu p_\rho p_\sigma p'_\nu C^{(pppp')} \\ & + p_\nu p_\rho p_\sigma p'_\mu D^{(pppp')} + \ldots) + (q_\mu q_\nu q_\rho q_\sigma A^{(qqqq)} + \ldots) \end{aligned} \tag{9}$$

(To obtain the omitted terms from the written ones, we replace the pairs qq by pp, p′p′, pp′, p′p, qp, pq, qp′, p′q; the tetrads qqpp by qqp′p′, qqpp′, qqp′p, qp′pp, p′qpp, qpp′p′, pqp′p′, ppp′p′, the tetrads pppp′ by p′p′p′p, qppp, qp′p′p′, pqqq, p′qqq, and the tetrads qqqq by pppp, p′p′p′p′.) The scalar coefficients A, . . . , $A^{(qq)}$, . . . , $A^{(qqpp)}$, . . . , $A^{(pppp')}$, . . . , $A^{(qqqq)}$, . . . are functions of the variables s and t.

The conditions of gauge invariance and invariance with respect to the CPT and CP transformations lead to a number of relations between the scalar coefficients in the expression for the tensor $T^{(1)}_{\mu\nu\rho\sigma}$. These transformations have the form

gauge invariance: $(q + p)_\mu T^{(1)}_{\mu\nu\rho\sigma} = 0,$

CPT transformation: $q \to q, \; p \leftrightarrow p', \; \mu \leftrightarrow \sigma, \; \nu \leftrightarrow \rho,$

CP transformation: $q \to q, \; p \to -p,$
$p' \to -p' \; \mu \leftrightarrow \nu, \; \rho \leftrightarrow \sigma.$

We have obtained the following relations ($\eta = t/s$):

$$A = -s[A^{(qp')} + A^{(p'p')} + (1 - 2\eta)A^{(pp')}],$$
$$A^{(qq)} = F^{(qq)} = s^{-1}A + A^{(qp')} + (1 - 2\eta)A^{(qp)},$$
$$B = -s[B^{(qp)} + B^{(pp)} + (1 - 2\eta)B^{(p'p)}],$$
$$B^{(qq)} = E^{(qq)} = s^{-1}B + B^{(qp')} + (1 - 2\eta)B^{(qp)},$$
$$C = -s[C^{(qp)} + C^{(pp)} + (1 - 2\eta)C^{(p'p)}],$$
$$C^{(qq)} = D^{(qq)} = s^{-1}C - C^{(qp')} - (1 - 2\eta)C^{(qp)},$$
$$A^{(pp)} = F^{(p'p')} = -(1 - 2\eta)^{-1}(A^{(qp)} + A^{(pp')}),$$
$$B^{(pp)} = E^{(pp)} = B^{(p'p')} = E^{(p'p')} = -(1 - 2\eta)^{-1}(B^{(qp')} + B^{(pp')}),$$
$$C^{(pp)} = D^{(pp)} = C^{(p'p')} = D^{(p'p')} = -(1 - 2\eta)^{-1}(C^{(qp')} + C^{(pp')}),$$
$$A^{(p'p)} = F^{(p)p'} = A^{(pp')} = F^{(p'p)} = A^{(qp)} + B^{(qp)} - C^{(qp)} - B^{(qp')} - sA^{(p'qqq)} + sB^{(qp'pp)} + s(1 - 2\eta)C^{(qpp'p')},$$
$$B^{(pp')} = E^{(pp')} = B^{(qp')} - C^{(qp')} + sA^{(qpp'p')} - sA^{(pqqq)} + s(1 - 2\eta)C^{(qp'pp)},$$
$$C^{(pp')} = D^{(pp')} = -[B^{(qp')} - C^{(qp')} + sA^{(qpp'p')} - sA^{(pqqq)} + s(1 - 2\eta)B^{(qp'pp)}],$$
$$B^{(p'p)} = E^{(p'p)}, \quad C^{(p'p)} = D^{(p'p)}, \quad A^{(qp)} = F^{(p'q)} = -A^{(pq)} = -F^{(qp')},$$
$$B^{(qp)} = E^{(p'q)} = -E^{(qp)} = -B^{(p'q)}, \quad C^{(qp)} = C^{(p'q)} = -D^{(qp)} = -D^{(p'q)},$$
$$A^{(qp')} = F^{(pq)} = -A^{(p'q)} = -F^{(qp)}, \quad B^{(qp')} = E^{(pq)} = -E^{(qp')} = -B^{(pq)},$$
$$C^{(qp')} = C^{(pq)} = -D^{(qp')} = -D^{(pq)},$$
$$A^{(qqpp)} = s^{-1}(B^{(qp)} + C^{(qp)} + A^{(qp')} + A^{(pp)}) - A^{(qpppp)} - (1 - 2\eta)A^{(qp'pp)},$$
$$B^{(qqpp)} = s^{-1}(A^{(qp)} + B^{(qp')} + C^{(qp')} + B^{(pp)}) - A^{(qp'p'p')} - (1 - 2\eta)B^{(qpp'p')},$$
$$C^{(qqpp)} = -s^{-1}(A^{(qp)} + B^{(qp')} + C^{(qp')} - C^{(pp)}) + A^{(qp'p'p')} + (1 - 2\eta)C^{(qpp'p')},$$
$$A^{(qqpp')} = s^{-1}(A^{(qp)} + B^{(qp)} - C^{(qp)}) - A^{(p'qqq)},$$
$$B^{(qqpp')} = s^{-1}(A^{(qp')} + B^{(qp')} - C^{(qp')}) - A^{(pqqq)},$$
$$C^{(qqpp')} = -s^{-1}(A^{(qp')} + B^{(qp')} - C^{(qp')}) + A^{(pqqq)},$$
$$A^{(qqp'p')} = s^{-1}A^{(p'p')} - A^{(qpp'p')} - (1 - 2\eta)A^{(qp'p'p')},$$
$$B^{(qqp'p)} = s^{-1}(B^{(p'p)} + C^{(qp)}) - C^{(qpp'p')} - (1 - 2\eta)A^{(qp'pp)},$$
$$C^{(qqp'p)} = s^{-1}(C^{(p'p)} - B^{(qp)}) + B^{(qpp'p')} + (1 - 2\eta)A^{(qp'pp)},$$
$$A^{(ppp'p')} = -s^{-1}A^{(p'p')} + A^{(qpp'p')} - (1 - 2\eta)A^{(p'p'p'p)},$$
$$B^{(ppp'p')} = E^{(ppp'p')} = -s^{-1}B^{(pp)} + B^{(qpp'p')} - (1 - 2\eta)A^{(p'p'p'p)},$$
$$C^{(ppp'p')} = D^{(ppp'p')} = -s^{-1}C^{(pp)} + C^{(qpp'p')} - (1 - 2\eta)A^{(p'p'p'p)},$$
$$F^{(ppp'p')} = -(1 - 2\eta)^{-1}[s^{-1}(B^{(p'p)} + C^{(p'p)}) - A^{(qp'pp)} + A^{(p'p'p'p)}],$$
$$A^{(qqqq)} = s^{-1}(A^{(qq)} + B^{(qq)} + C^{(qq)}) + A^{(pqqq)} + (1 - 2\eta)A^{(p'qqq)},$$

$$A^{(pppp)} = -s^{-1}(A^{(pp)} + B^{(pp)} + C^{(pp)}) + A^{(qppp)} - (1-2\eta)A^{(p'p'p'p)},$$
$$A^{(p'p'p'p')} = -(1-2\eta)^{-1}[s^{-1}(A^{(p'p')} + B^{(pp)} + C^{(pp)}) + A^{(pppp')} - A^{(qp'p'p')}],$$
$$A^{(pppp')} = D^{(p'p'p'p)} = B^{(pppp')} = C^{(p'p'p'p)},$$
$$A^{(p'p'p'p)} = B^{(p'p'p'p)} = C^{(pppp')} = D^{(pppp')}$$
$$A^{(pqqq)} = D^{(p'qqq)} = -B^{(pqqq)} = -C^{(p'qqq)},$$
$$A^{(p'qqq)} = D^{(pqqq)} = -B^{(p'qqq)} = -C^{(pqqq)}. \qquad (10)$$

(The relations between coefficients with the tetrads qqpp, qqp′p′, qqpp′, qqp′p and qp′pp, p′qpp, qpp′p′, pqp′p′ are the same as between the coefficients with the pairs pp, p′p′, pp′, p′p and qp, pq, qp′, p′q, and the relations between the coefficients with qppp and qp′p′p′ are the same as those between the coefficients with pqqq and and p′qqq.) Hereafter, we shall take as the independent coefficients the functions

$$A^{(qp)},\ B^{(qp)},\ C^{(qp)},\ A^{(qp')},\ B^{(qp')},\ C^{(qp')},\ A^{(p'p')},\ B^{(p'p)},\ C^{(p'p)},\ sA^{(pqqq)},\ sA^{(p'qqq)},\ sA^{(qppp)},\ sA^{(qp'p'p')},\ sA^{(qp'pp)},\ sB^{(qp'pp)},\ sC^{(qp'pp)},\ sA^{(qpp'p')},\ sB^{(qpp'p')},\ sC^{(qpp'p')},\ sA^{(pppp')},\ sA^{(p'p'p'p)}.$$

We give explicit expressions for the imaginary parts of the independent functions $A^{(qp)}$, $B^{(qp)}$, . . . obtained in the calculation of the spurs (8) (in the approximation $s, t \gg 4m^2$):

$$\mathrm{Im}\,A^{(qp)} = \mathrm{Im}\,B^{(qp)} = \mathrm{Im}\,C^{(qp)} = s(a_2/2 - a_3) = st(tF_2 + F_1)/2(s-t)^2 + F_0/4(s-t),$$
$$\mathrm{Im}\,A^{(qp')} = \mathrm{Im}\,B^{(qp')} = \mathrm{Im}\,C^{(qp')} = s(a_2/2 - a_3) + (a_0' - a_1')/2 = st(tF_2 + F_1)/2(s-t)^2 + (1/4(s-t) - 1/2s)F_0,$$
$$\mathrm{Im}\,A^{(p'p')} = 2s(b_4 - a_3) + s(b_2 - a_1) + (b_2' - a_1') = s^2(5s - 9t)(tF_2 + F_1)/6(s-t)^3 + (-1/4t + 1/4(s-t) - t/3(s-t)^2 + 1/2s)F_0,$$
$$\mathrm{Im}\,B^{(p'p)} = 2sc_4 - (s-t)c_2 + sa_1 + s(a_2 - a_0)/2 = -s(s^2 - 8st + 3t^2)(tF_2 + F_1)/6(s-t)^3 - (1/4t + 1/3(s-t) + t/3(s-t)^2)F_0,$$
$$\mathrm{Im}\,C^{(p'p)} = 2sc_4 - tc_2 - sa_1 + s(a_0 - a_2)/2 = -s(2s^2 + 5st - 3t^2)(tF_2 + F_1)/6(s-t)^3 - (7/12(s-t) + t/3(s-t)^2)F_0,$$
$$\mathrm{Im}\,sA^{(pqqq)} = \mathrm{Im}\,sA^{(p'qqq)} = s(a_3 - a_2) = s(s-2t)(tF_2 + F_1)/2(s-t)^2 - F_0/4(s-t),$$
$$\mathrm{Im}\,sA^{(qppp)} = \mathrm{Im}\,sA^{(qp'p'p')} = s(b_3 - b_2) = s^3(tF_2 + F_1)/2(s-t)^3 + (-1/8t + 3/8(s-t) + t/4(s-t)^2)F_0,$$
$$\mathrm{Im}\,sA^{(qp'pp)} = \mathrm{Im}\,sA^{(qpp'p')} = s(c_3 - c_2) = s^2t(tF_2 + F_1)/2(s-t)^3 + (-1/8t + 1/8(s-t) + t/4(s-t)^2))F_0,$$
$$\mathrm{Im}\,sB^{(qp'pp)} = \mathrm{Im}\,sC^{(qp'pp)} = \mathrm{Im}\,sB^{(qpp'p')} = \mathrm{Im}\,sC^{(qpp'p')} = sc_3 + s(a_1 - b_2 - c_2)/2 = s^2t(tF_2 + F_1)/2(s-t)^3 + (1/8t + 1/8(s-t) + t/4(s-t)^2)F_0,$$
$$\mathrm{Im}\,sA^{(pppp')} = 2s(c_4 - c_3) - s(b_3 - b_2) = -\frac{s^3(s-3t)}{2(s-t)^4}(tF_2 + F_1) + (1/24t - 5/24(s-t) + t/2(s-t)^2 + t^2/2(s-t)^3)F_0,$$
$$\mathrm{Im}\,sA^{(p'p'p'p)} = s(2e_4 - c_3) = \frac{s^3(s+t)}{2(s-t)^4}(tF_2 + F_1) + (1/24t + 13/24(s-t) + t/(s-t)^2 + t^2/2(s-t)^3)F_0, \qquad (11)$$

where $a_i, b_i, \ldots, F_i$ are given by formulas (A.1) and (A.2).

3. Singularities of the total scattering amplitude are represented in Fig. 2 as functions of two independent complex variables. In the real plane (s, t), the total amplitude breaks off at $2\,\mathrm{Im}\,(A_1 + A_{1,e})$ in the half-plane $s \geq 4m^2$, at $2\,\mathrm{Im}\,(A_2 + A_{3,e})$ in the half-plane $u \leq -4m^2$, and at $2\,\mathrm{Im}\,(A_3 + A_{2,e})$ in the half-plane $t \leq -4m^2$. The cuts in the complex plane s (for a given t) run from $4m^2$ to ∞ and from $-4m^2 + t$ to $-\infty$.

The analytic properties of the total scattering amplitude with respect to s are expressed by the dispersion relation (without subtraction)

$$A^{(i)}(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}\,A_1^{(i)}(s',t)}{s'-s}ds' - \frac{1}{\pi}\int_{-\infty}^{-4m^2+t}\frac{\mathrm{Im}\,A_2^{(i)}(s',t)}{s'-s}ds' + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}\,A_{1,e}^{(i)}(s',t_1)}{s'-s}ds' - \frac{1}{\pi}\int_{-\infty}^{-4m^2+t}\frac{\mathrm{Im}\,A_{3,e}^{(i)}(s',t_1)}{s'-s}ds', \qquad (12)$$

where $t_1 = s' - u$, or by the relation

$$A^{(i)} = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}\,A_1^{(i)}(s',t)}{s'-s}ds' + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}\,A_1^{(i_e)}(s',u)}{s'-s}ds' + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}\,A_1^{(i_c)}(u',t)}{u'+u}du' + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}\,A_1^{(i_a)}(u',-s)}{u'+u}du'. \qquad (12')$$

In Fig. 2 the contour of integration lies in the upper half-plane ($t \geq 0$) and, for $t > 4m^2$, partially enters the nonphysical region. Hence, no contribution comes from the contour of integration of the quantities $\mathrm{Im}\,A_3$ and $\mathrm{Im}\,A_{2,e}$. In (12) we understand by $\mathrm{Im}\,A_1^{(i)}$ the imaginary part of any of the independent scalar functions $A^{(qp)}$ and $B^{(qp)}$, and we understand by $\mathrm{Im}\,A_2^{(i)}$, $\mathrm{Im}\,A_{1,e}^{(i)}$, $\mathrm{Im}\,A_{3,e}^{(i)}$ the imaginary parts of the same independent functions in the expansion (9) for A_2, $A_{1,e}$, $A_{3,e}$. In (12′) all the functions under the integral sign are expressed in terms of independent coefficients of the expansion (9) for A_1 [which is possible owing to (6)].

4. Inserting in (12′) the expressions for the imaginary parts of the functions $A^{(qp)}$, $B^{(qp)}$, . . . (11), we find the expressions for the real parts of these functions. We note that for those terms in (11) which do not decrease (they remain constant) as $s \to \infty$, the subtraction should be carried out at the point $s = t$.

Using (A.3), we obtain, for $s, t \gg 4m^2$, the following expression for the real parts:

$$\operatorname{Re} A^{(qp)} = \frac{t}{4(s-t)^2} \ln^2 \frac{t}{s} + \frac{1}{2(s-t)} \ln \frac{t}{s},$$

$$\operatorname{Re} A^{(qp')} = \frac{t}{4(s-t)^2} \ln^2 \frac{t}{s} - \frac{s-2t}{2s(s-t)} \ln \frac{t}{s},$$

$$\operatorname{Re} A^{(p'p')} = \frac{s}{12} \frac{5s-9t}{(s-t)^3} \ln^2 \frac{t}{s} - \frac{3s^3 - 15s^2 t + 22st^2 - 6t^3}{6st(s-t)^2} \ln \frac{t}{s} - \frac{2}{3(s-t)},$$

$$\operatorname{Re} B^{(p'p)} = -\frac{1}{12} \frac{s^2 - 8st + 3t^2}{(s-t)^3} \ln^2 \frac{t}{s} - \frac{3s^2 - 10st + 3t^2}{6t(s-t)^2} \ln \frac{t}{s} + \frac{2}{3(s-t)},$$

$$\operatorname{Re} C^{(p'p)} = -\frac{1}{12} \frac{2s^2 + 5st - 3t^2}{(s-t)^3} \ln^2 \frac{t}{s} - \frac{7s-3t}{6(s-t)^2} \ln \frac{t}{s} - \frac{2}{3(s-t)},$$

$$\operatorname{Re} sA^{(pqqq)} = \frac{s-2t}{4(s-t)^2} \ln^2 \frac{t}{s} - \frac{1}{2(s-t)} \ln \frac{t}{s},$$

$$\operatorname{Re} sA^{(qppp)} = \frac{s^2}{4(s-t)^3} \ln^2 \frac{t}{s} - \frac{s^2 - 5st + 2t^2}{4t(s-t)^2} \ln \frac{t}{s} + \frac{1}{2(s-t)},$$

$$\operatorname{Re} sA^{(qp'pp)} = \frac{st}{4(s-t)^3} \ln^2 \frac{t}{s} - \frac{s(s-3t)}{4t(s-t)^2} \ln \frac{t}{s} + \frac{1}{2(s-t)},$$

$$\operatorname{Re} sB^{(qp'pp)} = \frac{st}{4(s-t)^3} \ln^2 \frac{t}{s} + \frac{s^2 - st + 2t^2}{4t(s-t)^2} \ln \frac{t}{s} + \frac{1}{2(s-t)},$$

$$\operatorname{Re} sA^{(pppp')} = -\frac{s^2}{4} \frac{s-3t}{(s-t)^4} \ln^2 \frac{t}{s} + \frac{s^3 - 8s^2 t + 25st^2 - 6t^3}{12t(s-t)^3} \ln \frac{t}{s} + \frac{s+t}{2(s-t)^2},$$

$$\operatorname{Re} sA^{(p'p'p'p)} = \frac{s^2}{4} \frac{s+t}{(s-t)^4} \ln^2 \frac{t}{s} + \frac{s(s^2 + 10st + t^2)}{12t(s-t)^3} \ln \frac{t}{s} + \frac{3s-t}{2(s-t)^2}. \qquad (13)$$

We now write the general expression for the scattering amplitude in the center-of-mass system:

$$A = 4\alpha^2 \{(e_1 e_2)(e_3 e_4)\, a + (e_1 e_4)(e_2 e_3)\, b + (e_1 e_3)(e_2 e_4)\, c + s^{-1}[(e_1 e_2)(pe_3)(pe_4) + (e_3 e_4)(p'e_1)(p'e_2)]\, d + s^{-1}[(e_1 e_4)(p'e_2)(pe_3) + (e_2 e_3)(p'e_1)(pe_4)]\, e + s^{-1}[(e_1 e_3)(p'e_2)(pe_4) + (e_2 e_4)(p'e_1)(pe_3)]\, f + s^{-2}(p'e_1)(p'e_2)(pe_3)(pe_4)\, g\}, \qquad (14)$$

where α is the fine-structure constant and

$$a = A + A_e + B_c + B_a, \quad b = B + C_e + A_c + C_a,$$

$$c = C + B_e + C_c + A_a,$$

$$d = s\,[A^{(pp)} + A_e^{(pp)} + \tfrac{1}{4}(B_c^{(qq)} + 2B_c^{(pp)} + B_c^{(pp')} + B_c^{(p'p)} + C_a^{(qq)} + 2C_a^{(pp)} + C_a^{(pp')} + C_a^{(p'p)} + 2C_a^{(qp)} + 2C_a^{(qp')})],$$

$$e = s\,[B^{(p'p)} - C_e^{(p'p)} + \tfrac{1}{4}(A_c^{(qq)} + A_c^{(pp)} + A_c^{(p'p')} + 2A_c^{(pp')} + A_a^{(qq)} + A_a^{(pp)} + A_a^{(p'p')} + 2A_a^{(pp')})],$$

$$f = s\,[C^{(p'p)} - B_e^{(p'p)} + \tfrac{1}{4}(C_c^{(qq)} + 2C_c^{(pp)} + C_c^{(pp')} + C_c^{(p'p)} + 2C_c^{(qp)} + 2C_c^{(qp')} + B_a^{(qq)} + 2B_a^{(pp)} + B_a^{(pp')} + B_a^{(p'p)})],$$

$$g = s^2\,[F^{(ppp'p')} + F_e^{(ppp'p')} + \tfrac{1}{16}(2B_c^{(qqpp)} + 2C_c^{(qqpp)} - A_c^{(ppp'p')} + 2B_c^{(ppp'p')} - 2C_c^{(ppp'p')} - F_c^{(ppp'p')} + 2A_c^{(qppp)} - 2A_c^{(qp'p'p')} + 2A_c^{(pqqq)} - 2A_c^{(p'qqq)} + A_c^{(qqqq)} + A_c^{(pppp)} + A^{(p'p'p'p)} + 4C_a^{(qqpp)} - A_a^{(ppp'p')} - 2B_a^{(ppp'p')} + 2C_a^{(ppp'p')} - F_a^{(ppp'p')} - 2A_a^{(qppp)} - 2A_a^{(qp'p'p')} - 2A_a^{(pqqq)} - 2A_a^{(p'qqq)} + A_a^{(qqqq)} + A_a^{(pppp)} + A_a^{(p'p'p'p')})]. \qquad (15)$$

It follows from (6) that the functions with the indices e, c, and a can be obtained from the corresponding functions without the indices by means of the substitutions $s \to s$, $t \to u$; $s \to -u$, $t \to t$; and $s \to -u$, $t \to -s$.

5. The differential cross section for photon-photon scattering is related to the scattering amplitude A by the expression

$$d\sigma = (16\pi)^{-2} \omega^{-2} |A|^2 do.$$

Inserting expression (14) in place of A and averaging over the polarizations of the initial photons and summing over the polarizations of the final photons, we obtain

$$d\sigma = (\alpha^4/2^6\pi^2)\, \omega^{-2} do\, [4|a|^2 + |b|^2 + |c|^2 + 2\operatorname{Re}(ab^* + ac^* + bc^*) + 2\cos^2\theta\,(|b|^2 + |c|^2 + \operatorname{Re}(ab^* + ac^* + bc^*)) + 4\sin^2\theta \operatorname{Re}(2ad^* - be^* - 2cf^*) - 4\sin^2\theta\cos\theta \operatorname{Re}(e+f)a^* + \cos^4\theta\,(|b|^2 + |c|^2) + 2\sin^4\theta\,(3|d|^2 + 2|e|^2 + 2|f|^2 + \operatorname{Re}(a+b+c)g^*) + 4\sin^2\theta\cos^2\theta \operatorname{Re}(bd^* - be^* + cd^*) - 4\sin^2\theta\cos^3\theta\,(bf^* + ce^*) - 8\sin^4\theta\cos\theta \operatorname{Re}(e+f)d^* + 4\sin^6\theta \operatorname{Re}(d-e-f)g^* + 2\sin^4\theta\cos^2\theta\,(|e|^2 + |f|^2 + 4\operatorname{Re} ef^*) + \sin^8\theta\,|g|^2]. \qquad (16)$$

If in the initial state the polarizations of the photons are orthogonal to each other and if the polarizations reverse as a result of the scattering, then the cross section has the form

$$d\sigma(1_\perp, 2_\parallel, 3_\parallel, 4_\perp) = (\alpha^4/2^6\pi^2)\, \omega^{-2} do\, [|b|^2\cos^2\theta + 2\operatorname{Re}(be^*)\sin^2\theta\cos\theta + |\epsilon|^2\sin^4\theta]. \qquad (17)$$

For small scattering angles ($m/\omega \ll \theta \ll 1$), the greatest contribution to the scattering amplitude (14) comes from terms containing the functions a, b, and c, which, in this case, are equal to (we neglect the contribution from the imaginary parts)

$$a = b \approx -\frac{4}{3}\ln\frac{t}{s}, \quad c \approx \ln^2\frac{t}{s} + \frac{8}{3}\ln\frac{t}{s},$$

the cross section is then expressed by the formula

$$d\sigma \approx (\alpha^4/\pi^2)\,\omega^{-2}do\,[(e_1e_3)^2(e_2e_4)^2\ln^4\theta + \tfrac{4}{3}(e_1e_3)(e_2e_4)((2-3\ln 2)(e_1e_3)(e_2e_4) - (e_1e_2)(e_3e_4) - (e_1e_4)(e_2e_3))\ln^3\theta]. \quad (18)$$

If we restrict ourselves to the first term in (18), we obtain the formula first found by Akhiezer.[5] In this case, the cross section averaged over the polarizations has the form

$$d\sigma \approx (4\alpha^4/\pi^2)\,\omega^{-2}do\,[\ln^4\theta - 4(\ln 2 - 1/3)\ln^3\theta]. \quad (19)$$

6. We shall also find an expression for zero-angle scattering cross section. In the center-of-mass system, the zero-angle scattering amplitude has the form

$$A = 4\alpha^2\,[(e_1e_2)(e_3e_4)\,a + (e_1e_4)(e_2e_3)\,b + (e_1e_3)(e_2e_4)\,c].$$

Using (A.1) and (A.2) for t = 0 [we now retain in (8) terms containing the electron mass] and determining the real parts of the scalar functions from the dispersion relations (12′), we obtain for a, b, and c the expressions

$$a = b \approx \frac{4}{3}\ln\frac{s}{m^2}, \quad c \approx \ln^2\frac{s}{m^2} - \frac{20}{3}\ln\frac{s}{m^2}$$

(we neglect the contribution from the imaginary parts).

In this case, the differential cross section is expressed by the formula

$$(d\sigma/do)_{\theta=0} \approx (\alpha^4/\pi^2)\,\omega^{-2}\,[(e_1e_3)^2(e_2e_4)^2\ln^4(\omega/m) - \tfrac{4}{3}(e_1e_3)(e_2e_4)((5-3\ln 2)(e_1e_3)(e_2e_4) - (e_1e_2)(e_3e_4) - (e_1e_4)(e_2e_3))\ln^3(\omega/m)]. \quad (20)$$

The cross section averaged over the polarizations has the form

$$(d\sigma/do)_{\theta=0} \approx (4\alpha^4/\pi^2)\,\omega^{-2}\,[\ln^4(\omega/m) - 4(4/3 - \ln 2)\ln^3(\omega/m)] \quad (21)$$

(the first term was obtained by Karplus and Neuman[6]).

7. Finally, we give expressions for the total cross section in the case of definite polarizations of the colliding and scattered photons (17). In (17) the real and imaginary parts of the functions b and e have the form

$$\mathrm{Re}\,b = \frac{x}{6}\frac{6-9x+3x^2+x^3}{(1-x)^2}\ln^2 x + \frac{2-2x-3x^2+5x^3+x^4}{6x^2}\ln^2(1-x) + \frac{1}{3}(3-5x-x^2)\ln x\ln(1-x) - \frac{4-4x-x^2}{3(1-x)}\ln x + \frac{2+x+x^2}{3x}\ln(1-x),$$

$$\mathrm{Re}\,e = \frac{1+4x+20x^2-36x^3+27x^4-8x^5}{24(1-x)^3}\ln^2 x + \frac{8+2x-6x^2-5x^3+5x^5}{24x^3}\ln^2(1-x) - \frac{3-3x+8x^2}{12}\ln x\ln(1-x) - \frac{6-23x+11x^2-10x^3+8x^4}{12x(1-x)^2}\ln x - \frac{6-x^2+4x^4}{6x(1-x)}\ln(1-x) + \frac{1+6x-3x^2}{6x(1-x)},$$

$$\mathrm{Im}\,b = \frac{x}{3}\frac{6-9x+3x^2+x^3}{(1-x)^2}\ln x + \frac{12-3x-5x^2-x^3}{3x}\ln(1-x) + \frac{3-3x+x^2}{3(1-x)},$$

$$\mathrm{Im}\,e = \frac{1+4x+20x^2-36x^3+27x^4-8x^5}{12(1-x)^3}\ln x + \frac{8+2x-6x^2+5x^3-5x^5}{12x^3}\ln(1-x) + \frac{8-13x+12x^2-4x^3+5x^4-4x^5}{6x^2(1-x)^2},$$

where $x = \sin^2 \frac{1}{2}\theta$.

Integrating (17) over x from 0 to 1 (we note that all divergences occurring here drop out), we obtain

$$\sigma(1_\perp, 2_\parallel, 3_\parallel, 4_\perp) = C\alpha^4/\omega^2,$$

$$C \approx \{2^2/\pi\}\{2[8\zeta(2) - 11\zeta(3) + 17\zeta(4) - 7\zeta(5) + \xi(3) - 3] + \pi^2[7 - \zeta(2) - 4\zeta(3)]\} \approx 27, \quad (22)$$

where $\zeta(s)$ is the Riemann zeta function and $\xi(s)$ is the series of Riemann functions (A.4).

In conclusion I express my deep gratitude to Professor A. I. Akhiezer for valuable advice and discussion.

APPENDIX

1. In (7) the following integrals are encountered:

$$F_0(s) = \int d^4v\,\delta(qv)\,\delta(v^2 - s + 4m^2) = 2\pi\theta(s-4m^2)\,r_1,$$

$$F_1(s) = \int \frac{\delta(qv)\,\delta(v^2-s+4m^2)}{vp-s}\,d^4v = -\frac{\pi}{s}\,\theta(s-4m^2)\ln\frac{1+r_1}{1-r_1},$$

$$F_2(s,t) = \int \frac{\delta(qv)\,\delta(v^2-s+4m^2)}{(vp-s)(vp'-s)}\,d^4v = \frac{\pi}{str_2}\,\theta(s-4m^2)\ln\frac{r_1+r_2}{|r_1-r_2|}$$

$$(r_1 = \sqrt{1-\gamma\eta}, \quad r_2 = \sqrt{1+\gamma(1-\eta)}, \quad \eta = t/s, \quad \gamma = 4m^2/t);$$

$$\int \frac{v_\mu\delta(qv)\,\delta(v^2-s+4m^2)}{vp-s}\,d^4v = p_\mu a_1', \quad \int \frac{v_\mu v_\nu\delta(qv)\,\delta(v^2-s+4m^2)}{vp-s}\,d^4v = (q_\mu q_\nu + s\delta_{\mu\nu})\,a_2' + p_\mu p_\nu b_2',$$

$$\int \frac{v_\mu\delta(qv)\,\delta(v^2-s+4m^2)}{(vp-s)(vp'-s)}\,d^4v = (p+p')_\mu\,a_1,$$

$$\int \frac{v_\mu v_\nu\delta(qv)\,\delta(v^2-s+4m^2)}{(vp-s)(vp'-s)}\,d^4v = (q_\mu q_\nu + s\delta_{\mu\nu})\,a_2 + (p_\mu p_\nu + p_\mu' p_\nu')\,b_2 + (p_\mu p_\nu' + p_\nu p_\mu')\,c_2,$$

$$\int \frac{v_\mu v_\nu v_\rho \delta(qv)\,\delta(v^2-s+4m^2)}{(vp-s)(vp'-s)}\, d^4v = [(q_\mu q_\nu + s\delta_{\mu\nu})(p+p')_\rho$$
$$+ \ldots]\, a_3 + (p_\mu p_\nu p_\rho + p'_\mu p'_\nu p'_\rho)\, b_3$$
$$+ (p_\mu p'_\nu p'_\rho + \ldots + p'_\mu p_\nu p_\rho + \ldots)\, c_3,$$
$$\int \frac{v_\mu v_\nu v_\rho v_\sigma \delta(qv)\,\delta(v^2-s+4m^2)}{(vp-s)(vp'-s)}\, d^4v = \Big[\frac{s^2}{3}(\delta_{\mu\nu}\delta_{\rho\sigma} + \ldots)$$
$$+ \frac{s}{3}(\delta_{\mu\nu} q_\rho q_\sigma + \ldots) + q_\mu q_\nu q_\rho q_\sigma\Big]\, a_4$$
$$+ [(p_\mu p_\nu + p'_\mu p'_\nu)(q_\rho q_\sigma + s\delta_{\rho\sigma}) + \ldots]\, b_4$$
$$+ [(p_\mu p'_\nu + p_\nu p'_\mu)(q_\rho q_\sigma + s\delta_{\rho\sigma}) + \ldots]\, c_4$$
$$+ (p_\mu p_\nu p_\rho p_\sigma + p'_\mu p'_\nu p'_\rho p'_\sigma)\, d_4$$
$$+ (p_\mu p'_\nu p'_\rho p'_\sigma + \ldots + p'_\mu p_\nu p_\rho p_\sigma + \ldots)\, e_4$$
$$+ (p'_\mu p'_\nu p_\rho p_\sigma + \ldots)\, f_4 \qquad \text{(A.1)}$$

(the omitted terms are obtained from the written ones by all possible permutations of the indices μ, ν, ρ, σ), where the quantities a'_1, a'_2, . . . are expressed in terms of the integrals F_0, F_1, and F_2:

$$a'_0 = F_1, \quad a'_1 = F_1 + F_0/s, \quad a'_2 = -F_0/2s - 2m^2F_1/s,$$
$$b'_2 = F_1 + 3F_0/2s + 2m^2F_1/s, \quad a_0 = F_2,$$
$$a_1 = (sF_2 + F_1)/2\,(s-t),$$
$$a_2 = -(tF_2 + F_1)\,/\,(s-t) - 4m^2F_2/s,$$
$$b_2 = s^2F_2/2\,(s-t)^2 + (2s-t)\,F_1/2\,(s-t)^2$$
$$-(s-2t)\,F_0/4st\,(s-t) + m^2sF_2/t\,(s-t),$$
$$c_2 = s\,(tF_2 + F_1)/2\,(s-t)^2 + F_0/4t\,(s-t)$$
$$-m^2\,(s-2t)\,F_2/t\,(s-t),$$
$$a_3 = -s\,(tF_2 + F_1)/2\,(s-t)^2 - F_0/4s\,(s-t)$$
$$-2m^2F_2/(s-t) - m^2F_1/s\,(s-t),$$
$$b_3 = s^3F_2/2\,(s-t)^3 + (3s\,(s-t) + t^2)\,F_1/2\,(s-t)^3$$
$$-(3s-2t)\,(s-3t)\,F_0/8st\,(s-t)^2$$
$$+3m^2s^2F_2/2t\,(s-t)^2 + m^2\,(3s-2t)\,F_1/2s\,(s-t)^2,$$
$$c_3 = s^2\,(tF_2 + F_1)/2\,(s-t)^3 + (s+t)\,F_0/8t\,(s-t)^2$$
$$-m^2s\,(s-4t)\,F_2/2t\,(s-t)^2 + m^2F_1/2\,(s-t)^2,$$
$$b_4 = -s^2\,(tF_2 + F_1)/3\,(s-t)^3 - (3s-t)\,F_0/12s\,(s-t)^2$$
$$-5m^2sF_2/3\,(s-t)^2 - 4m^4F_2/3t\,(s-t)$$
$$-m^2\,(2s-t)\,F_1/s\,(s-t)^2 + m^2\,(s-2t)\,F_0/3ts^2\,(s-t),$$
$$c_4 = -s\,(s+t)\,(tF_2 + F_1)/6\,(s-t)^3 - F_0/6\,(s-t)^2$$
$$-m^2\,(s+4t)\,F_2/3\,(s-t)^2 + 4m^4\,(s-2t)\,F_2/3st(s-t)$$
$$-m^2F_1/(s-t)^2 - m^2F_0/3st\,(s-t),$$
$$e_4 = s^3\,(tF_2 + F_1)/2\,(s-t)^4$$
$$+(s^2 + 2st - t^2)\,F_0/8t\,(s-t)^3$$
$$-m^2s^2\,(s-5t)\,F_2/2t\,(s-t)^3$$
$$-m^4s\,(s-2t)\,F_2/t^2\,(s-t)^2 + m^2\,(3s-t)\,F_1/2\,(s-t)^3$$
$$+m^2\,(s^2 - 2st + 2t^2)\,F_0/4st^2\,(s-t)^2. \qquad \text{(A.2)}$$

(We have given here the expressions for the coefficients used in the calculations.)

2. We now give the asymptotic integrals, for $s,\ t \gg 4m^2$, occurring in the first term of (12′):

$$\frac{1}{\pi}\int_{4m^2}^{\infty} \frac{s'^k\,[tF_2(s',t) + F_1(s')]}{(s'-t)^n\,(s'-s)}\, ds' \approx \frac{s^{k-1}}{2\,(s-t)^n}\ln^2\frac{t}{s},$$
$$\frac{1}{\pi}\int_{4m^2}^{\infty} \frac{F_0(s')}{(s'-t)^n\,(s'-s)}\, ds' \approx \frac{2}{(s-t)^n}\Big[\ln\frac{t}{s} - \sum_{k=1}^{n-1}\frac{(-1)^k}{k}\Big(\frac{s-t}{t}\Big)^k\Big]. \qquad \text{(A.3)}$$

3. We finally give the expressions for the integrals occurring in integration of (17):

$$\int_0^1 \ln^n\frac{1}{x}\,\frac{dx}{1-x} = n!\,\zeta(n+1),$$
$$\int_0^1 \ln^n\frac{1}{x}\,\ln\frac{1}{1-x}\,\frac{dx}{1-x} = n!\,\xi(n+1),$$
$$\xi(s) = \sum_{n=1}^{\infty}\frac{\zeta(s,\,n+1)}{n} \qquad \text{(A.4)}$$

and $\zeta(s, n)$ is the Riemann function of two arguments:

$$\zeta(s, n) = \sum_{k=0}^{\infty}\frac{1}{(k+n)^s}, \qquad \zeta(s) = \zeta(s, 1).$$

[1] Gell-Mann, Goldberger, and Thirring, Phys. Rev. 95, 1612 (1954).

[2] Bogolyubov, Medvedev, and Polivanov, Voprosy teorii dispersionnykh sootnoshenii (Problems in the Theory of Dispersion Relations), Fizmatgiz, 1958.

[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 (1959).

[4] H. Euler, Ann. Physik **26**, 398 (1936).

[5] A. I. Akhiezer, Physik Zeits. Sowjetunion **11**, 263 (1937).

[6] R. Karplus and M. Neuman, Phys. Rev **80**, 380 (1950).

Translated by E. Marquit
87