LINE SHAPE AND DISPERSION IN THE VICINITY OF AN ABSORPTION BAND, AS AFFECTED BY INDUCED TRANSITIONS

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The problem treated is that of the effect of a strong monochromatic electromagnetic field of frequency close to one of the characteristic frequencies of a system on the spectral composition of the radiation. The dielectric permittivity of the medium in the presence of the field is calculated.

1. The effect of an external electromagnetic field on the macroscopic characteristics of a medium has been treated in many papers. Karplus and Schwinger,[1] and also Basov and Prokhorov,[2] have calculated the absorption coefficient and the dielectric permittivity at the frequency of the applied field as affected by the change of the populations of the levels that is produced by the field (the "saturation effect"). A number of authors[3-6] have treated the change of shape of the spectral lines associated with one of the levels i or k when the system is in a field of frequency ω close to ω_{ik}. In particular it has been shown that at large amplitudes of the field the probability that the system is in the levels i and k oscillates with time. This leads to a splitting of the spectral lines associated with the levels i and k. In these papers, however, there was no study of the shape of the line of the transition $i \to k$ itself in the presence of a field with $\omega \sim \omega_{ik}$. Precisely this problem is the main one treated in the present paper.

As is well known, the necessity of taking induced transitions into account arises in systems with an inverted population of the levels. There has recently been special interest in systems with transitions that lie in the infrared and optical regions of the spectrum.[7-9] Here conditions arise which differ in a number of important respects from those in the microwave region. Namely, in all methods that have been proposed the inversion of the populations is produced owing to excitation with a broad spectrum (radiationless transitions,[7,9-17] or optical excitation by means of radiation with a broad spectral composition[7,8]). This means that the model of a monochromatic field used in [3-6] to describe the excitation of the system does not correspond to the actual conditions.

The second important difference is that the lifetimes of the levels considered are usually quite different. In the case of gaseous systems, for example, at small densities the lifetimes of the excited states are determined by spontaneous transitions, which have different probabilities for the different levels. In crystals the lifetime of the upper state can be determined by a spontaneous transition and that of the lower state by radiationless transitions or, for the ground state, by the probability of excitation. Moreover, as the result of various elastic processes the line widths may not correspond to the decay probabilities (Doppler effect, Weisskopf broadening mechanism, inhomogeneity of the crystal, etc.).

All of these circumstances must be included in the treatment. At first, however, we shall consider radiative processes, and shall take into account all the other causes of line broadening later on.

2. Let us consider an atom[1)] with nondegenerate levels $E_3 > E_2 > E_1$, E_j, E_m. It is obvious that the production of an inverted population of the levels E_3, E_2 is possible only if the probability $2\gamma_2$ of decay of level 2 is larger (in practice, much larger) than the probability $2\gamma_{32}$ of the spontaneous transition $3 \to 2$. Therefore our further calculation is made under the condition

$$\gamma_{32} \ll \gamma_2. \tag{1}$$

There is no restriction on the decay probability $2\gamma_3$ of level 3.

We shall assume that the atom is in a strong electromagnetic field of frequency $\omega_\lambda \sim \omega_{32}$

[1]For definiteness we shall speak of an atom, although all of the further treatment applies to any quantized system.

$= (E_3 - E_2)/\hbar$ and a weak field with a continuous spectrum, which must be considered for the calculation of the induced emission and absorption at frequencies ω_λ, ω_μ. At the initial time $t = 0$ let the atom be in level 3 and let there be in the radiation field n_λ, n_μ photons with frequencies ω_λ, ω_μ. As the result of the interaction with the field there can be various transitions accompanied by emission and absorption of photons ω_λ, ω_μ. We shall denote the probability amplitudes of the states of the system "atom + field" by $a(3, n_\lambda, n_\mu)$, $a(2, n_\lambda+1, n_\mu)$, etc. When Eq. (1) holds and n_μ is small we can take into account only the transitions shown in Fig. 1. In fact, under these conditions the probability amplitudes of the states formed through the emission and absorption of photons ω_μ [i.e., $a(3, n_\lambda-1, n_\mu+1)$, $a(2, n_\lambda, n_\mu+1)$, $a(3, n_\lambda+1, n_\mu-1)$, $a(2, n_\lambda+2, n_\mu-1)$] are small in comparison with $a(3, n_\lambda, n_\mu)$ and $a(2, n_\lambda+1, n_\mu)$. Therefore transitions of types

$$2, n_\lambda, n_\mu + 1 \to 3, n_\lambda, n_\mu + 1, n_{\mu'} - 1;$$
$$3, n_\lambda - 1, n_\mu + 1 \to 2, n_\lambda - 1, n_\mu + 1, n_{\mu'} + 1 \quad \text{etc.}$$

from these four states can be neglected. We note that the transition

$$2, n_\lambda + 1, n_\mu \to 3, n_\lambda + 1, n_\mu - 1$$

must be taken into account, since for large n_λ[2] the probability of this process is comparable with the probability of induced emission with the transition $3, n_\lambda, n_\mu \to 2, n_\lambda, n_\mu + 1$.

Within the framework of this approximation the system of perturbation-theory equations for the probability amplitudes $a(3, n_\lambda, n_\mu)$, $a(2, n_\lambda+1, n_\mu), \ldots$ can be integrated for arbitrary values of n_λ. The exact solutions of the system, valid for arbitrary t, are of the form

$$a(3, n_\lambda, n_\mu) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t},$$
$$a(2, n_\lambda + 1, n_\mu) = i c^*_{32\lambda} \sqrt{n_\lambda}\, e^{i\Omega_\lambda t} \left\{ \frac{A_1}{\alpha_2 + \gamma_3} e^{\alpha_1 t} + \frac{A_2}{\alpha_1 + \gamma_3} e^{\alpha_2 t} \right\},$$
$$a(3, n_\lambda - 1, n_\mu + 1) = -\frac{c^*_{32\mu}\sqrt{1+n_\mu}\, c_{32\lambda}\sqrt{n_\lambda}}{\alpha_1 - \alpha_2} \{A_1[\varphi_{11} - \varphi_{12}] + A_2[\varphi_{21} - \varphi_{22}]\},$$
$$a(2, n_\lambda, n_\mu + 1) = -i\frac{c^*_{32\mu}\sqrt{1+n_\mu}}{\alpha_1 - \alpha_2} \{A_1[(\alpha_1+\gamma_3)\varphi_{11} - (\alpha_2+\gamma_3)\varphi_{12}] + A_2[(\alpha_1+\gamma_3)\varphi_{21} - (\alpha_2+\gamma_3)\varphi_{22}]\} e^{i\Omega_\lambda t},$$
$$a(3, n_\lambda + 1, n_\mu - 1) = -\frac{c_{32\mu}\sqrt{n_\mu}\, c^*_{32\lambda}\sqrt{n_\lambda}}{\alpha_1 - \alpha_2} \times \left\{ A_1\left[\varphi_{11} - \frac{\alpha_1+\gamma_3}{\alpha_2+\gamma_3}\varphi_{21}\right] + A_2\left[\frac{\alpha_2+\gamma_2}{\alpha_1+\gamma_3}\varphi_{12} - \varphi_{22}\right]\right\} e^{i(\Omega_\lambda - \Omega_\mu)t},$$
$$a(2, n_\lambda + 2, n_\mu - 1) = i\frac{c_{32\mu}\sqrt{n_\mu}\, c^{*2}_{32\lambda} n_\lambda}{\alpha_1 - \alpha_2} \left\{ \frac{A_1}{\alpha_2 + \gamma_3}[\varphi_{11} - \varphi_{21}] + \frac{A_2}{\alpha_1+\gamma_3}[\varphi_{12} - \varphi_{22}]\right\} e^{i(2\Omega_\lambda - \Omega_\mu)t}. \quad (2)$$

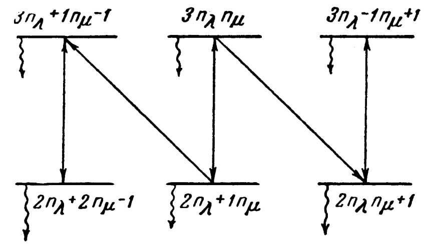

FIG. 1. Scheme of transitions. The wavy arrows denote spontaneous transitions to all lower levels.

Here we have introduced the following notations:

$$\Omega_\lambda = \omega_\lambda - \omega_{32}, \qquad \Omega_\mu = \omega_\mu - \omega_{32};$$

the coefficients $c_{32\lambda}$ and $c_{32\mu}$ are connected with the matrix elements of the interaction Hamiltonian:

$$\hbar c_{32\lambda}\sqrt{n_\lambda} = H_{2n_\lambda;\,3n_\lambda-1}, \qquad \hbar c_{32\mu}\sqrt{n_\mu} = H_{2n_\mu;\,3n_\mu-1} \quad \text{etc.};$$

A_1 and A_2 are constants of integration, which can be determined from the initial conditions; the functions φ_{11}, φ_{12}, φ_{21}, φ_{22} are given by

$$\varphi_{ij} = \frac{\exp\{[i(\Omega_\mu - \Omega_\lambda) + \alpha_i]t\} - \exp\{\alpha_j t\}}{i(\Omega_\mu - \Omega_\lambda) + \alpha_i - \alpha_j} \quad (i, j = 1, 2); \quad (3)$$

and finally, α_1 and α_2 are the roots of the characteristic equation:

$$\alpha_{1,2} = i\delta_{1,2} - \Gamma_{1,2} = -\frac{i}{2}\Omega_\lambda - \frac{1}{2}(\gamma_2 + \gamma_3) \mp \sqrt{(i\Omega_\lambda + \gamma_2 - \gamma_3)^2/4 - G^2},$$
$$G^2 = \pi c^3 \omega_{32}^{-2}\gamma_{32} N_\lambda, \quad (4)$$

where N_λ is the total number of photons of frequency ω_λ in unit volume, and δ_1, δ_2, $-\Gamma_1$, $-\Gamma_2$ are the real and imaginary parts of α_1 and α_2.

The formulas (1) — (4) enable us to obtain all needed characteristics of the radiation in the transition of the atom from an excited state.

3. With $a(3, n_\lambda, n_\mu)$ as an example, let us investigate the special features of the solutions that can arise for various relations between γ_2, γ_3, and N_λ. For simplicity let us consider the case of greatest practical interest, that of exact resonance, that is, the case in which the frequency of the external field coincides with that of the transition ($\Omega_\lambda = 0$). For $N_\lambda \to 0$ we have $\alpha_1 = -\gamma_2$, $\alpha_2 = -\gamma_3$, and Eq. (2) goes over into the well known solution of the problem of the purely spontaneous transition of the atom to the ground state. As N_λ increases the roots α_1, α_2 at first remain real and negative:

[2]If n_λ is sufficiently large, the probability of induced transitions exceeds the probability of spontaneous transitions and the populations of levels 3 and 2 are of the same order of magnitude.

$$-\alpha_{1,2}=\Gamma_{1,2}=\tfrac{1}{2}(\gamma_2+\gamma_3)\pm\sqrt{(\gamma_2-\gamma_3)^2/4-G^2}. \tag{5}$$

Under this condition

$$|a(3, n_\lambda, n_\mu)|^2=|A_1e^{-\Gamma_1 t}+A_2e^{-\Gamma_2 t}|^2$$

decreases monotonically with the time. The complexity of the damping curve is physically due to the "mixing" of the states 2 and 3 of the atom owing to the interaction with the field. For small $G^2/(\gamma_2-\gamma_3)^2$ we have

$$\Gamma_1=\gamma_2-G^2/(\gamma_2-\gamma_3), \qquad \Gamma_2=\gamma_3+G^2/(\gamma_2-\gamma_3), \tag{6}$$

that is, we can suppose that the induced transitions act along with the spontaneous transitions in changing the lifetime. With further increase of G^2, however, it turns out that the induced and spontaneous transitions are quite different from this point of view.

For $G^2=(\gamma_2-\gamma_3)^2/4$ we have $\alpha_{1,2} = -(\gamma_2+\gamma_3)/2$; that is, γ_2 and γ_3 have equal effects in determining the transition from the excited state. If $\gamma_2>\gamma_3$, as is so in the majority of cases, the decay occurs more rapidly than for $G = 0$. If, on the other hand, $\gamma_2<\gamma_3$, there is a lengthening of the lifetime.

When the external field is still stronger the radical becomes imaginary, and the real part no longer depends on the field:

$$\alpha_{1,2}=-\tfrac{1}{2}(\gamma_2+\gamma_3)\pm i\sqrt{G^2-(\gamma_2-\gamma_3)^2/4}, \tag{7}$$

and the time dependence of $|a(3, n_\lambda, n_\mu)|^2$ takes on the character of damped oscillations:

$$|a(3, n_\lambda, n_\mu)|^2=A\cos^2[\delta t+\psi]\,e^{-(\gamma_2+\gamma_3)t},$$
$$\delta^2=G^2-(\gamma_2-\gamma_3)^2/4. \tag{8}$$

We note that for $\gamma_2=\gamma_3$, i.e., in the only case considered in the papers cited earlier,[3-6] the oscillatory behavior begins at once at the smallest values of N_λ. At the same time, as will be shown below, a considerable saturation can be attained even in the "aperiodic case" of Eq. (5), if $\gamma_2\gg\gamma_3$.

Since the character of the solutions is determined by the sign of the radicand, the features we have noted will also appear in the other functions of Eq. (2).

4. In many cases the excitation is selective (for example, optical excitation), and we can assume that at the initial time $t = 0$ the atom is in the level 3. Then

$$A_1=-(\alpha_2+\gamma_3)/(\alpha_1-\alpha_2),$$
$$A_2=(\alpha_1+\gamma_3)/(\alpha_1-\alpha_2). \tag{9}$$

In the opposite case it is necessary to use also the solution under the initial condition $a(2, n_\lambda+1, n_\mu) = 1$ for $t = 0$, for which solution

$$A_1=-A_2=-ic_{32\lambda}\sqrt{n_\lambda}\,/\,(\alpha_1-\alpha_2). \tag{10}$$

The analysis here will be for the practically most interesting case, that of Eq. (9).

Let us consider the probability W_λ of induced emission of a photon of frequency ω_λ, and the integrated probability[3)] W_μ of spontaneous emission in the transition $3\rightarrow 2$. Calculations by the usual rules, by means of Eq. (2), lead to the following expressions:

$$W_\lambda=\frac{\gamma_2+\gamma_3}{\gamma_3}\,\frac{G^2}{\Omega_\lambda^2+(\gamma_2+\gamma_3)^2+G^2(\gamma_2+\gamma_3)^2/\gamma_2\gamma_3},$$
$$W_\mu=\frac{\gamma_{32}}{\gamma_3}\,\frac{\Omega_\lambda^2+(\gamma_2+\gamma_3)^2+G^2(\gamma_2+\gamma_3)/\gamma_2}{\Omega_\lambda^2+(\gamma_2+\gamma_3)^2+G^2(\gamma_2+\gamma_3)^2/\gamma_2\gamma_3}. \tag{11}$$

The dependence of W_λ and W_μ on $G^2/\gamma_2\gamma_3$ is shown in Fig. 2. For small N_λ the probability of induced emission increases in proportion to N_λ, and thereafter it reaches a limiting value (saturation effect)

$$W_\lambda=\gamma_2/(\gamma_2+\gamma_3), \quad N_\lambda\gg N_\lambda^0\equiv(\omega_{32}^2/\pi c^3)\,\gamma_2\gamma_3/\gamma_{32}. \tag{12}$$

If $\gamma_2\gg\gamma_3$, then $\gamma_2/(\gamma_2+\gamma_3)\sim 1$, i.e., practically every act of excitation leads to the induced emission of a photon of the frequency of the external field.

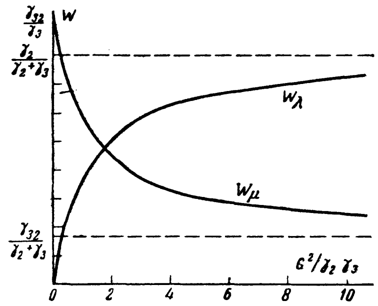

FIG. 2. Integrated probabilities of emission.

In Eq. (12) the quantity N_λ^0 obviously gives the number of photons per cm^3 for which W_λ reaches about half of its limiting value. Besides the well known dependence on the frequency ($\sim\omega_{32}^2$) we must also note that the larger the value of $\gamma_2\gamma_3/\gamma_{32}$ the greater the field intensity required to give saturation. The value of the parameter G^2 that corresponds to N_λ^0 is $\gamma_2\gamma_3$.

The integrated probability of spontaneous emission falls with increasing N_λ, and the larger the value of γ_2/γ_3 the larger is the amount by which this probability decreases (Fig. 2):

$$W_\mu=\gamma_{32}/\gamma_3 \quad (N_\lambda\ll N_\lambda^0),$$
$$W_\mu=\gamma_{32}/(\gamma_2+\gamma_3) \quad (N_\lambda\gg N_\lambda^0). \tag{13}$$

[3]The term probability is used here in denoting two different quantities: the transition probability per unit time (the Einstein coefficient $2\gamma_{ik}$), and the probability of emission of a photon (W_λ, W_μ), which is identical with the ordinary concept of probability.

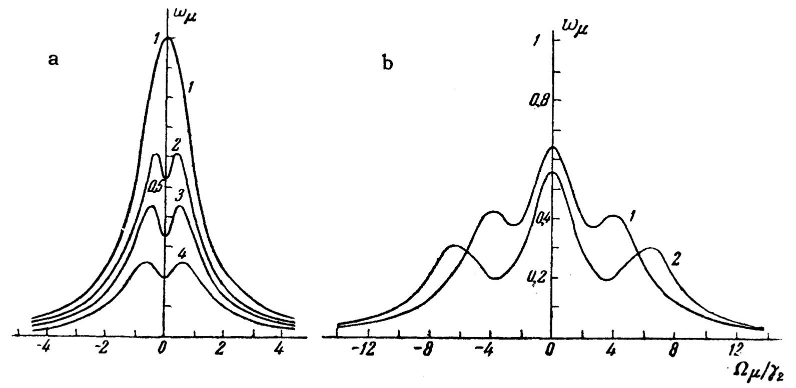

FIG. 3. Spectral density of the probability of spontaneous emission (in units $\gamma_{32}/\pi\gamma_3(\gamma_2+\gamma_3)$). Case a is for $\gamma_3 = \gamma_{32} = 0.1\,\gamma_2$; curve 1 is for G = 0, curve 2 $G^2 = 0.04\,\gamma_2^2$, curve 3 for $G^2 = 0.08\,\gamma_3^2$, and curve 4 for $G^2 = (\gamma_2-\gamma_3)^2/4$. Case b is for $\gamma_3 = \gamma_2$; curve 1 is for $G^2 = 4\,\gamma_2^2$ and curve 2 for $G^2 = 9\,\gamma_2^2$.

Thus as N_λ is increased there is a redistribution of the probabilities of the decay of the excited state between the two channels, one of which is the induced transition 3, n_λ, $n_\mu \to 2$, $n_\lambda+1$, n_μ, and the other the spontaneous transition 3, n_λ, $n_\mu \to 2$, n_λ, $n_\mu+1$. In the generators in the short-wave region that are described at the present time, $\gamma_2 \gg \gamma_3$, and with sufficiently powerful excitation these changes in the integrated probabilities should be observed.

5. As has already been pointed out, the transitions induced by the field cause changes of the spectral composition of the spontaneous radiation and of the shape of the absorption line. For the case in which Eq. (7) holds the cause of the change in the line shape is obvious: the probability of finding the atom in the excited states oscillates with the time, and it is easy to see that this leads to a splitting of the line. In the "aperiodic case" of Eq. (5) there are no oscillations, but the amplitudes of the different states of the system "atom + field" fall off with different damping constants Γ_1 and Γ_2, which also leads to a change of the shapes of the emission and absorption lines.

The spectral density (in the frequency scale) w_μ of the probability of spontaneous emission of a photon can be put in the form

$$w_\mu = \frac{\gamma_{32}\gamma_2}{\pi^2|\alpha_1-\alpha_2|^2}\int_{-\infty}^{\infty}\left|\frac{A_1}{x+\Omega_\mu-i\alpha_1}-\frac{A_2}{x+\Omega_\mu-i\alpha_2}\right|^2 \times\left\{\left|\frac{\alpha_1+\gamma_3}{x+\Omega_\lambda-i\alpha_1}-\frac{\alpha_2+\gamma_3}{x+\Omega_\lambda-i\alpha_2}\right|^2 + \frac{\gamma_3}{\gamma_2}G^2\left|\frac{1}{x+\Omega_\lambda-i\alpha_1}-\frac{1}{x+\Omega_\lambda-i\alpha_2}\right|^2\right\}dx. \quad (14)$$

It can be seen from this that w_μ will contain resonance terms with denominators of the forms

$$(\Omega_\mu-\Omega_\lambda)^2+4\Gamma_1^2,\quad (\Omega_\mu-\Omega_\lambda)^2+4\Gamma_2^2,$$
$$(\Omega_\mu+2\delta_1)^2+(\Gamma_1+\Gamma_2)^2,\quad (\Omega_\mu+2\delta_2)^2+(\Gamma_1+\Gamma_2)^2. \quad (15)$$

The coefficients of these terms, which depend on $\delta_{1,2}$, $\Gamma_{1,2}$, and Ω_λ, can be calculated by integrating Eq. (14). The general formulas for the coefficients are very cumbersome, however, and we shall not give them, but shall confine ourselves to a qualitative analysis of the line shape and numerical illustrations.

Let us first consider the case $\Omega_\lambda = \omega_\lambda - \omega_{32} = 0$, in which the line is symmetrical with respect to $\Omega_\mu = \omega_\mu - \omega_{32} = 0$. If the inducing field is not too large, so that $G^2 < (\gamma_2-\gamma_3)^2/4$, then $\delta_1 = \delta_2 = 0$, and $\Gamma_1 \neq \Gamma_2$ are determined by Eq. (5), from which it can be seen that $\Gamma_1 > \Gamma_2$. Consequently, $2\Gamma_1 > \Gamma_1+\Gamma_2 > 2\Gamma_2$, and of the four terms listed in Eq. (15) the one that decreases most sharply with increase of Ω_μ is the term with Γ_2. It can be shown that the coefficient of this term is negative and increases in absolute value with increasing G. Therefore the general picture of the change of the line shape reduces to the following (Fig. 3, a): for $G^2 \ll (\gamma_2-\gamma_3)^2/4$ there is an ordinary line of the dispersion shape with width $\gamma_2+\gamma_3$ and integrated emission probability γ_{32}/γ_2 (curve 1); as G increases a minimum appears in the center of the line (curves 2 — 4), and remains right up to the value $G^2 = (\gamma_2-\gamma_3)^2/4$.

Further increase of the external field leads to the appearance of imaginary parts $\delta_{1,2}$, and for $\Omega_\lambda = 0$ we have

$$\Gamma_1 = \Gamma_2 = (\gamma_2+\gamma_3)/2,$$
$$\delta_1 = -\delta_2 = \sqrt{G^2-(\gamma_2-\gamma_3)^2/4}. \quad (16)$$

Consequently, the emission line in this case consists of three components with equal widths $\gamma_2+\gamma_3$, separated from each other by distances $2\delta_1$. The splitting of the line becomes detectable, however, only for $\delta_1 \gtrsim \gamma_2+\gamma_3$, i.e., for

$$G^2 \gtrsim \gamma_2\gamma_3\left[\frac{3\gamma_2}{4\gamma_3}+\frac{1}{2}+\frac{3\gamma_3}{4\gamma_2}\right]. \quad (17)$$

Different values of G are required, depending on the value of γ_2/γ_3, but in any case the splitting of the line will be appreciable only when there is comparatively strong saturation, i.e., for $G^2 \gg \gamma_2\gamma_3$. Thus from the very start of the splitting the integrated intensity of the line is practically unchanged, and as G increases its two equal components move farther and farther apart. This can be seen well in curves 1 and 2 of Fig. 3, b. The former case corresponds to $N_\lambda = 4N_\lambda^0$, the latter to $N_\lambda = 9N_\lambda^0$. We note, finally, that at the indicated values of the field the oscillations of $a(3, n_\lambda, n_\mu)$ are still very strongly damped; in the former case the mean lifetime $1/(\gamma_2+\gamma_3)$ of an atom in level 3 is about $1/3$ of the period of oscillation, and in the latter case it is $2/3$ of the period.

With departure from resonance ($\Omega_\lambda \neq 0$) the line becomes extremely asymmetrical: the maximum of one of the side components approaches ω_{32}, and its magnitude increases sharply. At the same time the other two terms become smaller and the positions of their maxima go farther from ω_{32}.

Let us turn to the transitions induced by a weak field. For the initial conditions (9) the induced emission coefficient k_μ (dimensions cm^{-1}) is given by[4)]

$$k_\mu = \frac{\lambda^2}{4\pi^2}\frac{Q\gamma_{32}\gamma_2}{|\alpha_1-\alpha_2|^4}\int_{-\infty}^{\infty}\left\{\left|\frac{\alpha_2+\gamma_3}{x+\Omega_\mu-i\alpha_1}-\frac{\alpha_1+\gamma_2}{x+\Omega_\mu-i\alpha_2}\right|^2\right.$$

$$\times\left|\frac{\alpha_1+\gamma_3}{x+\Omega_\lambda-i\alpha_1}-\frac{\alpha_2+\gamma_3}{x+\Omega_\lambda-i\alpha_2}\right|^2$$

$$-G^4\left|\frac{1}{x+\Omega_\mu-i\alpha_1}-\frac{1}{x+\Omega_\mu-i\alpha_2}\right|^2\left|\frac{1}{x+\Omega_\lambda-i\alpha_1}\right.$$

$$\left.\left.-\frac{1}{x+\Omega_\lambda-i\alpha_2}\right|^2\right\}dx, \qquad (18)$$

where Q is the number of acts of excitation per unit volume and unit time. For the initial conditions (10) the formula for k_μ differs from Eq. (18) by a change of sign and replacement of γ_2 by γ_3 in the coefficient of the integral.

For $G^2 < (\gamma_2-\gamma_3)^2/4$ the nature of the change of the frequency dependence of k_μ is approximately the same as for the spontaneous emission line (Fig. 4, a). In the case $G^2 > (\gamma_2-\gamma_3)^2/4$, on the other hand, there is a decided difference between k_μ and w_μ: for sufficiently large G, k_μ is negative in a certain range of frequencies, i.e., there is absorption of a weak field of these frequencies (curves 3 and 4 in Fig. 4, b).

An interesting property of k_μ is that for $\omega_\mu \to \omega_\lambda$ it is not equal to the absorption coefficient for a strong field, $k_\lambda = \lambda^2 QW_\lambda/4N_\lambda$; k_λ is always larger than k_μ, as can be seen from Fig. 4, where k_μ/k_λ is plotted as the ordinate. We note that the discontinuity of the emission coefficient is of great importance for understanding the operation of quantum generators.

[4]The formula (18) is for the case of a gas and $k_\mu\lambda \ll 1$. By using the results of Ginzburg[18] one can easily extend the formula to the case of a medium with $\varepsilon \neq 1$ and small $k_\mu\lambda$. This comment also applies to the subsequent formulas for the dielectric permittivity.

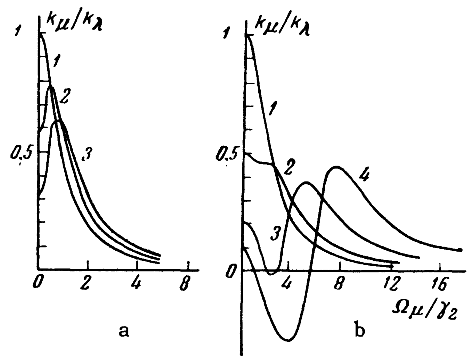

FIG. 4. Emission coefficient k_μ. Case a is for $\gamma_3 = \gamma_{32} = 0.1\,\gamma_2$; curve 1 is for $G^2 = 0$, curve 2 $G^2 = 0.08\,\gamma_2^2$, curve 3 for $G^2 = (\gamma_2-\gamma_3)^2/4$. Case b is for $\gamma_3 = \gamma_2$; curve 1 is for $G^2 = 0.2$, curve 2 for $G^2 = 4\,\gamma_2^2$, and curve 3 for $G^2 = 9\,\gamma_2^2$.

In a number of problems arising in the analysis of devices with negative resistance it is necessary to know the complex dielectric constant $\epsilon = \epsilon' + i\epsilon''$ of the medium. The imaginary part of ϵ is determined from the emission coefficient (for $k_\mu\lambda \ll 1$, cf. footnote [4)])

$$\varepsilon''(\omega_\mu) = -(\lambda/2\pi)\,k_\mu. \qquad (19)$$

For $\omega_\mu \neq \omega_\lambda$ the real part of ϵ can be found by means of the Kramers-Kronig formula:[19]

$$\varepsilon'(\omega_\mu) = 1 - \frac{\lambda}{2\pi^2}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{k_\mu(x)}{x-\omega_\mu}\,dx. \qquad (20)$$

Curves of $\epsilon' - 1$ for the cases considered above are shown in Fig. 5, a for $\omega_\mu > \omega_{32}$ [$\epsilon'(\Omega_\mu) - 1 = -\epsilon'(-\Omega_\mu) + 1$]. Like the emission coefficient, $\epsilon' - 1$ decreases in absolute value with increase of G. For $G^2 < (\gamma_2-\gamma_3)^2/4$ (Fig. 5, a) we note the fact that near ω_{32} the dispersion is positive and $\epsilon' - 1$ is zero at three points. For $\gamma_2 = \gamma_3$ and comparatively small G^2 there is negative dispersion (curves 1, 2 of Fig. 5, b). For sufficiently large G^2, when the splitting of the spontaneous emission line is appreciable, $\epsilon' - 1$ goes to zero at five points, and near ω_{32} the dispersion is still negative.

The dielectric permittivity at the frequency ω_λ must be determined separately because of the discontinuity at the point ω_λ. Calculating the energy of the interaction of the atom with the field by means of perturbed wave functions, we arrive at the following formulas:

$$\varepsilon'' = -\frac{\lambda^3}{8\pi^2}\,Q\,\frac{\gamma_2+\gamma_3}{\gamma_3}\,\frac{\gamma_{32}}{\Omega_\lambda^2 + (\gamma_2+\gamma_3)^2 + G^2(\gamma_2+\gamma_3)^2/\gamma_2\gamma_3}, \qquad (21)$$

$$\varepsilon' - 1 = -\varepsilon''\,\frac{\Omega_\lambda}{\gamma_2+\gamma_3}.$$

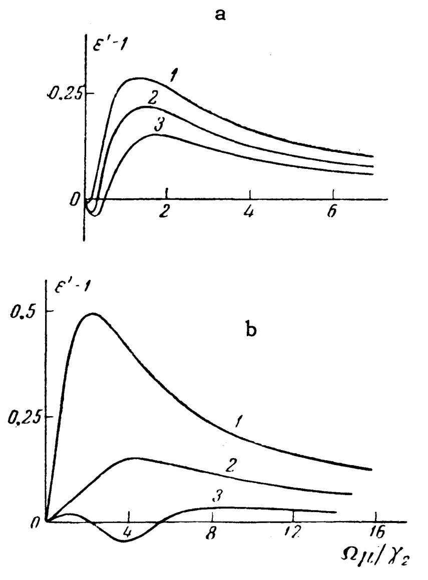

FIG. 5. Real part of the dielectric permittivity [in units $\gamma_{32}\lambda^3 Q/8\pi^2(\gamma_3+\gamma_2)^2$]. Case a is for $\gamma_3 = \gamma_{32} = 0.1\ \gamma_2$; curve 1 is for $G^2 = 0.04\ \gamma_2^2$, curve 2 for $G^2 = 0.08\ \gamma_2^2$, and curve 3 for $G^2 = 0.14\ \gamma_2^2$. Case b is for $\gamma_3 = \gamma_2$; curve 1 for $G^2 = 0$, curve 2 for $G^2 = 4\ \gamma_2^2$, and curve 3 for $G^2 = 9\ \gamma_2^2$.

If $\gamma_2 = \gamma_3$, Eq. (21) agrees with the formulas of Basov and Prokhorov[2] if we set $2\gamma_2 = 1/\tau$, where τ is the lifetime of the excited state.

In conclusion let us consider the extension of the formulas we have given to cases in which the widths of the levels are not due to spontaneous transitions only. Here we must distinguish between elastic and inelastic processes. If all inelastic processes, including also spontaneous transitions, lead to lifetimes τ_2 and τ_3 for levels 2 and 3, then in all of the formulas we must set $2\gamma_2 = 1/\tau_2$, $2\gamma_3 = 1/\tau_3$. Elastic processes (Doppler effect, inhomogeneity of the medium) lead to broadening of the lines on account of changes of the frequency ω_{32} of the transition. Since ω_{32} is involved in α_1, α_2, A_1, A_2, we must average the quantity in question over the possible values of ω_{32} with appropriate weights. For example, if the probability distribution for the various values of ω_{32} is of the form

$$\psi(\omega_{32}) = \frac{\Delta\nu/\pi}{(\omega_{32}-\overline{\omega}_{32})^2 + (\Delta\nu)^2}, \tag{22}$$

the probability of induced emission is

$$\overline{W}_\lambda = \int_{-\infty}^{\infty} W_\lambda \psi(\omega_{32})\, d\omega_{32}$$

$$= \frac{\Delta\nu + (\gamma_2+\gamma_3)\sqrt{1+G^2/\gamma_2\gamma_3}}{\gamma_3\sqrt{1+G^2/\gamma_2\gamma_3}}\,\frac{G^2}{\overline{\Omega}_\lambda^2 + [\Delta\nu + (\gamma_2+\gamma_3)\sqrt{1+G^2/\gamma_2\gamma_3}\,]^2},$$

$$\overline{\Omega}_\lambda = \omega_\lambda - \overline{\omega}_{32}. \tag{23}$$

In particular, it can be seen from Eq. (23) that for $\Delta\nu \gg \gamma_2 + \gamma_3$ the degree of saturation is determined by the parameter $G(\gamma_2+\gamma_3)/[\Delta\nu(\gamma_2\gamma_3)^{1/2}]$, which involves the characteristics of both the elastic and the inelastic processes.

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