ON RELATIVISTIC PERTURBATION THEORY FOR A COULOMB FIELD

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The relativistic perturbation series for a Coulomb field is summed. The relativistic perturbation series is represented in the form of a new series each term of which can be expressed linearly through an infinite nonrelativistic series. An integral equation defining the terms of this series is determined. All infrared divergences are collected together in a phase factor.

1. INTRODUCTION

THE absence of an analytical formula for the "incoming" ("outgoing")[1] wave function of a charged relativistic particle in the Coulomb field necessitates the use of various approximate expressions for this function whenever calculations are made. The first type of approximation consists in using only a finite number of terms in the expansion of the wave function in terms of states with given angular momentum and parity. Calculations of this type are called "exact" (see, for example, reference 1). If the series converges, this method allows us, in principle, to achieve any degree of accuracy if a sufficient number of terms is included. However, such calculations involve an enormous amount of numerical work and have, therefore, been carried out only for very few particular cases.

A second method makes use of the Born approximation, i.e., it is based on perturbation theory.[2,3] Since the convergence properties of the Born series are not known and, furthermore, each term of the series contains infrared divergences connected with the long range of the Coulomb field, there is notoriously little trust in the perturbation-theoretical approach. As a consequence, the limit of applicability of perturbation theory is extremely low ($\alpha ZE/p \ll 1$), which restricts the use of this method to the uninteresting region of small Z.

The third method makes use of the Furry-Sommerfeld-Maue (FSM) function, which has a convenient analytic form.[4] However, this function must in general be regarded as containing only the first two terms (the zeroth and the first) of the expansion of the exact function in powers of αZ.

It is shown in the present paper that the summation of the nonrelativistic perturbation series leads to the known expression for the nonrelativistic wave function. All infrared divergences are collected together in a phase factor and, therefore, do not affect the physical processes. Furthermore, it turns out that the relativistic perturbation series can be represented in the form of a series in powers of αZ, each term of which is expressed linearly through the infinite nonrelativistic series, and hence through a known function (see the remarks above). No further infrared divergences appear aside from those collected together in a phase factor in the nonrelativistic series. Thus all infrared divergences are separated out in the form of this phase factor also in the case of the complete relativistic wave function. The resulting expansion remains valid for small momenta, in contrast to the Born expansion.

2. SUMMATION OF THE PERTURBATION SERIES IN THE NONRELATIVISTIC CASE

In order to cope with the infrared divergences, one usually considers first a potential of the form $eZe^{-\lambda r}/r$ and then lets λ go to zero.[2] The Schrödinger equation for the "outgoing" wave function of a charged particle in this potential field has the following form in momentum space ($\hbar = c = 1$):

$$\varphi_0(\mathbf{p}, \mathbf{f}) = \delta(\mathbf{q}_{fp}) + 2E\beta \frac{1}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{1}{\mathbf{q}_{fs}^2 + \lambda^2} \varphi_0(\mathbf{p}, \mathbf{s})\, d^3s;$$
$$\beta = \alpha Z/2\pi^2, \qquad \alpha = e^2 = 1/137, \qquad \mathbf{q}_{fs} = \mathbf{f} - \mathbf{s}, \quad (1)$$

where Z is the charge of the nucleus ($Z > 0$ for attraction), and E and **p** are the energy and momentum of the particle.[2)]

[1)]We shall use the term "outgoing" ("incoming") for functions which have the asymptotic form of the sum of a plane wave and a diverging (converging) wave.

[2)]This equation cooresponds to the ordinary Schrödinger equation for $E = m$; however, in the following we shall regard E as the relativistic energy of the particle: $E^2 = p^2 + m^2$. We note that **p** is the momentum of the particle in the usual sense only in the asymptotic region, where the field vanishes, because E is the total energy of the particle.

Using the method of successive approximations, we can write (1) in the form of a series in powers of $2E\beta$:

$$\varphi_0(\mathbf{p}, \mathbf{f}) = \delta(\mathbf{q}_{fp}) + \sum_{n=1}^{\infty} (2E\beta)^n \varphi_0^n(\mathbf{p}, \mathbf{f}), \tag{2}$$

$$\varphi_0^n(\mathbf{p}, \mathbf{f}) = \int \frac{d^3 s_1}{(\mathbf{q}_{ps_1}^2 + \lambda^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{s_1 s_2}^2 + \lambda^2)} \cdot \int \frac{d^3 s_2}{(\mathbf{s}_2^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{s_2 s_3}^2 + \lambda^2)} \cdots \int \frac{d^3 s_{n-1}}{(\mathbf{s}_{n-1}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{s_{n-1} f}^2 + \lambda^2)} \frac{1}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon}. \tag{2a}$$

Starting with the first term, we apply successively the following relation to (2a) (see reference 5, Appendix A):

$$\int \frac{d^3 s}{(\mathbf{q}_{sr}^2 - a^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{sk}^2 + \lambda^2)} = \int_0^1 \frac{dx}{\Lambda} \int \frac{1}{2} \frac{\partial}{\partial \Lambda} \frac{1}{\mathbf{q}_{fr_x}^2 - \Lambda^2} \frac{d^3 s}{\mathbf{q}_{sk}^2 + \lambda^2} = \pi^2 i \int_0^1 \frac{dx}{\Lambda} \frac{1}{\mathbf{q}_{kr_x}^2 - (\Lambda + i\lambda)^2},$$

$$\Lambda^2 = (p^2 - r^2 x)(1 - x) + a^2 x + i\varepsilon(1 - x), \quad \mathbf{r}_x = \mathbf{r}x, \tag{3}$$

Then (2a) can be written in the form

$$\varphi_0^n(\mathbf{p}, \mathbf{f}) = \left(\frac{\pi^2 i}{p}\right)^n \int_0^1 \frac{dy_1}{\Lambda_1} \int_0^1 \frac{dy_2}{\Lambda_2} \cdots \int_0^1 \frac{dy_n}{\Lambda_n} \frac{1}{2\pi^2 i} \frac{\partial}{\partial(p\Lambda_n)} \frac{1}{\mathbf{q}_{fp_n}^2 - (p\Lambda_n)^2}, \tag{4}$$

where

$$p^2 \Lambda_k^2 = (p^2 - p_{k-1}^2 y_k)(1 - y_k) + (\Lambda_{k-1} + i\lambda)^2 y_k + i\varepsilon(1 - y_k),$$

$$\mathbf{p}_k = \mathbf{p} y_1 y_2 \ldots y_k,$$

or

$$\Lambda_k^2 = (1 - y_1 y_2 \ldots y_k)^2 - \eta^2 y_1 y_2 \ldots y_k + (2i\eta\Lambda_1 - \eta^2) y_2 y_3 \ldots y_k + (2i\eta\Lambda_{k-1} - \eta^2) y_k + i\varepsilon(1 - y_k),$$

$$\eta = \lambda / p.$$

Introducing the new variables

$$x_k = 1 - y_1 y_2 \ldots y_k, \qquad dy_k = -dx_k/(1 - x_{k-1}),$$

we obtain

$$\varphi_0^n(\mathbf{p}, \mathbf{f}) = \left(\frac{\pi^2 i}{p}\right)^n \int_0^1 \frac{dx_1}{(1 - x_1)\Lambda_1} \int_{x_1}^1 \frac{dx_2}{(1 - x_2)\Lambda_2} \cdots \int_{x_{n-1}}^1 \frac{dx_n}{(1 - x_n)\Lambda_n} F(\mathbf{f}, \mathbf{p}; x_n),$$

$$F(\mathbf{f}, \mathbf{p}; x_n) = \frac{1}{2\pi^2 i}(1 - x_n) \frac{\partial}{\partial(p\Lambda_n)} \frac{1}{\mathbf{q}_{fp_n}^2 - (p\Lambda_n)^2},$$

$$\Lambda_k^2 = x_k^2 - \eta^2 x_k + (2i\eta\Lambda_1 - \eta^2)\frac{1 - x_n}{1 - x_1} + \ldots + (2i\eta\Lambda_{n-1} - \eta^2)\frac{1 - x_n}{1 - x_{n-1}} + i\varepsilon \frac{x_n - x_{n-1}}{1 - x_{n-1}}. \tag{5}$$

The following relation will be important for the subsequent discussion:

$$\Lambda_n |_{x_n = x_{n-1}} = \Lambda_{n-1} + i\eta. \tag{6}$$

Let us now introduce a quantity a such that

$$\eta \ll a \ll 1. \tag{7}$$

We split up all integrals in (5) into a part with $x_k < a$ and a part with $x_k > a$ and neglect a as compared to unity and η as compared to a, according to (7). We then obtain

$$\varphi_0^n(\mathbf{p}, \mathbf{f}) = \left(\frac{\pi^2 i}{p}\right)^n \{A_0 B_n + A_1 B_{n-1} + \ldots + A_k B_{n-k} + \ldots + A_n B_0\}, \tag{8}$$

$$B_k = B_k(a) = \int_a^1 \frac{dx_1}{(1 - x_1)x_1} \cdots \int_{x_{k-1}}^1 \frac{dx_k}{(1 - x_k)x_k} F(x_k) = \int_a^1 \frac{dx_1}{(1 - x_1)x_1} B_{k-1}(x_1), \tag{8a}$$

$$B_0(x) = F(x) = -\frac{1}{2\pi^2} \frac{\partial}{\partial\varepsilon} \frac{1 - x}{(\mathbf{q}_{fp} + \mathbf{p}x)^2 - (|\mathbf{p}|x + i\varepsilon)^2}, \tag{8b}$$

$$A_k = \int_0^a \frac{dx_1}{\Lambda_1} \int_{x_1}^a \frac{dx_2}{\Lambda_2} \cdots \int_{x_{k-1}}^a \frac{dx_k}{\Lambda_k}, \qquad A_0 = 1, \tag{8c}$$

$$\Lambda_k^2 = x_k^2 + 2i\eta(\Lambda_1 + \Lambda_2 + \ldots + \Lambda_{k-1}) - k\eta^2. \tag{8d}$$

The function $B_0(x)$ goes over into $\delta(\mathbf{q}_{fp})$ for $x, \epsilon \to 0$.

In (8b) we have replaced $(px)^2 + i\epsilon$ by $(px + i\epsilon)^2$, which allowed us to include the zero order term (2) [corresponding to the case when x is exactly zero in (8b)] in (8) without changing the rules for bypassing the poles.

Expression (8) agrees with the n-th term of the product of two series. We can therefore write (2) in the form

$$\varphi_0(\mathbf{p}, \mathbf{f}) = \sum_{k=0}^{\infty} \alpha^k B_k \sum_{m=0}^{\infty} \alpha^m A_m = B(\alpha, \mathbf{p}, \mathbf{f}, a) A(\alpha, \eta, a), \tag{9}$$

$$B(\alpha, \mathbf{p}, \mathbf{f}, a) = \sum_{k=0}^{\infty} \alpha^k B_k, \tag{9a}$$

$$A(\alpha, \eta, a) = \sum_{m=0}^{\infty} \alpha^m A_m. \tag{9b}$$

Here $\alpha = \pi^2 i 2E\beta/p = ie^2ZE/p = i\xi$.

It is easy to find an analytic expression for the function (9a). By integrating (8a) by parts, starting with the last term, and using $F(1) = 0$, we find

$$B_k = -\frac{1}{k!}\int_a^1 \left(\ln\frac{x}{1-x} - \ln a\right)^k dF(x), \qquad k = 0, 1, \ldots \tag{10}$$

Thus we obtain for (9a)

$$B(\alpha, \mathbf{p}, \mathbf{f}, a) = -\int_a^1 \exp\left[\alpha\left(\ln\frac{x}{1-x} - \ln a\right)\right] dF(x)$$

$$= -a^{-\alpha}\int_a^1 x^\alpha (1-x)^{-\alpha}\, dF(x). \tag{11}$$

Integrating (11) by parts and using the fact that

$$\alpha\int_0^a x^{\alpha-1}(1-x)^{-\alpha-1}F(x)dx = F(0)a^\alpha,$$

with an accuracy up to terms of order a, we find

$$B(\alpha, \mathbf{p}, \mathbf{f}, a) = a^{-\alpha}\alpha\int_0^1 x^{\alpha-1}(1-x)^{-\alpha-1}F(x)dx \tag{12}$$

Let us now consider (9b). We change the variables in (8c) to $z_k = x_k + \Lambda_k$, taking into account (8d). Considering the fact that (6) implies

$$z_k|_{x_k=x_{k-1}} = z_{k-1} + i\eta,$$

we obtain ($\gamma = i\eta$)

$$A_k = A_k(\gamma) = \int_\gamma^{2a}\frac{dz_1}{z_1}\int_{z_1+\gamma}^{2a}\frac{dz_2}{z_2}\cdots\int_{z_{k-1}+\gamma}^{2a}\frac{dz_k}{z_k} = \int_\gamma^{2a}\frac{dz}{z}A_{k-1}(z+\gamma). \tag{13}$$

As in the case of expression (8a), we integrate by parts, starting with the last term, and use the relation $|z_k| \geq |\gamma|$. We then obtain for (13) (see Appendix A)[3]

$$A_k = b_0\frac{1}{k!}\ln^k\frac{2a}{\gamma} + b_2\frac{1}{(k-2)!}\ln^{k-2}\frac{2a}{\gamma}$$

$$+ \ldots + b_n\frac{1}{(k-n)!}\ln^{k-n}\frac{2a}{\gamma} + \ldots + b_k, \tag{14a}$$

$$b_k = b_k(x=1)$$

$$= \int_0^{x=1}\frac{dy}{y}\left\{\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}[\ln^n(1+y)]b_{k-n-1}\left(\frac{y}{1+y}\right) - b_{k-1}(y)\right\}, \tag{14b}$$

$$b_0 = 1, \qquad b_1 = 0, \qquad b_2 = -\frac{\pi^2}{12}. \tag{14c}$$

We note that (14a) represents the general term of the product of two series, in analogy to (8). We therefore find for (9b)

$$A(\alpha, \eta, a) = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\left(\ln^k\frac{2a}{\gamma}\right)\sum_{m=0}^{\infty}\alpha^m b_m = a^\alpha\exp\left(\alpha\ln\frac{2p}{i\lambda}\right)b(\alpha), \tag{15}$$

$$b(\alpha) = \sum_{m=0}^{\infty}\alpha^m b_m. \tag{15a}$$

In Appendix A we shall make an estimate of the radius of convergence of (15a) and show that it is at least larger than unity. Substituting (15) and (12) in (9), we obtain finally

$$\varphi_0(\mathbf{p}, \mathbf{f}) = e^{\pi\xi/2}\, i\xi b(i\xi)$$

$$\times\int_0^1 x^{i\xi-1}(1-x)^{-i\xi-1}F(x)dx\exp\{i\xi\ln(2p/\lambda)\}. \tag{16}$$

In going over to coordinate space,

$$\psi(\mathbf{p}, \mathbf{r}) = \int e^{i\mathbf{f}\mathbf{r}}\varphi(\mathbf{p}, \mathbf{f})d^3f, \tag{17}$$

we have

$$\psi_0(\mathbf{p}, \mathbf{r}) = e^{\pi\xi/2}\, i\xi\, b(i\xi)$$

$$\times\int_0^1 x^{i\xi-1}(1-x)^{-i\xi}e^{i(pr-\mathbf{pr})x}\,dx\, e^{i\mathbf{pr}}\exp(i\xi\ln(2p/\lambda)). \tag{18}$$

The known expression for the "outgoing" nonrelativistic function of a charged particle in the Coulomb field has the form

$$\psi_0(\mathbf{p}, \mathbf{r}) = Ne^{i\mathbf{pr}}{}_1F_1(i\xi, 1; i\rho), \qquad \rho = pr - \mathbf{pr}, \tag{19a}$$

$$N = e^{\pi\xi/2}\Gamma(1 - i\xi) \tag{19b}$$

or

$$\psi_0(\mathbf{p}, \mathbf{r}) = e^{\pi\xi/2}\, i\xi\frac{1}{\Gamma(1+i\xi)}\int_0^1 x^{i\xi-1}(1-x)^{-i\xi}e^{i(pr-\mathbf{pr})x}\,dx e^{i\mathbf{pr}}. \tag{19c}$$

Since the functions (18) and (19c) satisfy the same equation and boundary conditions and have the same normalization (the incident wave has unit amplitude), it follows that $b(i\xi)$ must differ from $1/\Gamma(1 + i\xi)$ only by a phase factor within its radius of convergence. Comparing the first few terms of the expansions of $b(i\xi)$ [formula (14c)] and of $1/\Gamma(1 + i\xi)$ (reference 6), we obtain

$$b(i\xi) = e^{-i\xi C}/\Gamma(1 + i\xi), \tag{20}$$

where C is the Euler constant. Thus the series (2) differs from the function (19a) by the phase factor

$$M' = \exp\{i\xi\ln(2p/\lambda)\}\exp\{-i\xi C\}.$$

However, if we agree to regard the normalization factor (19b) as defined only by its modulus, we can leave out the phase factor in (20). Then the difference between (18) and (19c) is given by the infrared phase factor

[3] We note that, if we set equal to zero all λ in (2a) except the first, the result will be convergent and $A_k = (k!)^{-1}\ln^k(2a/\gamma)$, i.e., this leads to a violation of the normalization of the wave function.

$$M=\exp\{i\xi\ln(2p/\lambda)\}. \tag{21}$$

By analytic continuation of (20) into the region in which convergence is guaranteed, we find that the radius of convergence of $b(i\xi)$ is infinite, since the right-hand side of (20) has no poles. For an estimate of the radius of convergence of (9a), we write (12) in coordinate space [see formula (18)] and replace the oscillatory exponent by unity, assuming $\mathbf{r}=0$. Then we obtain

$$\alpha\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}dx=\frac{\alpha\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}=\frac{\pi\alpha}{\sin\pi\alpha}. \tag{22}$$

The right-hand side of this expression has its first pole at $\alpha=1$. It follows from this that the radius of convergence of (9a) is equal to unity. Thus the over-all radius of convergence of (2) is also equal to unity.

The perturbation series (2) for the Coulomb field is a majorant for potentials with shorter range than the Coulomb potential. For example, it is easily seen that the modulus of the general term (2a) decreases as λ increases. Since the series (9a), (9b), and hence (2) are absolutely convergent for small but finite λ, it can be asserted that the perturbation series for the Yukawa potential has a radius of convergence which is not smaller than unity.

It is easy to verify that the replacement of $i\epsilon$ by $-i\epsilon$ converts the "outgoing" function into the "incoming" one. Indeed, since the series is convergent, it follows from (2a) that

$$\varphi_0(\mathbf{p},\mathbf{f})=\varphi_0(-\mathbf{p},-\mathbf{f});\qquad \varphi_0^*(\mathbf{p},\mathbf{f};i\varepsilon)=\varphi_0(\mathbf{p},\mathbf{f};-i\varepsilon). \tag{23}$$

Taking the complex conjugate of both sides of (17) and using (23), we easily obtain the well-known relation

$$\psi_0^{in}(\mathbf{p},\mathbf{r})=\{\psi_0^{out}(-\mathbf{p},\mathbf{r})\}^*. \tag{23a}$$

In this way we have obtained the result that the perturbation series for the "incoming" and "outgoing" wave functions of a nonrelativistic charged particle in the Coulomb field converges for $\xi=\alpha ZE/p\leq 1$ and diverges for $\xi>1$. It follows, in particular, that the series diverges for arbitrarily small values of Z if $p=0$.

The limit of unity for the radius of convergence is due only to a single term in (19c), namely, the function $\Gamma(1-i\xi)$ in the normalization factor, which has a pole for $i\xi=1$. All the other factors in (19c) have an infinite radius of convergence. If we use a finite number of terms of the perturbation series (2), but separate out the expansion of the normalization factor (19b) and replace it by its exact value, we can therefore extend the radius of convergence of the remaining expansion to values which are not limited by the condition $\xi\leq 1$.

Furthermore, it is easily seen from (19a) that the expansion in terms of ξ contains only the normalization factor (19b) [in (18) also the phase factor]. The expansion of the confluent hypergeometric function has the form

$$F(i\xi;1;i\rho)=1+i\xi\sum_{k=1}^{\infty}\frac{1}{k\cdot k!}(i\rho)^k+\ldots+(i\xi)^n\sum_{k=n}^{\infty}C_k(i\rho)^k+\ldots,$$

i.e., each ξ^n stands together with a factor ρ^{n+k} $k\geq 0$. Hence the expansion is really in terms of αZ, where each term has a finite value and is proportional to $(\alpha Z)^n$ for arbitrary values of the momentum. It follows that the wave function can be obtained with an accuracy up to terms of order $(\alpha Z)^n$, independently of the momentum, in the following way: take n successive Born approximations, leave out the expansion of the phase and normalization factors and multiply by the exact value of the normalization factor (19b).

3. DISCUSSION OF THE RELATIVISTIC PERTURBATION THEORY

This section is a direct continuation and generalization of the discussion presented in an earlier paper of the author.[5] We write the Dirac equation for the "outgoing" wave in momentum space:

$$\varphi(\mathbf{p},\mathbf{f})=\delta(\mathbf{q}_{fp})u(\mathbf{p})+\beta\frac{m-i\hat{f}}{\mathbf{f}^2-\mathbf{p}^2-i\varepsilon}\int\frac{\gamma_n}{\mathbf{q}_{fs}^2+\lambda^2}\varphi(\mathbf{p},\mathbf{s}), \tag{24}$$

$$(i\hat{p}+m)u(\hat{p})=0,\qquad \hat{f}=\boldsymbol{\gamma}\cdot\mathbf{f}+\gamma_4 f_4,$$

$$f_0=E,\qquad \beta=\alpha Z/2\pi^2. \tag{24a}$$

Let us set

$$\varphi(\mathbf{p},\mathbf{f})=\{\varphi_0(\mathbf{p},\mathbf{f})+\Phi(\mathbf{p},\mathbf{f})\}u(\mathbf{p}). \tag{25}$$

Using (1), (24a) and the identity $(\tilde{q}=\boldsymbol{\alpha}\cdot\mathbf{q})$

$$(m+i\hat{f})\gamma_4=2E+\tilde{q}_{fp}+\gamma_4(m+i\hat{p}),$$

$$\beta\frac{\tilde{q}_{fp}}{\mathbf{f}^2-\mathbf{p}^2-i\varepsilon}\int\frac{1}{\mathbf{q}_{fs}^2+\lambda^2}\varphi_0(\mathbf{p},\mathbf{s})\,d^3s=\frac{1}{2E}\tilde{q}_{fp}\varphi_0(\mathbf{p},\mathbf{f})\equiv\varphi_{\mathrm{I}}(\mathbf{p},\mathbf{f}), \tag{26}$$

we substitute (25) in (24) and obtain the following equation[4)] for $\Phi(\mathbf{p},\mathbf{f})$:

[4)]In obtaining (26) we have used $\mathbf{q},\delta(\mathbf{q})=0$; thus the expansion of φ_{I} in powers of ξ begins with ξ. We note that our discussion is valid for any static potential, where the function $\varphi_0+\varphi_{\mathrm{I}}$ will be the analogue of the FSM function.

$$\Phi(\mathbf{p}, \mathbf{f}) = \varphi_I(\mathbf{p}, \mathbf{f}) + \beta \frac{m - i\hat{f}}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{\gamma_4}{\mathbf{q}_{fs}^2 + \lambda^2} \Phi(\mathbf{p}, \mathbf{s})\, d^3s. \tag{27}$$

Writing (27) in the form of a series in powers of αZ, we have

$$\varphi(\mathbf{p}, \mathbf{f}) = \{\varphi_0(\mathbf{p}, \mathbf{f}) + \alpha Z \varphi_1(\mathbf{p}, \mathbf{f}) + (\alpha Z)^2 \varphi_2(\mathbf{p}, \mathbf{f}) + \ldots\} u(\mathbf{p}), \tag{28}$$

where $\varphi_0(\mathbf{p}, \mathbf{f})$ is defined by (1) and

$$\varphi_1(\mathbf{p}, \mathbf{f}) = \frac{1}{2p\xi} \tilde{q}_{fp} \varphi_0(\mathbf{p}, \mathbf{f}), \qquad \xi = \frac{\alpha Z E}{p}; \tag{28a}$$

$$\varphi_2(\mathbf{p}, \mathbf{f}) = \frac{1}{2\pi^2} \frac{(m - i\hat{f})\gamma_4}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{\varphi_1(\mathbf{p}, \mathbf{s})}{\mathbf{q}_{fs}^2 + \lambda^2} d^3s, \tag{28b}$$

$$\varphi_n(\mathbf{p}, \mathbf{f}) = \frac{1}{2\pi^2} \frac{(m - i\hat{f})\gamma_4}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{\varphi_{n-1}(\mathbf{p}, \mathbf{s})}{\mathbf{q}_{fs}^2 + \lambda^2} d^3s. \tag{28c}$$

The series (28) can also be obtained directly from the expansion of (24) in powers of ξ. For this purpose we must expand the numerators in each term of the resulting series, using the identity

$$\prod_{n=1}^{k} (m - i\hat{s}_n)\gamma_4 = (2E)^k + \sum_{n=1}^{k} (2E)^{k-n} (m - i\hat{s}_1)\gamma_4 \ldots (m - i\hat{s}_{n-1})\gamma_4 \tilde{q}_{s_n p}.$$

We must then collect all terms with the same matrix structure of the numerator and different powers of 2E. Then (24) appears in the form of the series (28), where $\varphi_0(\mathbf{p}, \mathbf{f})$ is given by the series (2). The first three terms of (28) were obtained by this method in reference 5. According to the discussion of the paper just mentioned (see footnote 3), the terms of the series (28), where φ_0 is replaced by the series (2), will not contain any additional infrared divergences besides those contained in φ_0 [this also follows directly from the convergence of the integrals (28c) for $\lambda \to 0$]. Since all terms in (28) are linearly expressed in terms of φ_0, all infrared divergences can be separated out of the complete function (25) in the form of the phase factor (21). It can also easily be shown that φ_0 and φ_1 give a finite contribution for $p \to 0$. Let us assume that the same holds also for φ_{n-1}. Replacing $\mathbf{f}$ by $p\mathbf{f}_1$ and $\mathbf{s}$ by $p\mathbf{s}_1$ ($\lambda, \epsilon \to 0$) in (28c), we see that $\varphi_n \sim 1/p^3$ and hence $\varphi_n d^3f = \varphi_n p^3 d^3f_1$ is finite. Thus the expansion (28) is valid even for $p \to 0$.

The integrals appearing in (28b, c) can be transformed to a simpler form. Let us consider, for example, the function (28b), the first correction to the FSM function. Following reference 5, we can write (28b) in the form

$$\varphi_2(\mathbf{p}, \mathbf{f}) = \frac{1}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \{2\mathbf{p} \cdot \mathbf{J} + \tilde{q}_{fp} \tilde{J}\}, \tag{29}$$

$$\mathbf{J}(\mathbf{p}, \mathbf{f}) = \frac{1}{2p\xi} \frac{1}{2\pi^2} \int \frac{\mathbf{q}_{sp} \varphi_0(\mathbf{p}, \mathbf{s})}{\mathbf{q}_{fs}^2 + \lambda^2} d^3s, \qquad \tilde{J} = \boldsymbol{\alpha} \cdot \mathbf{J}, \tag{29a}$$

The vector (29a) lies in the plane $(\mathbf{p}, \mathbf{f})$, because of the relation $\mathbf{J}(\mathbf{p}, \mathbf{f}) = -\mathbf{J}(-\mathbf{p}, -\mathbf{f})$ [see formula (23)]. Hence we can write

$$\mathbf{J}(\mathbf{p}, \mathbf{f}) = a\mathbf{q}_{fp} + b\mathbf{p}. \tag{29b}$$

Substituting (29b) in (29), we obtain

$$\varphi_2(\mathbf{p}, \mathbf{f}) = a + b \frac{\mathbf{p}^2 + \tilde{f}\tilde{p}}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon}. \tag{30}$$

A representation for the functions a and b is given in Appendix B. It has the form

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = -\frac{N}{4p\xi\pi^3} \oint^{(0^+1^+)} \left(\frac{-x}{1-x}\right)^{i\xi} \times \left[\int_0^\infty \begin{Bmatrix} p \\ -i\lambda \end{Bmatrix} \frac{d\lambda'}{[(\mathbf{q}_{fp} + \mathbf{p}x)^2 - (px + i\lambda')^2]^2}\right] dx, \tag{31}$$

where N is given by (19b). Formula (31) can be reduced to a single quadrature; however, the expression quoted above is more convenient for the calculation of matrix elements.

Both integrals (31) can be evaluated by retaining only the zeroth order term in the expansion in powers of αZ. However, in this case it is simpler to use (29a) directly. Indeed, the zeroth order term of (31) is obtained by substituting the series (2) in (29a) and discarding all terms except the first nonvanishing term. In this way we find

$$\mathbf{J}_0(\mathbf{p}, \mathbf{f}) = \int \frac{\mathbf{q}_{sp} d^3s}{(\mathbf{q}_{fs}^2 + \lambda^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{sp}^2 + \lambda^2)}. \tag{29'}$$

The integral (29′) has been computed in a number of papers. Using the results of Gavrila,[3] we obtain[5)]

$$a_0 = \frac{\pi^2 p}{2\Delta} \left\{\pi \frac{\mathbf{pq}}{pq} + \frac{\mathbf{fp}}{fp} i \ln \frac{f + p + i\lambda}{-f + p + i\lambda} + i \ln \frac{q^2}{f^2 - (p + i\lambda)^2}\right\},$$

$$b_0 = \frac{\pi^2 q}{2\Delta} \left\{\pi + \frac{\mathbf{fq}}{fq} i \ln \frac{f + p + i\lambda}{-f + p + i\lambda} + \frac{\mathbf{pq}}{pq} i \ln \frac{q^2}{f^2 - (p + i\lambda)^2}\right\},$$

$$\Delta = f^2 p^2 - (\mathbf{fp})^2, \qquad \mathbf{q} = \mathbf{f} - \mathbf{p}. \tag{31'}$$

Expression (29a′) represents the zeroth order term in the expansion of (29a) in powers of αZ. In accordance with what has been said in Sec. 1, we must simply multiply (29′) by the normalization factor (19b) in order to take account of all terms of the expansion of (29′) in terms of the parameter $\xi = \alpha ZE/p$. We note that in reference 5 an expression for (29) is given which is equivalent to (30)

[5)]The expressions (30), (31′) were obtained in a somewhat different manner by Johnson and Mullin.[7]

and (31), but which is more convenient for the calculations.

Analogous expressions can also be obtained for the higher-order terms in the expansion (28). These involve a correspondingly larger number of quadratures. They are more compact than the corresponding terms in the Born approximation, do not contain infrared divergences, and remain valid for small energies.

The functions (28), with φ_0 given by its exact value (16), can be combined with the higher Born approximations in carrying out calculations. One must only see to it that the phase factors (or their expansions) are identical for all terms.

In order to exclude the infrared divergences in calculations employing a Born series for the wave function, we must reorder this series into the series (28) and then multiply each term of the resulting series (28) by the complex conjugate of the phase factor (21) up to the power to which the expansion in the corresponding term is carried out. When both expansions are multiplied out, the divergences must cancel out. This recipe can easily be tested on the example of formulas (10) and (12) in reference 5.

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APPENDIX A

Let

$$A_k(x)=\sum_{m=0}^{k}\frac{1}{m!}b_{k-m}(x)\ln^m\frac{2a}{x},\qquad A_0=1;\qquad \text{(A.1)}$$

$$b_{n+1}(x)=\int_x^{2a}\frac{dz}{z}\left\{\sum_{l=0}\frac{(-1)^l}{l!}\ln^l\left(1+\frac{\gamma}{z}\right)b_{n-l}(z+\gamma)-b_n(z)\right\},$$

$$b_0=1. \qquad \text{(A.2)}$$

We show that (13) leads to the same expressions for A_{k+1} and b_{k+1}. We have

$$A_{k+1}(x)=\int_x^{2a}\frac{dz}{z}A_k(z+\gamma);\qquad |z|\geqslant|x|\geqslant|\gamma|. \qquad \text{(A.3)}$$

With the help of the identity

$$\ln[2a/(z+\gamma)]=\ln(2a/z)-\ln(1+\gamma/z),$$

we obtain

$$A_{k+1}(x)=\int_x^{2a}\frac{dz}{z}\sum_{m=0}^{k}\sum_{n=0}^{m}\frac{1}{n!}$$

$$\times\ln^n\frac{2a}{z}\cdot\left[\frac{(-1)^{m-n}}{(m-n)!}\ln^{m-n}\left(1+\frac{\gamma}{z}\right)\right]b_{k-m}(z+\gamma).$$

Changing the order of summation and replacing $m-n$ by l, we find

$$A_{k+1}(x)=\int_x^{2a}\sum_{n=0}^{k}\frac{1}{n!}\ln^n\frac{2a}{z}$$

$$\times\sum_{l=0}^{k-n}\frac{(-1)^l}{l!}\ln^l\left(1+\frac{\gamma}{z}\right)b_{k-n-l}(z+\gamma)\frac{dz}{z}. \qquad \text{(A.4)}$$

Noting that

$$db_{n+1}(z)$$

$$=-\frac{dz}{z}\left\{\sum_{l=0}^{n}\frac{(-1)^l}{l!}\left[\ln^l\left(1+\frac{\gamma}{z}\right)\right]b_{n-l}(z+\gamma)-b_n(z)\right\}, \qquad \text{(A.5)}$$

we can write (A.4) in the form

$$A_{k+1}(x)=-\int_x^{2a}\sum_{n=0}^{k}\frac{1}{n!}\ln^n\frac{2a}{z}\cdot db_{k-n+1}(z)$$

$$-\int_x^{2a}\sum_{n=0}^{k}b_{k-n}(z)\cdot\frac{1}{n!}\ln^n\frac{2a}{z}\frac{dz}{z}. \qquad \text{(A.6)}$$

Integrating the first term in (A.6) by parts, we obtain expressions which coincide with (A.1) and (A.2) for $k\to k+1$. Making the transformation of variables $\gamma/z=y$ in (A.2), we obtain (14b).

We show now that $|b_{k+1}(x)|<x^k$ and $b_k<1$, i.e., that the radius of convergence of (15a) is larger than unity. These inequalities are satisfied by b_1 and b_2. Assume that they are fulfilled for b_k. Using the inequality

$$(n!)^{-1}\ln^n(1+y)<\ln^n(1+y)\leqslant y^n,$$

we obtain from (14b)

$$|b_{k+1}(x)|<\int_0^x\frac{dy}{y}\sum_{n=0}^{k}y^ny^{k-n}=\int_0^x dy\,ky^{k-1}=x^k.$$

Thus the inequality is fulfilled for all k.

APPENDIX B

Using the representation of the hypergeometric function in the form of a contour integral,[4] we obtain

$$\varphi_0(\mathbf{p},\mathbf{s})=\frac{N}{2\pi i}\oint^{(0^+,1^+)}\frac{dx}{x}\left(\frac{-x}{1-x}\right)^{i\xi}\frac{1}{2\pi^2 i}\frac{\partial}{\partial B}\frac{1}{(\mathbf{q}_{sp}+\mathbf{p}x)^2-(B+i\varepsilon)^2}, \qquad \text{(B.1)}$$

where $B=px$. Substituting (B.1) in (29a) and noting that

$$\frac{1}{2\pi^2 i}\frac{\partial}{\partial B}\int\frac{\mathbf{q}_{sp}\,d^3s}{(q_{fs}^2+\lambda^2)\left[(\mathbf{q}_{sp}+\mathbf{p}x)^2-(B+i\varepsilon)^2\right]}$$

$$=\nabla_B\frac{-p}{2\pi^2 i}\int\frac{d^3s}{(q_{fs}^2+\lambda^2)\left[(\mathbf{q}_{sp}+\mathbf{B})^2-(B+i\varepsilon)^2\right]}$$

$$=\nabla_B\frac{-p}{2}\frac{1}{P}\ln\frac{P+B+i\lambda}{-P+B+i\lambda}=p\nabla_B\int_\lambda^\infty\frac{d\lambda'}{P^2-(B+i\lambda')^2}, \qquad \text{(B.2)}$$

where $\mathbf{P} = \mathbf{q}_{fp} + \mathbf{B}$, we find

$$\mathbf{J} = \frac{N}{4\pi^2\xi}\frac{1}{2\pi}\oint\limits^{(0^+,\,1^+)} dx\left(\frac{-x}{1-x}\right)^{i\xi}\nabla_B\int\limits_{\lambda}^{\infty}\frac{d\lambda'}{P^2-(B+i\lambda')^2}. \qquad \text{(B.3)}$$

By evaluating the gradient, we obtain the expressions (30) and (31).

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