QUANTUM THEORY OF SOUND ABSORPTION BY METALS IN A MAGNETIC FIELD

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Submitted to JETP editor December 19, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 1446-1451 (May, 1961)

We have studied the influence of electron scattering on the absorption of sound by metals in strong magnetic fields for the case of large spatial dispersion. We considered the case where the electron scattering is caused by their interaction with neutral impurities.

GUREVICH, Firsov, and the present author[1] studied the deformation absorption of sound in metals for the case of a strong magnetic field $\Omega \gg kT$ and a long mean free path of the conduction electrons (Ω is the electron Larmor frequency, T the temperature, and k Boltzmann's constant).* We showed that the theory under those conditions should essentially be a quantum-theoretical one and that when the magnetic field changes the sound absorption may show gigantic oscillations.

The cause of these oscillations is the following one. In the case $\Omega \gg kT$ in the region of the tail of the Fermi distribution, which is responsible for the transport properties, only narrow ranges of k_z-values (the component of the quasi-momentum in the direction of the magnetic field **H**) turn out to be allowed, while for $\Omega \ll kT$ all values of k_z are allowed. The position of these ranges on the Fermi surface is determined by the value of H and when H is changed other values of k_z become allowed. On the other hand, when the electron mean free path is sufficiently large those electrons which move in phase with the sound wave will play the main role in the absorption of sound. Other electrons can absorb sound only during collisions with scatterers, and their contribution to the absorption is proportional to the small quantity $1/\kappa l$ (reference 2) (κ is the sound wave vector and l the average mean free path of the conduction electrons). When the magnetic field is changed the position of the allowed k_z-intervals changes, and if the angle between the directions of κ and **H** is considerably different from a right angle the electrons moving in phase with the sound wave turn out to be sometimes in the region of the tail of the Fermi distribution and sometimes outside it. This leads to the gigantic oscillations in the absorption coefficient.

*We use a system of units where $\hbar = c = 1$.

Assuming existence of an electron relaxation time, we showed that collisions are not important and that the oscillations remain gigantic for $\omega < \nu$, provided the inequality

$$(\kappa l)^2 \gg \zeta / \Omega,$$

where ζ is the chemical potential of the metal, ω the sound frequency, and ν the collision frequency of the electrons with the scatterers, is satisfied.

The present paper is devoted to a study of the influence of scattering of the electrons on the absorption of sound by metals for the case of a strong magnetic field $\Omega \gg kT$ and a strong spatial dispersion $\kappa l \gg 1$ ($l = v_F/\nu$; v_F is the Fermi velocity). We consider the case where the transverse electrical fields which occur when the crystal is deformed do not play an important role, so that we can neglect them. We assume that the electron scattering is caused by their interaction with randomly distributed fixed centers, the radius of action of which is small compared with their average mutual distance and with the electron wave length. For the sake of simplicity we restrict our consideration to that of an isotropic quadratic electron spectrum and do not take the electron spin into account.

Let the electron Hamiltonian when there is no sound be $\mathcal{H} = \mathcal{H}_0 + V$, where

$$\mathcal{H}_0 = (\mathbf{p} - e\mathbf{A})^2/2m, \qquad V = \sum_j V_j \equiv \sum_j V(\mathbf{r} - \mathbf{r}_j); \quad (1)$$

here **A** is the vector potential of the magnetic field **H**, **p** the electron quasi-momentum operator, m the effective mass, and V_j the potential of the interaction between an electron and a scattering center at $\mathbf{r}_j$.

The perturbation caused by the interaction of an electron with the sound wave is taken to be of the form

$$\mathcal{H}'(t) = \tfrac{1}{2}(Ue^{-i\omega t} + U^+e^{i\omega t}), \qquad U = U_0 e^{i\kappa r}, \quad (2)$$

U_0 is a Hermitian operator which, generally speaking, is a function of the electron quasi-momentum operator. In the case of inductive sound absorption the perturbation has, for instance, the form $A_\mu(t)\,J_\mu$ where J_μ is the operator of the μ-th component of the current density, $\mathbf{A}(t) = [\mathbf{H} \times \mathbf{u}(t)]$, where $\mathbf{u}(t)$ is the displacement vector in the sound wave.

In the case of deformation absorption

$$\mathcal{H}'(t) = \Lambda_{ik}\, u_{ik}\,(t), \tag{3}$$

where $u_{ik} = \frac{1}{2}(\partial u_i/\partial x_k + \partial u_k/\partial x_i)$ is the deformation tensor of the crystal, and where the tensor Λ_{ik} in general depends on the quasi-momentum $\mathbf{p}$. In the present paper we shall consider the deformation absorption of sound by metals with a low carrier concentration (such as bismuth) for which Λ_{ik} = const.

The energy absorbed by the electrons per unit time is given in the first Born approximation in $\mathcal{H}'$ by the formula[2,4]

$$Q = \frac{\pi\omega}{2}\sum_{ab}\langle|U_{ba}|^2\,\delta(\omega_{ab}+\omega)\,[f_a(1-f_b) - f_b(1-f_a)]\rangle. \tag{4}$$

Here a and b characterize the stationary states of the Hamiltonian $\mathcal{H}$, $\omega_{ab} = E_a - E_b$ is the energy difference, and f_a is the probability of finding an electron in the state a. Since the scattering of electrons by fixed centers is elastic and temperature-independent, f_a is simply the Fermi function of the argument $(E_a - \zeta)/kT$. The pointed brackets $\langle\ldots\rangle$ indicate here and henceforth an average over the positions of the scattering centers $\mathbf{r}_j$.

One easily writes Eq. (4) in the form

$$Q = \frac{\pi\omega}{2}\int_{\Omega/2}^{+\infty} dE\,[f(E) - f(E+\omega)]\,I(E), \tag{5}$$

$$I(E) = \sum_{ab}\langle|U_{ba}|^2\,\delta(E - E_b + \omega)\,\delta(E - E_a)\rangle. \tag{6}$$

Using the identity

$$\delta(x) = \frac{1}{2\pi i}\lim_{\varepsilon\to+0}\left(\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right)$$

and writing (6) in operator form we find that I(E) is a linear combination of four terms

$$\frac{1}{(2\pi)^2}\,\mathrm{Sp}\left\langle\left\{\frac{1}{E-\mathcal{H}\pm i\varepsilon}\,U^+\,\frac{1}{E+\omega-\mathcal{H}\pm i\varepsilon}\,U\right\}\right\rangle, \tag{7}$$

which can conveniently be evaluated in the $\mathcal{H}_0$ representation. The eigenfunctions of the Hamiltonian $\mathcal{H}_0$ are characterized by the magnetic quantum number n, the component of the wave vector along the direction of the magnetic field k_z, and X, the coordinate center of rotation of the electron.[5] The totality of these quantum numbers will be denoted by indices $\alpha, \beta, \ldots$

To evaluate (7) we need the exact Green's function

$$G_{\beta\alpha}(u) = \left\langle\left(\beta\left|\frac{1}{u-\mathcal{H}}\right|\alpha\right)\right\rangle, \tag{8}$$

averaged over the positions of the scatterers ($u = E + i\epsilon$). We have shown[6,7] that G(u) can be written as a series

$$G(u) = \frac{1}{u-\mathcal{H}_0} + \sum_j \frac{1}{u-\mathcal{H}_0}\langle T^j(u)\rangle\frac{1}{u-\mathcal{H}_0} + \sum_{j\neq k}\frac{1}{u-\mathcal{H}_0}\left\langle T^j(u)\frac{1}{u-\mathcal{H}_0}T^k(u)\right\rangle\frac{1}{u-\mathcal{H}_0} + \ldots, \tag{9}$$

$$T^j_{\beta\alpha}(E+i\varepsilon) = \Psi^*_\beta(\mathbf{r}_j)\,\Psi_\alpha(\mathbf{r}_j)\,\frac{2\pi f/m}{1+iK(E)f}. \tag{10}$$

Here $\Psi_\alpha(\mathbf{r})$ is an eigenfunction of $\mathcal{H}_0$ and f is the amplitude for the scattering of a zero-energy free electron by the potential V_j,

$$K(E) = \sum_{n=0}^{N}\frac{m\Omega}{[2m\,[E - \Omega(n+1/2)]]^{1/2}}, \tag{11}$$

where N is that integer which leads to one imaginary term in the sum (11). $T^j_{\beta\alpha}(E_\alpha + i\epsilon)$ is the amplitude for the transition from the state α to β when an electron is scattered by the j-th center.

In the series (9) each term of the expansion in powers of T is a sum over all possible scatterer numbers j, k, $l, \ldots$ except for terms where any two adjacent numbers are the same. One can show that cross terms of the kind

$$\frac{1}{u-\mathcal{H}_0}T^j(u)\frac{1}{u-\mathcal{H}_0}T^k(u)\frac{1}{u-\mathcal{H}_0}T^j(u)\frac{1}{u-\mathcal{H}_0}\times T^k(u)\frac{1}{u-\mathcal{H}_0}$$

correspond to the simultaneous scattering of an electron by two centers j and k, and give a small contribution to G(u) if $f \ll \lambda$; $f \ll a$, and $\Omega \ll E$, where λ is the electron wavelength and a the mean distance between the scatterers. We shall therefore drop such terms.

We introduce the operator for the scattering by the j-th center, averaged over the positions of all the other centers:

$$\tau^j(u) = T^j(u) + T^j(u)\left[\left\langle\frac{1}{u-\mathcal{H}}\right\rangle - \frac{1}{u-\mathcal{H}_0}\right]T^j(u) + T^j(u)\left[\left\langle\frac{1}{u-\mathcal{H}}\right\rangle - \frac{1}{u-\mathcal{H}_0}\right]T^j(u)\left[\left\langle\frac{1}{u-\mathcal{H}}\right\rangle - \frac{1}{u-\mathcal{H}_0}\right]T^j(u) + \ldots. \tag{12}$$

To evaluate the matrix element $\tau^j_{\beta\alpha}(u)$ we must know the asymptotic expression of the function

$$q(\mathbf{r},\mathbf{r}';u) = \frac{2\pi i}{m}\sum_{\alpha\beta}\Psi^*_\beta(\mathbf{r}')\left(\beta\left|\left\langle\frac{1}{u-\mathcal{H}}\right\rangle - \frac{1}{u-\mathcal{H}_0}\right|\alpha\right)\Psi_\alpha(\mathbf{r}) \tag{13}$$

as $|\mathbf{r}' - \mathbf{r}| \sim r_0$, where r_0 is the action radius of a scattering center. It is fairly obvious that at such small distances this quantity is independent of the coordinates (one can choose the gauge of the vector potential **A** in such a way). If we denote it by $q(E)$ and sum the geometric series obtained from (12) we find

$$\tau^j_{\beta\alpha}(E + i\varepsilon) = \Psi^*_\beta(\mathbf{r}_j)\,\Psi_\alpha(\mathbf{r}_j)\,2\pi f/m[1 + i\,[K(E) + q(E)]f]. \quad (14)$$

One can easily express the Green's function $G_{\alpha\beta}(u)$ in terms of $\tau^j(u)$:

$$G_{\beta\alpha}(u) = \frac{\delta_{\alpha\beta}}{u - E_\alpha} + \sum_j \frac{1}{u - E_\beta}\langle\tau^j_{\beta\alpha}(u)\rangle\frac{1}{u - E_\alpha} + \sum_{j\neq k}\sum_\delta \frac{1}{u - E_\beta}\left\langle\tau^j_{\beta\delta}(u)\frac{1}{u - E_\delta}\tau^k_{\delta\alpha}(u)\right\rangle\frac{1}{u - E_\alpha} + \ldots. \quad (15)$$

There are now no longer any two numbers of scattering centers j, k, *l*, ... equal in any term of the expansion in powers of $\tau(u)$. The average over $\mathbf{r}_j$ of the product of several different $\tau^j(u)$ is thus equal to the product of the averages. Taking into consideration that

$$\sum_j \langle\Psi^*_\beta(\mathbf{r}_j)\,\Psi_\alpha(\mathbf{r}_j)\rangle = n_s\delta_{\alpha\beta},$$

where n_s is the number of scatterers per unit volume, and that according to the sum rule the quantity

$$-2\mathrm{Im}\sum_j\langle\tau^j_{\alpha\alpha}(E + i\varepsilon)\rangle = -(4\pi f n_s/m)\mathrm{Im}\,[1 + i\,(K + q)f]^{-1} = \nu(E) \quad (16)$$

is the total probability that an electron of energy E is scattered per unit time, we get

$$G_{\beta\alpha}(E + i\varepsilon) = \delta_{\alpha\beta}\,[E - E_\alpha + i\nu(E)/2]^{-1}. \quad (17)$$

We have dropped $\mathrm{Re}\,\Sigma_j\langle\tau^j_{\alpha\alpha}\rangle$ which leads to a renormalization of the energy.

We can now use (17) to verify that the asymptotic expression of $q(\mathbf{r}, \mathbf{r}'; u)$ at small distances is indeed independent of the coordinates. A direct evaluation leads to the equation

$$q(E) = K\,[E + i\nu(E)/2] - K(E), \quad (18)$$

the form of the function $K(E)$ was given in the foregoing by (11). Substituting this expression for $q(E)$ into (16) we get the following equation for $\nu(E)$:*

$$\nu(E) = \nu_0(E)(2mE)^{-1/2}\mathrm{Re}K\,[E + i\nu(E)/2], \quad (19)$$

where

$$\nu_0(E) = 4\pi f^2(2E/m)^{1/2}n_s$$

is the scattering probability when there is no magnetic field. In the denominator of (19) we neglected the quantity Kf as compared to unity.

The equation we have obtained gives us immediately the value of $\nu(E)$ provided the condition

$$\Omega \ll (E\nu_0)^{1/2} \quad (20)$$

is satisfied. Indeed, in that case

$$K(E + i\nu/2) \approx (2mE)^{1/2}$$

and (19) leads to the equation $\nu(E) = \nu_0(E)$. If, however, condition (20) is not satisfied, $\nu(E)$ increases when $E \to \Omega(n + 1/2)$.

In the following we shall be interested in the case $\nu_0 \ll kT$. Condition (20) is then replaced by the less restrictive one:

$$\Omega \ll (\zeta kT)^{1/2}, \quad (21)$$

which is easily satisfied in metals. We shall thus assume that $\nu(E) \approx \nu_0$; the increase of $\nu(E)$ in energy ranges which are small compared to kT will not show up in the final results because of the averaging over the Fermi distribution.

We shall not consider the matrix elements $U_{\alpha'\alpha}$. If $\boldsymbol{\kappa} \parallel \mathbf{H}$,

$$(n'X'k'_z|\,U\,|nXk_z) = U_0\delta_{n'n}\delta_{X'X}\delta_{k'_z,\,k_z+\varkappa}.$$

If $\boldsymbol{\kappa}$ makes an angle with **H** the matrix elements of U which are nondiagonal in n will be different from zero. One sees, however, easily that if

$$\varkappa_\perp R \ll 1, \quad (22)$$

where $\kappa_\perp$ is the component of $\boldsymbol{\kappa}$ perpendicular to **H**, and R the Larmor radius of the electron, these matrix elements are smaller than the diagonal ones in the ratio $(\kappa_\perp R)^{|n'-n|}$. The same occurs in the semi-classical approximation $\Omega \ll \zeta$ under condition (22) for an arbitrary electron spectrum and an arbitrary dependence $\Lambda_{ik} = \Lambda_{ik}(\mathbf{p})$. For the sake of simplicity we consider the case $\boldsymbol{\kappa} \parallel \mathbf{H}$.

If we use the Green's function (17) and the transition amplitude (14) we can write each of the terms of (7) as a power series

$$\mathrm{Sp}\left\{\frac{1}{E - H + i\varepsilon}U^+\frac{1}{E + \omega - H - i\varepsilon}U\right\} = \sum_{\alpha\alpha'}\frac{1}{E - E_\alpha + i\nu/2}U^+_{\alpha\alpha'}\frac{1}{E + \omega - E_{\alpha'} - i\nu/2} \times\left\{U_{\alpha'\alpha} + \sum_j\sum_{\beta\beta'}\left\langle\tau^j_{\alpha'\beta'}\frac{1}{E + \omega - E_{\beta'} - i\nu/2} \times U_{\beta'\beta}\frac{1}{E - E_\beta + i\nu/2}\tau^j_{\beta\alpha}\right\rangle + \ldots\right\}. \quad (23)$$

An estimate of the second term within the braces for the case $\kappa l \gg 1$ shows that except for a small neighborhood around $E = \Omega(n + 1/2)$ it is of the

*This equation is similar to the one obtained by Bychkov[8] for $\mathrm{Im}\,\Sigma_\alpha G_{\alpha\alpha}(u)$.

order $U_0/\kappa l$, i.e., the series (23) is an expansion in the small parameter $1/\kappa l$.* We have thus

$$I(E)=\frac{1}{(2\pi)^2}\sum_{\alpha\alpha'}|U_{\alpha'\alpha}|^2\frac{\nu(E)}{(E-E_\alpha)^2+\nu^2/4}\frac{\nu(E)}{(E-E_{\alpha'}+\omega)^2+\nu^2/4}, \tag{24}$$

and summing over X and k_Z and neglecting ω we get

$$I(E)\approx\frac{V_0}{(2\pi)^3}|U_0|^2(8m)^{1/2}m\Omega\sum_{n=0}^{N-1}\frac{\nu(E)[E-\Omega(n+1/2)]^{-1/2}}{2\varkappa^2m^{-1}[E-\Omega(n+1/2)]+\nu^2}, \tag{25}$$

where V_0 is the volume of the crystal. It is clear that in the case

$$(\varkappa l)^2\gg\zeta/\Omega \tag{26}$$

it is sufficient to restrict oneself in (25) to the one term for which $[E-\Omega(n+1/2)]$ is a minimum. When H changes, this quantity and with it the energy Q absorbed by the electrons, oscillates strongly and this causes the gigantic oscillations in the absorption of sound.[1]

In the opposite case when

$$(\zeta/\Omega)\gg(\varkappa l)^2\gg 1, \tag{27}$$

many values of n are important in Eq. (25) and to evaluate I(E) we can use the Poisson summation formula

$$\sum_{n=0}^{\infty}\varphi(n)=\frac{1}{2}\varphi(0)+\sum_{k=-\infty}^{+\infty}\int_0^\infty dn\varphi(n)\exp(2\pi ikn).$$

If we use this formula we get easily the final expression for the sound absorption coefficient

$$\Gamma(H)=\Gamma_0\left\{1+\varkappa l\left(\frac{\Omega}{\zeta}\right)^{1/2}\sum_{n=1}^{\infty}(-1)^nA_n\frac{\cos[2\pi n\zeta/\Omega-\pi/4]}{\pi\sqrt{n}}\right\},$$

$$A_n=\frac{2\pi^2nkT/\Omega}{\mathrm{sh}\,(2\pi^2nkT/\Omega)},\qquad \Gamma_0=\frac{m^2|\Lambda_{ik}u_{ik}|^2}{2\pi\rho|\mathbf{u}|^2s\varkappa}, \tag{28}$$†

where ρ is the crystal density, s the sound velocity, and Γ_0 the sound absorption coefficient when there is no magnetic field.

*In the case $\varkappa\perp\mathbf{H}$, the second and later terms in (23) are not small compared to the first one and one must solve the integral equation obtained from (23) to evaluate I(E).

†sh = sinh.

In all cases those electrons from the region of the tail of the Fermi distribution which satisfy the condition

$$\varkappa l_z\lesssim 1, \tag{29}$$

play the main role in the absorption of sound; in Eq. (29) $l_Z=v_Z/\nu$, v_Z is the z component of the velocity. In the case (26) there are, depending on the magnitude of the magnetic field either a group of electrons satisfying condition (29) or there are no such electrons on the Fermi surface. This also causes the gigantic oscillations of the absorption coefficient. In the case (27) there is always a number of electron groups satisfying condition (29). The relative amplitude of the oscillations is in that case thus small and is, as is usual in such cases, proportional to the square root of the number of these groups.

In conclusion I express my gratitude to Yu. A. Bychkov and L. D. Landau for valuable discussions.

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Translated by D. ter Haar
247