# *DISPERSION EQUATION FOR AN ORDINARY WAVE MOVING IN A PLASMA PERPENDICULAR TO AN EXTERNAL MAGNETIC FIELD*

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A general qualitative investigation is carried out on the dispersion equation for an ordinary wave moving in a plasma perpendicular to an external magnetic field. The regularities found are illustrated by some results of a numerical solution of the dispersion equation.

## INTRODUCTION

It is known that two types of waves can exist in a homogeneous, unbounded plasma located in a homogeneous external magnetic field $\mathbf{H}_0$. The direction of propagation of these waves is perpendicular to the field $\mathbf{H}_0$. The first of these types is a purely transverse wave with an electric vector polarized along the field $\mathbf{H}_0$ (ordinary wave), and the second consists of waves in which the electric vector is polarized perpendicular to $\mathbf{H}_0$ (extraordinary and plasma waves). The frequency $\omega$ and the propagation constant k of these waves are related by the dispersion equations, which are obtained with the aid of Maxwell's equations and a linearized kinetic equation for the electrons (without consideration of collisions). Depending on the character of the problem of the dispersion equation considered, one can determine the propagation constant as a function of the frequency or, conversely, the frequency as a function of the propagation constant.

In the present work, we shall be interested in the dispersion equation for the ordinary wave:[1-6]

$$D(k, \omega) = k^2 - \frac{\omega^2}{c^2} + \frac{\omega_0^2\omega}{2c^2\omega_H \sin(\pi\omega/\omega_H)} \times \int_0^{2\pi} \exp\left\{-\frac{Tk^2}{m\omega_H^2}(1-\cos\tau)\right\}\cos\frac{\omega}{\omega_H}(\tau-\pi)\,d\tau = 0. \quad (1)$$

Here $\omega_H = eH_0/mc$ is the Larmor frequency, $\omega_0 = \sqrt{4\pi Ne^2/m}$ is the plasma frequency, T is the temperature of the electrons in energy units. By means of transformations similar to those used by Gross,[7] Eq. (1) can be written in the form

$$D(k, \omega) = k^2 - \frac{\omega^2}{c^2} + \frac{\omega_0^2\omega}{c^2\omega_H} \sum_{n=-\infty}^{\infty} \frac{\zeta_n(Tk^2/m\omega_H^2)}{\omega/\omega_H - n} = 0, \quad (1')$$

where $\zeta_n(x) = e^{-x} I_n(x)$, $I_n(x)$ is the Bessel function of imaginary argument. In some cases, the second form of writing is the more convenient.

Equation (1) was studied in a number of researches.[2-4,6] The characteristic feature of the methods of investigation employed has been the preliminary expansion of the equation in some small parameter; usually, it is assumed that $Tk^2/m\omega_H^2 \ll 1$. However, the behavior of the roots of Eq. (1) and of the simplified equations have a number of essential differences. Equation (1) loses meaning for $\omega = n\omega_H$. In this case, the motion of the electrons over the Larmor curves falls into resonance with the vibrations in the field of the wave. The simplified equations, in the first place, do not take into account resonances of higher orders, and in the second place lead to formulas which are inapplicable even in the vicinity of the first resonance, since $k^2$ increases without limit in this region. Finally, because of the non-uniformity of the expansion in the small parameter, it is not possible to study the problem of the total number of roots and their distribution over the complex plane. Therefore, analysis of Eq. (1) in the works mentioned is incomplete, and is clearly incorrect in a number of cases. We have, therefore, set up the problem in the present work of giving a sufficiently detailed and mathematically rigorous study of the dispersion equation (1). At the same time we shall give the results of a numerical solution of Eq. (1) which illustrates the regularities that have been established.

## 1. PRELIMINARY INVESTIGATION OF THE DISPERSION EQUATION

We shall begin our investigation of the dispersion equation (1) with a case in which it is required to find the propagation constant for a given frequency. We introduce the dimensionless variables:

$$s = k^2c^2/\omega^2 = N^2, \qquad \alpha = \omega/\omega_H,$$
$$\beta = \omega_0/\omega_H, \qquad \gamma = T/mc^2,$$

($N^2$ is the square of the index of refraction). Then we have, in place of (1) and (1′),

$$\mathscr{D}(s, \alpha, \beta, \gamma) = s - 1 + \frac{\beta^2}{2\alpha \sin \alpha\pi} \int_0^{2\pi} \exp\{-s\alpha^2\gamma(1-\cos\tau)\} \cos\alpha(\tau - \pi)\, d\tau = 0, \tag{2}$$

$$\mathscr{D}(s, \alpha, \beta, \gamma) = s - 1 + \frac{\beta^2}{\alpha} \sum_{n=-\infty}^{\infty} \frac{\zeta_n(s\alpha^2\gamma)}{\alpha - n} = 0. \tag{2′}$$

We shall seek the roots of Eq. (2) $s = s(\alpha, \beta, \gamma)$ for all possible positive values of $\alpha$ ($\alpha \neq n$). Real roots are considered in Secs. 2 and 3, and complex roots in Sec. 4. Every positive root $s$ of Eq. (2) determines a pair of real roots of Eq. (1), $k = \pm\sqrt{s}\,\omega/c$, and every negative root determines a pair of purely imaginary roots $k = \pm i\sqrt{-s}\,\omega/c$; finally, every complex root of Eq. (2), together with its corresponding complex conjugate root, determines four complex roots of Eq. (1) which can be written in the form $k = p \pm iq$, $k = -p \pm iq$.

A comparison of the signs of the functions $\mathscr{D}(s, \alpha, \beta, \gamma)$ at $s = 0$ and $s = \pm\infty$ plays an important role in the investigation of the real roots of Eq. (2). By direct calculation, we find that

$$\mathscr{D}(0, \alpha, \beta, \gamma) = \beta^2/\alpha^2 - 1. \tag{3}$$

On the other hand, the behavior of the function $\mathscr{D}$ as $|s| \to \infty$ is characterized by the following asymptotic formula:

$$\mathscr{D}(s, \alpha, \beta, \gamma) \approx s + \sqrt{\pi/8}\,\beta^2 e^{-2s\alpha^2\gamma}/\alpha \sin\alpha\pi \sqrt{-s\alpha^2\gamma}. \tag{4}$$

Thus

$$\lim_{s\to+\infty} \mathscr{D}(s, \alpha, \beta, \gamma) = +\infty, \tag{5}$$

$$\lim_{s\to-\infty} \mathscr{D}(s, \alpha, \beta, \gamma) = +\infty \operatorname{sign}(\sin\alpha\pi). \tag{6}$$

In the study of the behavior of the roots in the vicinity of cyclotron resonances, it is convenient to make use of the equation in the form (2′). It should be noted that only two terms of the series play an important role for small $\gamma$ and $\alpha \sim n$: the zero term and the n-th term. The first of these is the only nonvanishing term of the series for $s\gamma \to 0$, the second approaches infinity for $\alpha \to n$. Therefore, one can make use of a simplified equation of the form

$$\zeta_n(sn^2\gamma) = (\alpha - n)\, n\beta^{-2}\,[1 - s - \beta^2 n^{-2}\zeta_0(sn^2\gamma)] \tag{7}$$

for the qualitative investigation of the behavior of the roots $\alpha \sim n$.

Equation (1) is obtained by means of the nonrelativistic kinetic equation; therefore, it is reasonable to consider only small values of the parameter $\gamma$. We compare Eq. (2) with the degenerate equation which is obtained from (2) for $\gamma = 0$ ($T = 0$):

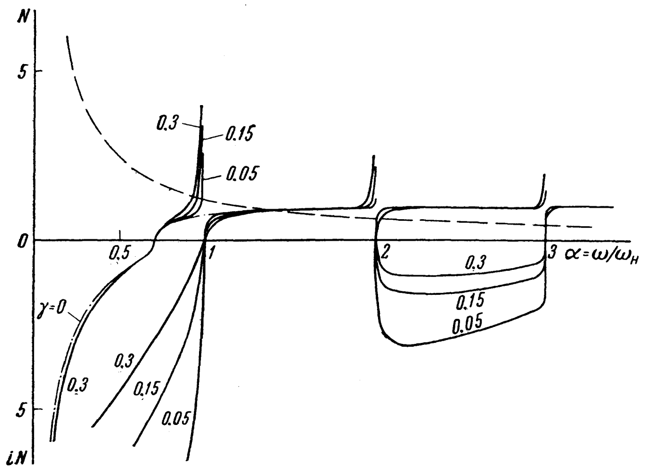


FIG. 1. Frequency dependence of the index of refraction for $\beta = \sqrt{0.5}$ and for various values of $\gamma$.

$$\mathscr{D}(s, \alpha, \beta, 0) = s - 1 + \beta^2/\alpha^2 = 0. \tag{8}$$

According to this equation, there exist traveling waves in the plasma at zero temperature for $\alpha > \beta$ ($\omega > \omega_0$); for $\alpha < \beta$ ($\omega < \omega_0$), propagation of an ordinary wave is impossible (in the first case, the root of Eq. (8) is positive, in the second, it is negative). This conclusion does not carry over completely to Eq. (2); however, a number of peculiarities of the solutions of this equation also depend significantly on the relation between $\alpha$ and $\beta$. Therefore, proceeding to the investigation of Eq. (2), we consider separately the cases $\alpha > \beta$ and $\alpha < \beta$.

## 2. INVESTIGATION OF THE REAL ROOTS OF EQUATION (2) FOR $\alpha > \beta$ ($\omega > \omega_0$)

In the region of change of the parameter $\alpha$ under consideration, we have

$$\mathscr{D}(0, \alpha, \beta, \gamma) < 0. \tag{9}$$

Taking into account (5) and (9), we find that Eq. (2) necessarily has a positive root for $\alpha > \beta$. As $\gamma \to 0$, this root approaches the solution of Eq. (8). We therefore compare the signs of the function $\mathscr{D}$ for $s = -\infty$ [Eq. (6)] and $s = 0$ [the inequality (9)]. If $\sin\alpha\pi > 0$, then the signs are opposite. For such $\alpha$, Eq. (2) also has a negative root which approaches $-\infty$ as $\gamma \to 0$. If now $\sin\alpha\pi < 0$, then the signs of the function $\mathscr{D}$ are identical for $s = -\infty$ and $s = 0$. In this case, Eq. (2) has no negative roots for sufficiently small values of $\gamma$. By considering the equation in the form (2′), we can

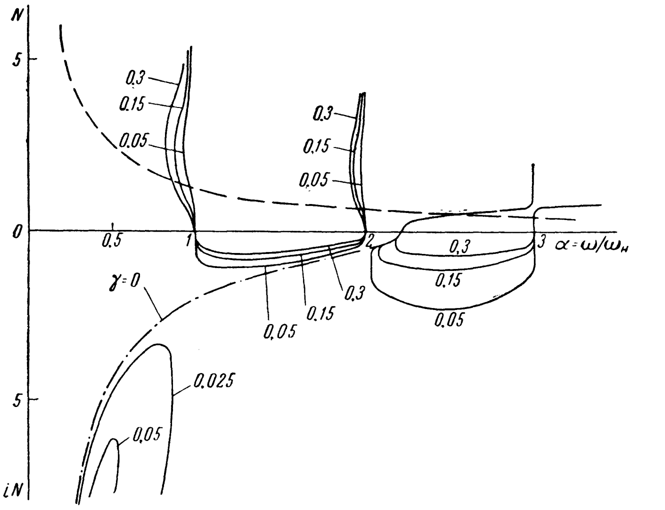


FIG. 2. Frequency dependence of the index of refraction for $\beta = \sqrt{5}$ and for various values of $\gamma$.

investigate the behavior of the roots in the vicinity of the resonances $\alpha \sim n$. As $\alpha \to n - 0$, the negative root approaches $+\infty$; as $\alpha \to n + 0$, it approaches zero. The negative root approaches zero as $\alpha \to n$.

Results of a numerical solution of Eq. (2) are shown in Figs. 1 and 2. The index of refraction $N = \sqrt{s}$ is plotted along the positive ordinate, corresponding to the positive root; the quantity $iN = \sqrt{-s}$, corresponding to the negative root, is plotted along the negative ordinate. We see that the positive root is almost everywhere close to the root of the degenerate equation (8) for $\alpha > \beta$. The drawings give a graphic picture of the behavior of the roots in the vicinity of the resonance points. Finally, the peculiarities of the dependence of the roots on the parameter $\gamma$ are very well seen from the drawings.

## 3. INVESTIGATION OF THE REAL ROOTS OF EQUATION (2) FOR $\alpha < \beta$ ($\omega < \omega_0$)

By comparing the signs of the expressions $\mathscr{D}(\pm\infty, \alpha, \beta, \gamma)$ and $\mathscr{D}(0, \alpha, \beta, \gamma)$ for $\alpha < \beta$, we arrive at the following conclusions:

1. If $\sin \alpha\pi > 0$, then Eq. (2) has an even number of positive and negative roots.

2. If $\sin \alpha\pi < 0$, then Eq. (2) has an even number of positive and an odd number of negative roots.

Comparison of Eq. (2) with Eq. (8) shows that Eq. (2) generally does not have positive roots for small values of $\gamma$, and for frequencies far from the resonance. So far as the negative roots are concerned, such a root will be unique if $\sin \alpha\pi < 0$. As $\gamma \to 0$, this root approaches the solution of Eq. (8). In the case in which $\sin \alpha\pi > 0$, Eq. (2) has two negative roots; as $\gamma \to 0$, one of these approaches the solution of Eq. (8), the other approaches $-\infty$.

By considering the equation in the form (2′), we can investigate the behavior of the roots in the vicinity of the resonances. The picture in the given case is seen to be more complicated than for $\alpha > \beta$. It is most important to note the following.

1. In the vicinity of each resonance $\alpha \sim n$, there is a region $\bar{\alpha}_n(\beta, \gamma) < \alpha < n$ in which Eq. (2) has two positive roots. As $\alpha \to n - 0$, one of these approaches 0, and the other approaches $+\infty$.

2. In the vicinity of each resonance $\alpha \sim n$, there is a region in which Eq. (2) no longer has real roots. For resonances of odd order, this region is located to the left of the point $\alpha = n$: $\tilde{\alpha}_n(\beta, \gamma) < \alpha < \bar{\alpha}_n(\beta, \gamma)$; for resonances of even order, it is located to the right of the point $\alpha = n$: $n < \alpha < \tilde{\alpha}_n(\beta, \gamma)$.

We shall not describe the remaining peculiarities of behavior of roots in the vicinity of the cyclotron resonances in any detail; the general picture can be seen from Fig. 2. We only note that the number of real roots of Eq. (2) in the region $\alpha > \beta$ does not exceed 2, and in the region $\alpha < \beta$, it does not exceed 3.

## 4. COMPLEX ROOTS OF EQUATION (2)

In the investigation of complex roots of Eq. (2), we make use in principle of an argument which goes as follows; if the function $F(s)$ is analytic everywhere in the region $G$ except for a finite number of singular points of the type of poles, and does not vanish on the boundary of the region $\Gamma$, then the change of $\arg F(s)$ in passing about the contour in a positive direction, divided by $2\pi$, is equal to the difference between the number of zeros and poles of the function $F(s)$ in the region $G$:

$$N - P = \frac{1}{2\pi} \operatorname{var}(\arg F(s)) \tag{10}$$

(each zero and pole is considered as many times as its multiplicity).

The function $\mathscr{D}(s, \alpha, \beta, \gamma)$ has no finite singularities in the plane of the complex variable $s$; therefore, for it there will only be the number of zeros on the left side of Eq. (10).

Let us consider a circle of sufficiently large radius $R$ in the complex plane of the variable $s$; the center of the circle is at the origin of the coordinates. If Eq. (2) has roots on the bounding circle, then one can either go around these roots by means of a local deformation of the curve, or decrease the radius $R$ slightly. For the calculation of the change of $\arg \mathscr{D}$ along the boundary of the region, we make use of the asymptotic formula

(4) for the function $\mathscr{D}$ as $|s| \to \infty$, which is valid for both real and complex values of s in the interval $-3\pi/2 < \arg s < 3\pi/2$. It is evident from this formula that, in crossing the right semicircle, $\arg \mathscr{D}$ changes by a quantity of the order of $\pi$, and in crossing the semicircle on the left, by a quantity of the order of $4\alpha^2\gamma R$. The total variation of $\arg \mathscr{D}$ on the contour would be a quantity of the order of $4\alpha^2\gamma R$ which increases without limit as $R \to \infty$. This means that Eq. (2) has an infinite (denumerable) set of roots.

Since the number of real roots is finite (not larger than 3), the number of complex roots must be infinitely great. It is evident from the asymptotic formula (4) that for $s \to \infty$ the roots are grouped beside the imaginary axis in the second and third quadrants. If the parameters $\alpha$ and $\beta$ are fixed, while $\gamma$ approches 0, then all complex roots will tend to infinity.

## 5. DETERMINATION OF THE FREQUENCY AS A FUNCTION OF THE PROPAGATION CONSTANT FROM THE DISPERSION EQUATION (1)

We now consider Eq. (1) from another point of view, in which it is required to find the frequency $\omega = \omega(k)$ from this equation for a given real propagation constant k. Such a problem can easily be investigated with the help of the results obtained above.

To find the real roots of Eq. (1), it is necessary to find in the plane $\alpha$, N the points of intersection of the line $N = kc/\omega_H\alpha$ (the dashed curve in the drawings) with the curve of the function $N = \sqrt{s(\alpha, \beta, \gamma)}$. To each such point $\alpha^*$, $N^*$ there corresponds a pair of roots of Eq. (1): $\omega = \pm\omega_H\alpha^*$. The analysis carried out in Secs. 2 and 3 shows that the number of points of intersection will be infinitely large. If we enumerate the abcissas of these points in increasing order ($\alpha_1 < \alpha_2 < \alpha_3 < \ldots$), then their distribution along the $\alpha$ axis will possess the following features (see Figs. 1 and 2):

1. The values of $\alpha_n$ lie within the limits $n-1 < \alpha_n < n$ $(n = 1, 2, 3, \ldots)$.
2. One of the abcissas $\alpha_{n_0}$ is close to the abcissa of the point of intersection of the lines $N = kc/\omega_H\alpha$ и $N = \sqrt{1-\beta^2/\alpha^2}$, i.e. $\alpha_{n_0} \approx \sqrt{\beta^2 + (kc/\omega_H)^2}$.
3. For $1 \le n < n_0$, the quantity $\alpha_n \approx n$.
4. For $n_0 < n < \infty$, the value of $\alpha_n \approx n-1$.

Thus to each real value of k there corresponds an infinite number of real roots of Eq. (1): $\pm\omega(k) = \pm\omega_H\alpha_n$, and $\omega_n \approx \sqrt{\omega_0^2 + k^2c^2}$, and $\omega_n \approx n\omega_H$ for $1 \le n < n_0$; $\omega_n \approx (n-1)\omega_H$ for $n_0 < n < \infty$.

We shall now show that Eq. (1) does not possess any complex roots $\omega = \omega(k)$. For this purpose, we consider circles $C_m$ of radius $R_m = \omega_H(m + 1/2)$ in the plane of the complex variable $\omega$; here m is a sufficiently large integer. On circles of such a type we have

$$\left|\sum_{n=-\infty}^{\infty} \frac{\zeta_n(Tk^2/m\omega_H^2)}{\omega/\omega_H - n}\right| < A,$$

where A is some constant independent of the number m. Consequently, for sufficiently large values of m,

$$\frac{1}{2\pi}\operatorname{var}(\arg D)\Big|_{C_m} = \frac{1}{2\pi}\operatorname{var}\left(\arg\left[-\frac{\omega^2}{c^2}\right]\right)\Big|_{C_m} = 2.$$

On the other hand, inside the circle $C_m$, the function $D(k, \omega)$ has $2m + 2$ real roots and $2m$ poles of first order at the points $\omega = \pm n\omega_H$ $(n = 1, 2, \ldots, m)$. Since, in accord with (10), the total number of zeros of the function D inside the circle $C_m$ must be equal to $2m + 2$, this means that the complex roots $\omega = \omega(k)$ are absent from Eq. (1).

## CONCLUSIONS

The investigation that has been carried out makes it possible to draw the following conclusions on the peculiarities of the solutions of the dispersion equation (1):

1. For $\omega > \omega_0$, $\omega \ne \omega_H n$ Eq. (1) always has a pair of real roots $\pm k(\omega)$.
2. For all $\omega < \omega_0$, regions exist in the neighborhood of the resonance frequencies $\omega \sim n\omega_H$ in which Eq. (1) has two pairs of real roots. Outside these regions Eq. (1) has no real roots for $\omega < \omega_0$.
3. For $\omega < \omega_0$, regions exist in the vicinity of the resonance frequencies in which Eq. (1) has neither real nor purely imaginary roots [the real roots are lacking in Eq. (2)].
4. For each value $\omega \ne n\omega_H$, Eq. (1) has an infinite set of quartets of complex roots:

$$k = p_n(\omega) \pm iq_n(\omega), \qquad k = -p_n \pm iq_n.$$

5. For any real value of k, Eq. (1) has an infinite set of pairs of real roots $\pm\omega_n(k)$. It has no complex roots $\omega = \omega(k)$.

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