SOVIET PHYSICS JETP VOLUME 13, NUMBER 5 NOVEMBER, 1961

THE PARTICLE MASS IN THE ONE-DIMENSIONAL MODEL WITH FOUR-FERMION COUPLING

V. G. VAKS and A. I. LARKIN

Submitted to JETP editor December 3, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 1392-1398 (May, 1961)

The methods of the theory of superconductivity are used to find the mass of a particle in renormalizable theories with vanishing bare mass and involving a weak coupling constant. Only the zero solution for the mass is possible in electrodynamics and in the one-dimensional Thirring model; for the one-dimensional model of two interacting fields a nonvanishing solution is obtained. A finite expression for the charge has been obtained for this model.

1. INTRODUCTION

It has been proposed by Nambu[1] and the authors[2] that the mass of particles has the same origin as the gap in the energy spectrum of the excitations in a superconductor. The mass is not introduced into the original Lagrangian but arises as a result of "pairing" of particles, which leads to a rearrangement of the vacuum state. As in the theory of superconductivity the mass is equal to zero in any order of perturbation theory but appears in the exact solution. However, the four-fermion interaction considered in the above-mentioned papers leads to strong divergences, which makes quantitative discussion impossible. It is therefore of interest to verify the indicated conjectures on the example of renormalizable theories, where the calculations are possible for small coupling constants.

It turns out that in electrodynamics and in the one-dimensional Thirring model[3] the resultant homogeneous equation for the mass has only the zero solution. However in the one-dimensional model of two interacting fields proposed by Ansel'm[4] there exists in addition to the trivial zero solution also a nonzero solution for the mass. The massless solution turns out to be unstable in this case.

2. ELECTRODYNAMICS AND THE THIRRING MODEL

Abrikosov, Landau, and Khalatnikov[5] have considered the question of the particle mass in electrodynamics in the approximation $e_1^2 < 1$. If $d_l(k^2) = 0$ then the electron Green's function has the form $G(p) = (\hat{p} - m(p^2))^{-1}$. In the case of a vanishing bare mass the quantity $m(p^2)$ obeys the homogeneous equation

$$m(\xi) = \frac{3e_1^2}{4\pi}\int_{\xi}^{L}\frac{m(z)\,dz}{1+(e_1^2/3\pi)(L-z)}, \qquad (1)$$

where $\xi = \ln(p^2/m^2)$, $L = \ln(\Lambda^2/m^2)$, Λ is the cut-off momentum and e_1 is the bare charge.

Equation (1) has only the zero solution. Indeed, the general solution of the differential equation corresponding to Eq. (1) has the form

$$m(\xi) = m(L)\,[1 + (e_1^2/3\pi)(L-\xi)]^{-9/4}. \qquad (2)$$

Substituting Eq. (2) into Eq. (1) we obtain $m(\xi) = 0$. The same result is obtained in meson theory for a coupling constant $g_0^2 \ll 1$.

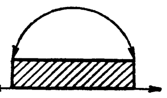
Let us consider the one-dimensional Thirring model. The Lagrangian has the form

$$\mathcal{L}(x) = -u^+\sigma p u - \lambda(u^+\sigma^r u)(u^+\sigma^r u). \qquad (3)$$

Here u is a two-component spinor, $\sigma^r = (\boldsymbol{\sigma}, 1)$, the $\boldsymbol{\sigma}$ are the Pauli matrices, and $\sigma p = \sigma_z p_z - p_0$. This model has an exact solution.[3,6] We shall restrict ourselves to the case of small λ and will take into account only terms of order $[\lambda \ln(\Lambda^2/p^2)]^n$, throwing away terms of the form $\lambda[\lambda \ln(\Lambda^2/p^2)]^n$. The totality of diagrams with two incident and two outgoing lines will be referred to as the vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2; p_3, p_4)$. As was shown by Ansel'm[7] and Maĭer and Shirkov[8] in the case when the momenta p_i are of the same order Γ does not get renormalized, but simply equals the zero-order expression $4\lambda\sigma^r_{\alpha\gamma}\sigma^r_{\beta\delta}$.

We shall look for the mass of the particle by using the method used in the theory of superconductivity for finding the energy spectrum.[9] We introduce the functions $G(x-y) = \langle Tu(x)u^+(y)\rangle$ and $F^+(x-y) = \langle Tu^+(x)u^+(y)\rangle$. The Dyson equations for G and F^+ are of the form

$$G(p) = G_0(p)\,[1 + \Sigma_{11}(p)\,G(p) + \Sigma_{20}(p)\,F^+(p)],$$
$$F^+(p) = \widetilde{G}_0(-p)\,[\Sigma_{11}(p)\,F^+(p) + \Sigma_{02}(p)\,G(p)]. \qquad (4)$$

$\Sigma_{02} =$ 

Here $\Sigma_{11}(p)$ represents, as usual, the totality of compact diagrams with one incident and one outgoing line, Σ_{20} and Σ_{02} representing similar totalities with two incident and two outgoing lines respectively. In our approximation the quantity $\Sigma_{11}(p)$ is equal to zero.[7] The matrix $\Sigma_{20}(p) = \Sigma_{02}^{+}(-p)$ is of the form

$$\Sigma_{\alpha\beta}^{20}(p) = \sigma_{\alpha\beta}^{y}\Delta^{*}(p^2).$$

From Eq. (4) we obtain

$$G(p) = -i\sigma' p/(p^2 + |\Delta|^2 - i\delta),$$
$$F^{+}(p) = -\sigma_y\Delta/(p^2 + |\Delta|^2 - i\delta), \tag{5}$$

where

$$\sigma' p = \sigma_z p_z + p_0, \qquad \delta \to +0.$$

In each of the diagrams forming Σ_{02} should enter an odd number of lines representing F or F^{+}; in the asymptotic region $p^2 \gg \Delta^2$ it is sufficient to limit oneself to a single line with F^{+}. The equation for Σ_{02} is shown graphically in the figure. It is of the form

$$\Sigma_{\alpha\beta}^{02}(p) = \int K_{\alpha\beta\gamma\delta}(p, -p;\ p', -p')\, F_{\gamma\delta}^{+}(p')\frac{dp'}{(2\pi)^2}, \tag{6}$$

where $K_{\alpha\beta\gamma\delta}(p_1, p_2; p_3, p_4)$ is a four-pole term irreducible with respect to the separation of the lines p_1, p_2 from p_3, p_4.[10] The matrix $K_{\alpha\beta\gamma\delta}$ is of the form $K_{\alpha\beta\gamma\delta} = -4i\lambda\sigma_{\alpha\gamma}^{r}\sigma_{\beta\delta}^{r}K(p_1, p_2; p_3, p_4)$. Substituting this expression into Eq. (6) we obtain

$$\Delta(p) = \int K(p, -p;\ p', -p')\frac{\Delta(p')}{p'^2 + |\Delta|^2}\frac{dp'}{(2\pi)^2}. \tag{7}$$

Nambu[1] and the authors[2] used for K the zeroth order approximation K = 1. In this case the following equation is obtained for Δ:

$$\Delta\left(1 - \frac{\lambda}{4\pi}\ln\frac{\Lambda^2}{|\Delta|^2}\right) = 0. \tag{8}$$

This equation has for arbitrary λ the nonzero solution $|\Delta| = \Lambda\exp(-2\pi/\lambda)$. However in this model such an approximation leads to a qualitatively false result. In the expression for K one must take into account all terms of the order $[\lambda\ln(\Lambda^2/p^2)]^n$. The quantity K was found to within this accuracy by Ansel'm:[7]

$$K(p, -p;\ p', -p') = 1 - (\lambda/4\pi)\ln(\Lambda^2/p_1^2), \tag{9}$$

where $p_1^2 = \max\{p^2, p'^2\}$.

After substitution of Eq. (9) into Eq. (7) and the introduction of the logarithmic variables

$$\xi = \ln\frac{p^2}{m^2}, \qquad z = \ln\frac{p'^2}{m^2}, \qquad L = \ln\frac{\Lambda^2}{m^2}, \qquad m^2 = |\Delta|^2,$$

we obtain the equation

$$\Delta(\xi) = \frac{\lambda}{4\pi}\int_0^L \Delta(z)\,dz - \left(\frac{\lambda}{4\pi}\right)^2\left[(L - \xi)\int_0^\xi \Delta(z)\,dz + \int_\xi^L (L - z)\,\Delta(z)\,dz\right]. \tag{10}$$

Differentiating Eq. (10) with respect to ξ we get

$$\Delta'(\xi) = (\lambda/4\pi)^2\int_0^\xi \Delta(z)\,dz. \tag{11}$$

The solution of Eq. (11) is of the form $\Delta(\xi) = \Delta_0 \times \cosh(\lambda\xi/4\pi)$. After substitution of $\Delta(\xi)$ into Eq. (10) we arrive at the following equations for Δ_0:

$$\Delta_0\exp\left(-\frac{\lambda}{4\pi}\ln\frac{\Lambda^2}{|\Delta_0|^2}\right) = 0, \tag{12}$$

which has only the zero solution.

In the model under discussion this result is quite natural. In order for a mass to appear it is necessary for two massless particles to form a bound state. But in this model there is no physical scattering of two particles — in the exact solution of the two-particle problem there is no reflected wave, only the phase changes.[3] It is therefore natural that in this case the interaction does not lead to the formation of a bound state. Formally this finds its reflection in the fact that in the Thirring model the exact vertex part is momentum independent, whereas a bound state should have a corresponding pole in Γ. Analogous considerations apply in electrodynamics — in the photon Green's function $d_t = [1 + (e_1^2/3\pi)(L - \xi)]^{-1}$ there appears no pole corresponding to a bound state.

3. THE VERTEX PART IN THE ANSEL'M MODEL

Let us consider the one-dimensional four-fermion model with two two-component fields u and v:[4]

$$\mathscr{L}(x) = -u^{+}\sigma p u - v^{+}\sigma p v + \lambda_1(u^{+}\sigma_r u)^2 + \lambda_2(v^{+}\sigma_r v)^2 + \lambda_3(u^{+}\sigma_r u)(v^{+}\sigma_r v) + \lambda_4(u^{+}\sigma_\mu u)(v^{+}\sigma_\mu v). \tag{13}$$

Here the summation over μ is of the form $\sigma_\mu \times \sigma_\mu = \sigma_z \times \sigma_z - 1 \times 1$, the remaining notation being the same as in Eq. (3). One could take in the original Lagrangian $\lambda_1 = \lambda_2 = \lambda_4 = 0$, however, this would not result in a simplification since terms of this form appear in Γ in subsequent approximations.

We introduce the two-row "isotopic" matrices τ_1, τ_2, determined by the relations $\tau_1 u = u$, $\tau_2 v = v$, $\tau_1 v = \tau_2 u = 0$. Then the vertex part is of the following form:

$$\Gamma = i\{\sigma^r\times\sigma^r\,[(\tau_1\times\tau_1)\,4\alpha_1 + (\tau_2\times\tau_2)\,4\alpha_2 + (\tau_1\times\tau_2)\,\alpha_3] + (\sigma^\mu\times\sigma^\mu)(\tau_1\times\tau_2)\,\alpha_4\}. \tag{14}$$

We shall solve the problem in the same approximation as was used above. When the fact that the Green's function G is given in the asymptotic region by its lowest order approximation is taken into account, one gets by the methods outlined by Ansel'm[4] and Dyatlov, Sudakov and Ter-Martirosyan[10] the following equations* for the α_i:

$$\alpha_1(\xi) = \lambda_1 + \frac{1}{8\pi}\int_\xi^L \alpha_3^2(z)\,dz,$$

$$\alpha_2(\xi) = \lambda_2 + \frac{1}{8\pi}\int_\xi^L \alpha_3^2(z)\,dz,$$

$$\alpha_3(\xi) = \lambda_3 - \frac{1}{\pi}\int_\xi^L \alpha_3(z)\,[\alpha_3(z) + \alpha_4(z) - 2\alpha_1(z) - 2\alpha_2(z)]\,dz,$$

$$\alpha_4(\xi) = \lambda_4 + \frac{1}{\pi}\int_\xi^L \alpha_3(z)\left[\frac{\alpha_3(z)}{2} + \alpha_4(z) - 2\alpha_1(z) - 2\alpha_2(z)\right]dz. \tag{15}$$

Here $\xi = \ln(p^2/m^2)$ and all momenta entering or leaving the vertex part are of order p. Introducing $\lambda = \lambda_1 + \lambda_2$, $\alpha = \alpha_1 + \alpha_2$ and differentiating Eq. (15) we arrive at the following equations:

$$\frac{d\alpha}{d\xi} = -\frac{\alpha_3^2}{4\pi}, \qquad \frac{d\alpha_3}{d\xi} = \frac{\alpha_3}{\pi}(\alpha_3 + \alpha_4 - 2\alpha),$$

$$\frac{d\alpha_4}{d\xi} = -\frac{\alpha_3}{\pi}\left(\frac{\alpha_3}{2} + \alpha_4 - 2\alpha\right). \tag{16}$$

Dividing the second and third of Eqs. (16) by the first and adding the resultant equations we get, after Eq. (15) is taken into account,

$$d(\alpha_3 + \alpha_4)/d\alpha = -2,$$

$$\alpha_3 + \alpha_4 + 2\alpha = \lambda_3 + \lambda_4 + 2\lambda \equiv 4C_1. \tag{17}$$

Further, by quadrature we obtain

$$\alpha_3^2 - 16\alpha^2 + 32\,C_1\alpha = \lambda_3^2 + 8\lambda\,(\lambda_3 + \lambda_4) \equiv 16\,C_2. \tag{18}$$

Substituting Eq. (18) into Eq. (16) we get

$$\int\frac{d\alpha}{\alpha^2 - 2C_1\alpha + C_2} = -\frac{4\xi}{\pi}. \tag{19}$$

Let $D^2 = |C_2 - C_1^2|$. Then for $C_2 - C_1^2 = \frac{1}{16}(2\lambda_3 + \lambda_4 - 2\lambda)(2\lambda - \lambda_4) > 0$ we have

$$\alpha = D\,\mathrm{tg}\,\varphi + C_1, \quad \alpha_3 = \pm 4D/\cos\varphi, \quad \alpha_4 = 4C_1 - 2\alpha - \alpha_3;$$

$$\varphi = 4D\pi^{-1}(L-\xi) + \varphi_0, \qquad D\,\mathrm{tg}\,\varphi_0 = \tfrac{1}{4}(2\lambda - \lambda_3 - \lambda_4). \tag{20a†}$$

For $C_2 - C_1^2 < 0$ we have

$$\alpha = D\,\mathrm{th}\,\varphi + C_1, \qquad \alpha_3 = \pm 4D/\mathrm{ch}\,\varphi,$$

$$\alpha_4 = 4C_1 - 2\alpha - \alpha_3; \qquad \varphi = 4D\pi^{-1}(L-\xi) + \varphi_0,$$

$$D\,\mathrm{th}\varphi_0 = \tfrac{1}{4}(2\lambda - \lambda_3 - \lambda_4). \tag{20b}$$

Finally, for $C_2 - C_1^2 = 0$ we have

$$\alpha_3 = \lambda_3[1 \pm \lambda_3\pi^{-1}(L-\xi)]^{-1}, \quad \alpha = \lambda \pm \tfrac{1}{4}(\alpha_3 - \lambda_3),$$

$$\alpha_4 = 4\lambda \pm \lambda_3 - 2\alpha - \alpha_3. \tag{20c}$$

The upper sign in the last formula corresponds to the case $2\lambda - \lambda_4 = 0$, the lower to $2\lambda_3 + \lambda_4 - 2\lambda = 0$. We note that for $\lambda_3 = 0$ the quantities α, α_4 and α_3 are equal to their zeroth order values λ, λ_4 and 0, and, as in the Thirring model, no mass is produced. This case is obtained when the four-fermion interaction is considered to be due to a heavy vector boson with an interaction Lagrangian $(e_1u^+\sigma_\mu u + e_2v^+\sigma_\mu v)A_\mu$.

When the mass m is equal to zero the expressions (20) for Γ are valid for arbitrary momenta p. Then in the case (20a) Γ has poles at the points $\varphi + \varphi_0 = \pi/2 + n\pi$. This is an indication of the appearance of bound states and, consequently, of instability of the massless solution. Let us note that in these same cases there is no "zero charge," the renormalized charge remaining finite also in the limit $L \to \infty$. The behavior of Γ in the case (20c) is determined by the sign of λ_3. In what follows we limit ourselves for the sake of simplicity to the case $\lambda_4 = 2\lambda$, $\lambda_3 = -4\lambda$. Then

$$\alpha = \alpha_4/2 = -\alpha_3/4 = \lambda\,[1 - 4\lambda\pi^{-1}(L-\xi)]^{-1}, \tag{21}$$

and for $\lambda > 0$ the massless solution is unstable.

4. MASS OF THE PARTICLE IN THE ANSEL'M MODEL

We shall seek the particle mass by the same method as was used in Sec. 2. We introduce the functions $G = \langle Tu(x)u^+(y)\rangle$, $F^+ = \langle Tv^+(x)u^+(y)\rangle$ and write out the Dyson equations for these functions:

$$G(p) = G_0(p)\,[1 + \Sigma_{20}(p)\,F^+(p)],$$

$$F^+(p) = \widetilde{G}_0(-p)\,\Sigma_{02}(p)\,G(p). \tag{22}$$

We look for $\Sigma_{20}(p) = \Sigma_{02}^+(-p)$ in the form $\Sigma_{20} = \sigma_y\Delta(p^2)$; then G and F^+ are given by Eqs. (5), and the equation for Σ is of the form of Eq. (6). The irreducible four-pole term K has in this case the form

$$K_{\alpha\beta\gamma\delta}(p,-p;\,p',-p') = i\sigma^\mu_{\alpha\gamma}\sigma^\mu_{\beta\delta}K_1 + i\sigma^r_{\alpha\gamma}\sigma^r_{\beta\delta}K_2;$$

$$K_1(\eta) = \lambda_4 + \frac{1}{2\pi}\int_\eta^L [(\alpha_3(z) + \alpha_4(z))^2 - 4\alpha(z)\,\alpha_3(z)]\,dz,$$

$$K_2(\eta) = \lambda_3 + \frac{2}{\pi}\int_\eta^L \alpha_3(z)\,\alpha(z)\,dz, \tag{23}$$

*Equations (15) differ from the analogous equations of Ansel'm[4] in which a calculational error was commited. The error, however, did not affect the qualitative results.

†tg = tan, ch = cosh, th = tanh.

where $\eta = \ln(p_1^2/m^2)$, $p_1^2 = \max\{p^2, p'^2\}$. The equation for Δ, analogous to Eq. (7), is written as follows:

$$\Delta(p^2) = 2i\int \frac{\Delta(q^2)dq}{q^2+m^2(q^2)}[K_1(p,-p;\ q,-q) + 2K_2(p,-p;\ q,-q)]. \quad (24)$$

Equation (24) becomes in the special case (21)

$$\Delta(\xi) = \frac{9\lambda}{4\pi}\int_0^L \Delta(z)\,dz + \frac{3}{4\pi}\Big(\alpha(\xi)\int_0^\xi \Delta(z)\,dz + \int_\xi^L \alpha(z)\,\Delta(z)\,dz\Big). \quad (25)$$

The general solution of the differential equation corresponding to Eq. (25) may be written as follows:

$$\Delta(\xi) = C_1[1-4\lambda\pi^{-1}(L-\xi)]^{-1/4} + C_2[1-4\lambda\pi^{-1}(L-\xi)]^{-3/4}. \quad (26)$$

Substituting Eq. (26) into Eq. (25) we obtain

$$C_2 = -\tfrac{1}{3}(1-4\lambda\pi^{-1}L)^{1/2}C_1, \qquad C_2 = 0. \quad (27)$$

The condition for the existence of a nonzero solution has the form

$$1 - 4\lambda\pi^{-1}L = 0. \quad (28)$$

From Eq. (28) we get

$$m = \Lambda\exp(-\pi/8\lambda). \quad (29)$$

Since terms of order λ were neglected in Eq. (28) in comparison with unity, the expression for m is accurate only to within a numerical factor multiplying the exponential. Correspondingly in the asymptotic region $p^2 \gg m^2$ the quantity $\Delta(p^2)$ is proportional to $[\ln(p^2/m^2)]^{-1/4}$. We note that in this case the approximation of Nambu and the authors,[1,2] i.e., the taking into account of only the first terms in expression (23) for K, results in a numerical inaccuracy only: the factor $\pi/8$ in Eq. (29) is replaced by $\pi/6$.

In the original Lagrangian (13) one may introduce a mass term $m_0(u^+\sigma_y v^+ + v\sigma_y u)$. Then on the right side of Eq. (25) the additional term m_0 will appear. By solving the resultant inhomogeneous equation we obtain the connection between the physical mass m and the bare mass m_0:

$$m\left(1-\frac{4\lambda}{\pi}\ln\frac{\Lambda^2}{m^2}\right)^{3/4} = m_0. \quad (30)$$

In the limit as m_0 tends to zero the mass m tends to a finite value, determined by Eq. (29). Although the value m = 0 is a solution of the homogeneous equation (25), this value is not reached by this limiting procedure.

In the case under consideration the mass appeared as a result of the pairing of particles described by the different fields u and v. Let us consider another possibility, namely when particles of the same type are paired: u with u or v with v. The resultant massive particle would then be identical to its antiparticle, i.e., these would be Majorana particles. In that case one must consider in addition to the function G the functions $F_1^+ = \langle Tu^+(x)u^+(y)\rangle$ and $F_2^+ = \langle Tv^+(x)v^+(y)\rangle$. The calculations are analogous to those described above. Instead of Eq. (27) for Δ we obtain the general expression

$$\Delta(\xi) = C_1[1-4\lambda\pi^{-1}[L-\xi)]^{1/4} + C_2[1-4\lambda\pi^{-1}(L-\xi)]^{-5/4}. \quad (31)$$

Substituting Eq. (31) into the appropriate integral equation we get

$$C_2 = \tfrac{1}{5}(1-4\lambda\pi^{-1}L)^{3/2}C_1, \qquad C_1 = 0. \quad (32)$$

Equation (32) possesses only the zero solution $C_1 = C_2 = 0$, i.e., the mass is equal to zero and no Majorana-type pairing occurs.

In the one-dimensional model under consideration it is possible to form out of the spinors u, u^+, v, v^+ in addition to $u\sigma_y v$ and $u\sigma_y u$ a number of other scalars, which would in the three-dimensional case transform like components of a vector or tensor (for example $u\sigma_x v$ and $u^+\sigma_y v$). This leads to the possibility of pairing of a type different from that considered above. It turns out that all these pairings are absent if the constants are so related as to result in Eq. (21); for different relations among the constants some of these pairings do appear.

When the relation (29) is taken into account the expression (21) for the vertex part becomes

$$\alpha(p^2) = \pi/4\ln(p^2/m^2). \quad (33)$$

Consequently, in the asymptotic region $p^2 \gg m^2$ the effective interaction (33) depends only on the observable mass m and does not contain the bare constant λ or the cut-off parameter Λ.

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Translated by A. M. Bincer
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