ASYMPTOTIC FORM OF THE VERTEX PART IN ELECTRODYNAMICS

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The vertex part in electrodynamics is found for a certain range of momentum values.

THE vertex part in electrodynamics $\Gamma_\sigma(p, q, l)$ was obtained by Sudakov[1] in the region $l^2 \gg p^2$, $q^2 \gg m^2$, $e^2 \ln(l^2/m^2) \ll 1$. By using the renormalization group method Blank and Shirkov[2] generalized this result to the region in which vacuum polarization plays a role. They considered in addition to the terms of order $\sim e^2 \ln(l^2/p^2) \ln(l^2/q^2)$ treated by Sudakov also terms of order $e^2 \ln(l^2/m^2)$, $e^2 \ln(p^2/m^2)$, and $e^2 \ln(q^2/m^2)$, however assumed, as is customary, that

$$e^2 d_t(l^2) = e^2\left[1 - \frac{e^2}{3\pi}\ln(-l^2/m^2)\right]^{-1} \ll 1,$$

as well as that $e^2 \ln(l^2/p^2) \ll 1$ and $e^2 \ln(l^2/q^2) \ll 1$. As a result of an insufficiently correct application of the method the result obtained by Blank and Shirkov,[2] also cited by Bogolyubov and Shirkov,[3] turned out to be erroneous. Since the problem may be of some interest as far as the method is concerned we present in this note a correct version of the Blank and Shirkov result. We make use of the method and notation of Sudakov.[1]

At first we set the longitudinal part of the photon Green's function $d_l(k^2)$ equal to zero. Then the skeleton diagram of lowest order in $\Gamma_\sigma(p, q, l)$ has the form

$$\Gamma_\sigma^{(2)} = \frac{e^2}{\pi i}\int \gamma_\mu \frac{1}{\hat{p}-\hat{k}}\gamma_\sigma \frac{1}{\hat{q}-\hat{k}}\gamma_\nu \frac{dk}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \times\left(1 - \frac{e^2}{3\pi}\ln\frac{-k^2}{m^2}\right)^{-1}. \quad (1)$$

It was shown by Abrikosov[4] that the longitudinal term $\sim k_\mu k_\nu$ in Eq. (1) gives, after integration in the region $|k^2| < |l^2|$, only singly logarithmic terms $\sim e^2 \ln(l^2/p^2)$, $e^2 \ln(l^2/q^2)$; whereas the region $|k^2| > |l^2|$ gives no logarithms at all. One may therefore drop this term in Eq. (1) and restrict the integration to the region $|k^2| < |l^2|$.

For the sake of definiteness let $l^2 < 0$; the opposite case will be obtained by analytic continuation. Following Sudakov we introduce the variables u, v, x:

$$k = pu + qv + k_\perp, \qquad k_\perp^2 / l^2 = x,$$

and transform $\Gamma_\sigma^{(2)}$ to the form

$$\Gamma_\sigma^{(2)} = \frac{3i}{4\pi}\gamma_\sigma \int \frac{du}{u}\frac{dv}{v}\frac{dx}{x - uv - i\delta}[\beta_l^{-1} - \ln(x - uv)]^{-1}. \quad (2)$$

Here

$$\delta \to +0, \qquad \beta_l = \beta[1 - \beta \ln(-l^2/m^2)]^{-1}, \qquad \beta = e^2/3\pi,$$

and the relevant region of integration is given by

$$\alpha_1 = |q^2/l^2| \ll |u| \ll 1, \qquad \alpha_2 = |p^2/l^2| \ll |v| \ll 1,$$

$$0 < x < \min\{|u|, |v|\}.$$

We integrate first over x. For $uv < 0$ the singular point $x = uv$ lies outside the contour of integration, for $uv > 0$ we deform the contour of integration into the complex x-plane so as to pass the point $x = uv$ from below along a semicircle of radius uv. Taking into account that $\beta_l \ll 1$ we find

$$\Gamma_\sigma^{(2)} = \frac{3i}{4\pi}\gamma_\sigma \int \frac{du}{u}\int \frac{dv}{v}\left[\frac{\theta(uv)\, i\pi}{\beta_l^{-1} - \ln uv} + f(|uv|)\right] = -\frac{3}{2}\gamma_\sigma \int_{\alpha_1}^{1}\frac{du}{u}\int_{\alpha_2}^{1}\frac{dv}{v}(\beta_l^{-1} - \ln uv)^{-1}, \quad (3)$$

where $\theta(x) = 0$ for $x < 0$, $\theta(x) = 1$ for $x > 0$, and where $f(x)$ is a certain function which drops out of the answer after integration over an even interval of u or v. According to Sudakov,[2] in the 2n-th order relevant contributions to Γ_σ come from those diagrams in which all virtual photon lines surround the point at which the real photon l is emitted. Each such term, in a manner analogous to that described above, is reduced to the form

$$\gamma_\sigma\left(-\frac{3}{2}\right)^n \int \frac{du_1}{u_1}\frac{du_2}{u_2}\cdots\frac{du_n}{u_n}\frac{dv_{i_1}}{v_{i_1}}\frac{dv_{i_2}}{v_{i_2}}\cdots\frac{dv_{i_n}}{v_{i_n}} \times(\beta_l^{-1} - \ln u_1 v_1)^{-1}(\beta_l^{-1} - \ln u_2 v_2)^{-1}\cdots(\beta_l^{-1} - \ln u_n v_n)^{-1}, \quad (4)$$

where the integration is over the region

$$\alpha_1 \ll u_1 \ll u_2 \ldots \ll u_n \ll 1,$$

$$\alpha_2 \ll v_{i_1} \ll v_{i_2} \ldots \ll v_{i_n} \ll 1.$$

If diagrams with all possible distributions of photon lines are combined, then the inequality for the v_{i_k} of Eq. (4) is supplemented by all other possible inequalities among the v_i; consequently one may integrate over all v_i from α_2 to unity independently. If one further symmetrizes over u_i one arrives at the result

$$\Gamma_\sigma^{(2n)} = \frac{\gamma_\sigma}{n!}\left[-\frac{3}{2}\int\limits_{\alpha_1}^{1}\frac{du}{u}\int\limits_{\alpha_2}^{1}\frac{dv}{v}(\beta_l^{-1}-\ln uv)^{-1}\right]^n \equiv \gamma_\sigma\frac{(-J)^n}{n!}. \quad (5)$$

The physical reason which makes the above simplifications possible is connected with the identity and "infrared" smallness of the momenta of the relevant quanta. From Eq. (5) we obtain

$$\Gamma_\sigma = \gamma_\sigma \exp(-J),$$

$$J = \frac{3}{2}\Big[\Big(\beta^{-1}-\ln\frac{xy}{z}\Big)\ln\Big(\beta^{-1}-\ln\frac{xy}{z}\Big) + (\beta^{-1}-\ln z)\ln(\beta^{-1}-\ln z) - (\beta^{-1}-\ln x)\ln(\beta^{-1}-\ln x) - (\beta^{-1}-\ln y)\ln(\beta^{-1}-\ln y)\Big]. \quad (6)$$

Here $z = -l^2/m^2$, $x = -p^2/m^2$, $y = -q^2/m^2$; $xy/z \gg 1$ in accordance with the assumed approximation

$$\beta\ln\frac{z}{x},\ \beta\ln\frac{z}{y} \ll 1, \qquad \beta\ln z,\ \beta\ln x,\ \beta\ln y \lesssim 1.$$

The result (6) is easily generalized to an arbitrary $d_l(k^2) \neq 0$. At that one should use for the functions G and Γ_μ in internal parts of the diagrams their asymptotic expressions

$$G(p) = \frac{1}{\hat{p}}\alpha^{-1}(-p^2/m^2), \qquad \Gamma_\mu(p_1, p_2, k) = \gamma_\mu\alpha(-f^2/m^2),$$

$$|f^2| = \max\{|p_1^2|, |p_2^2|, |k^2|\},$$

$$\alpha(x) = \exp\Big(\frac{e^2}{4\pi}\int\limits_{\ln x}^{\infty} d_l(\xi)\,d\xi\Big). \quad (7)$$

It is easy to see that the change in Γ_μ according to Eq. (7) is compensated by a corresponding change in the function G at all vertices except for the point at which the external photon l is emitted. Therefore each term in Eq. (5) gets multiplied by $\alpha(-l^2/m^2)$ so that the answer is given by

$$\Gamma_\sigma = \gamma_\sigma\exp\Big(-J + \frac{e^2}{4\pi}\int\limits_{\ln z}^{\infty} d_l(\xi)\,d\xi\Big). \quad (8)$$

The connection between Eqs. (8) and (6) corresponds to a usual gauge transformation of Γ_σ as a result of a change in the gauge of $d_l(k^2)$.[6]

The renormalization invariance of expressions (6) and (8) is easily verified directly from the equations of the renormalization group [Eq. (3) of Blank and Shirkov[2] or Eq. (42.33) of Bogolyubov and Shirkov[3]]. For $e^2 \ln z \ll 1$ we obtain from Eqs. (6) and (8) Sudakov's result

$$\Gamma_\sigma = \gamma_\sigma\exp\Big(-\frac{e^2}{2\pi}\ln\frac{z}{x}\ln\frac{z}{y}\Big). \quad (9)$$

We note in conclusion that the exponential form of the expressions (6) and (8) indicates that it must be possible also to obtain the answer by the renormalization group method. The correct prescription is as follows: making use of a renormalization invariant expression for

$$\frac{e^2}{3\pi}d_t(k^2) = \beta\Big(1-\beta\ln\frac{-k^2}{m^2}\Big)^{-1},$$

one must calculate the correction to Γ_σ in lowest order, and then go over in the usual manner[3] from this "perturbation theory" formula to the exponential expressions (6) and (8). A proof of this prescription in the general case would make it possible to solve in a simple manner certain other problems, such as for example the taking into account in the vertex part $\Gamma_\sigma(p, q, l)$ under discussion of "singly logarithmic" terms of order $e^2\ln(l^2/p^2)$ and $e^2\ln(l^2/q^2)$.

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