QUANTUM THEORY OF ACOUSTIC OSCILLATIONS OF AN ELECTRON-ION PLASMA IN A MAGNETIC FIELD

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Conditions for the existence of acoustic excitations are studied on the basis of the quantum dispersion equations for electron-ion plasma oscillations. It is shown that in strong magnetic fields longitudinal ultrasonic vibrations with a wave vector perpendicular to the magnetic field degenerate into ionic vibrations, inasmuch as the screening radius becomes infinite in this case. The decay frequency of ultrasonic waves moving along the magnetic field, as well as across it, is calculated. The decay frequency thus determined is found to oscillate, depending on the magnetic field strength.

1. Classical theory of longitudinal low-frequency oscillations of an electron-ion plasma in a magnetic field with a Maxwellian energy particle distribution in the ground state was considered in the work of Stepanov.[1] An attempt at the construction of a quantum theory was made in the research of Yakovlev and Kalyush;[2] however, since they did not take into account the quantum energy of the orbital motion of the particles in the magnetic field in the ground state, their results did not differ essentially from the results of the work of Stepanov. Moreover, the limits of applicability of the resultant formulas were not given in reference 2, and the damping was not calculated. The final results, with the exception of quite insignificant temperature-dependent corrections, are, in point of fact, the consequence of the hydrodynamic approximation in the description of the motion of ions interacting according to a screening law. Finally, it should be remarked that the assumptions made therein for the calculation of the acoustic branch at low wave numbers, for waves propagating across the magnetic field, contradict the result obtained.

The purpose of the present research was to construct a theory of low-frequency oscillations of the electron-ion plasma which would take into account the quantization of the energy of the orbital motion of charged particles in the magnetic field, and also to explain the dependence of the ultrasonic attenuation coefficient on the direction of the magnetic field.

2. We pause briefly to consider the specific case of absorption of ultrasound in a metal, which we shall approximate in what follows by an electron-ion plasma. We shall represent schematically three mutually interacting subsystems: the ultrasonic wave, the electrons, and the lattice (ions). The energy of the regular motion of the ultrasonic wave can dissipate via the channels shown in Fig. 1. For metals, the relaxation time τ_3 is always large in comparison with τ_1 and τ_2; therefore, in what follows, we shall not take it into account.

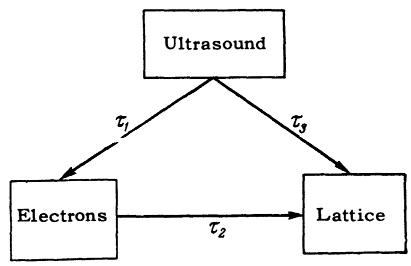

FIG. 1

Two limiting cases can be separated, depending on the ratio of the ultrasonic wavelength λ and the mean free path of the electron l, which is determined by the collisions with the lattice. The first case occurs when $\tau_2 \gg \tau_1$. Here, the slowest process is the transfer of energy obtained by the electrons from the ultrasonic wave to the lattice. This case corresponds to a wavelength $\lambda \gg l$. The electrical conductivity of the metal plays the principal role in such a case. In the second case, $\tau_2 \ll \tau_1$. This situation is realized when $\lambda \ll l$. Here, the slowest process is the process of energy transfer from the ultrasonic wave to the electron; therefore, collisions of electrons with the lattice do not play any role and can be neglected.

In the second case, we can consider the electron-ion system in the plasma approximation. The mechanism of energy transfer from the regular collective motion of the plasma to the individual, random motion of the electrons was first established by Landau[3] in the case of damping of plasma waves. Further investigation of this mechanism in the study of ultrasonic damping was undertaken in the researches of Silin[4] and Kittel.[5] The effect of the magnetic field on the damping was not studied in these researches. An attempt is made below to study the effect of the magnetic field on the ultrasonic absorption coefficient within the framework of the plasma model of the metal.

3. The quantum dispersion relation was found earlier for longitudinal oscillations in a system of particles of one kind.[6] The result is trivially generalized to the case of several kinds of particles interacting with one another by Coulomb's law. For the case of an electron-ion plasma, the dispersion equation has the form

$$1 = \lim_{\gamma\to 0}\frac{G(q)}{2\pi^2\hbar^2\alpha^2}\sum_{nn'}F^2_{nn'}(q_x)\left\{\int dk_z\frac{f_{01}(E^{(1)}_{k_z+q_z,n'})-f_{01}(E^{(1)}_{k_z,n})}{E^{(1)}_{k_z+q_z,n'}-E^{(1)}_{k_z,n}-\hbar\omega+i\hbar\gamma}+\frac{z^2}{2}\int dk_z\frac{f_{02}(E^{(2)}_{k_z+q_z,n'})-f_{02}(E^{(2)}_{k_z,n})}{E^{(2)}_{n_z+q_z,n'}-E^{(2)}_{k_z,n}-\hbar\omega+i\hbar\gamma}\right\}, \tag{1}$$

where

$$G(q)=4\pi e^2/q^2,\qquad \alpha^2=c\hbar/eH,\qquad \omega_{jc}=e_jH/m_jc,$$
$$E^{(j)}_{k_z,n}=\hbar^2k_z^2/2m_j+\hbar\omega_{jc}(n+{}^1/_2)$$

($j = 1, 2$; 1 refers to electrons and 2 refers to ions), $e_1 = -e$, $e_2 = Ze$;

$$F_{nn'}(q_x)=(n!/n'!)^{1/2}\exp\{-\alpha^2q_x^2/4\}(-\alpha q_x/\sqrt{2})^{n'-n}L_n^{n'-n}(\alpha^2q_x^2/2),$$
$$L_n^{n'-n}(x)=(1/n!)\,e^{-x}x^{n'-n}\frac{d^n}{dx^n}(e^xx^{n'}),\qquad n'\geqslant n,$$

f_{01} is the degenerate Fermi function, f_{02} is the Maxwellian distribution function of the ions.

For simplicity, we shall neglect the chaotic motion of the ions, although consideration of this motion does not present any difficulties. In the description of the ions, this approximation is equivalent to the hydrodynamic approximation. In such an approximation, we have

$$1 = \lim_{\gamma\to 0}\frac{G(q)}{2\pi^2\hbar^2\alpha^2}\sum_{nn'}F^2_{nn'}(q_x)\int dk_z\frac{f_{01}(E^{(1)}_{k_z+q_z,n'})-f_{01}(E^{(1)}_{k_z,n})}{E^{(1)}_{k_z+q_z,n'}-E^{(1)}_{k_z,n}-\hbar\omega+i\hbar\gamma}+\frac{\omega_{02}^2}{\omega^2-\omega_{2c}^2}\sin^2\vartheta+\frac{\omega_{02}^2}{\omega^2}\cos^2\vartheta,$$
$$\omega_{02}^2=4\pi e_2^2N_{02}/m_2,\qquad \cos\vartheta=\cos(\mathbf{q}\,\mathbf{H}), \tag{2*}$$

where N_{02} is the mean number density of the ions.

From Eq. (2), under the condition

$$\omega^2\gg\omega_{2c}^2 \tag{3}$$

it follows that

$$1=-k_0^2(q_x,\omega,q_z)/q^2+\omega_{02}^2/\omega^2+i\varepsilon''(q_x,\omega,q_z), \tag{4}$$

where

$$\varepsilon''=\frac{G(q)}{2\pi\hbar^2\alpha^2}\sum_{nn'}F^2_{nn'}(q_x)\int f_{01}(F^{(1)}_{k_z,n})\times\{\delta(E^{(1)}_{k_z-q_z,n'}-E^{(1)}_{k_z,n}+\hbar\omega)-\delta(E^{(1)}_{k_z+q_z,n'}-E^{(1)}_{k_z,n}-\hbar\omega)\}\,dk_z,$$
$$-k_0^2=\frac{2e^2}{\pi\hbar^2\alpha^2}\,\mathrm{P}\sum_{nn'}F^2_{nn'}(q_x)\int dk_z\frac{f_{01}(E^{(1)}_{k_z+q_z,n'})-f_{01}(E^{(1)}_{k_z,n})}{E^{(1)}_{k_z+q_z,n'}-E^{(1)}_{k_z,n}-\hbar\omega}. \tag{4a}$$

The symbol P means that the integral is taken in the sense of the principal value.

Neglecting damping entirely, we find from (4) that

$$\omega^2=\omega_{02}^2q^2/[q^2+k_0^2(q_x,\omega,q_z)], \tag{5}$$

ω_{02} is the Langmuir frequency of the ions, while ω_{01} is the Langmuir frequency of the electrons; the quantity k_0^{-1} can be interpreted as the screening radius. Introducing the longitudinal complex dielectric constant of the electrons in the magnetic field

$$\varepsilon=\varepsilon'+i\varepsilon''=1+k_0^2(q_x,\omega,q_z)/q^2+i\varepsilon''(q_x,\omega,q_z), \tag{6}$$

we can interpret Eq. (4) by a clear cut elementary model.

Actually, let us consider the oscillations of the ions, which are described in the Fourier representation by the equation of motion

$$\omega\mathbf{v}_q=(Ze/m_2)\mathbf{q}\varphi, \tag{7}$$

by the equation for the potential φ

$$q^2\varphi(\mathbf{q},\omega)=(4\pi Ze/\varepsilon(q_x,\omega,q_z))\,\rho(\mathbf{q},\omega) \tag{8}$$

(ρ is the number density of the ions), and by the equation of continuity

$$N_{02}\mathbf{q}\cdot\mathbf{v}_q+\omega\rho=0. \tag{9}$$

From Eqs. (7) — (9) we find the frequency:

$$\omega^2=\omega_{02}^2/\varepsilon(q_x,\omega,q_z). \tag{10}$$

If we assume that $\epsilon'\gg\epsilon''$, we immediately obtain Eq. (4) from (10) with the aid of (6).

For the existence of an acoustic branch of the vibrations, we must find such a solution for ϵ which has the following asymptotic behavior:

$$\lim_{q\to 0}\varepsilon'\sim\mathrm{const}/q^2.$$

Thus the role of the electronic system reduces to screening the Coulomb field of the point ions and to damping of the acoustic oscillations. In the absence of a magnetic field, the plasma is isotropic; if the ion velocities are small in comparison with

*(qH) = q·H

the electron velocities, then the screening is spherically symmetric. The screening is anisotropic in the presence of a magnetic field. As is seen from the definition, k_0 depends upon the angle ϑ between the vectors $\mathbf{q}$ and $\mathbf{H}$, and on the frequency ω. Inasmuch as the sound velocity is determined from k_0, it is clear from (5) that the magnetic field leads to space-time dispersion of the acoustic oscillations and to anisotropy in the sound velocity.

We shall consider below the dependence of k_0 on the parameters of the system in different limiting cases, since the general formula for k_0 is too complicated for investigation.

4. We shall consider first the conditions under which we can neglect both the space and the time dispersions of the sound velocity and its anisotropies. In the case of a completely degenerate Fermi function, the expression for k_0^2 can be written in the form

$$-k_0^2 = \frac{2m_1e^2}{\pi\hbar^2\alpha}\sum_{nn'} F_{nn'}^2(q_1)\,\frac{1}{q_2} \times \left\{\ln\frac{\Omega - \sqrt{2}\,q_2(n_0-n')^{1/2} + q_2^2/2 - (n'-n)}{\Omega + \sqrt{2}\,q_2(n_0-n')^{1/2} + q_2^2/2 - (n'-n)} + \ln\frac{\Omega + \sqrt{2}\,q_2(n_0-n)^{1/2} - q_2^2/2 - (n'-n)}{\Omega - \sqrt{2}\,q_2(n_0-n)^{1/2} - q_2^2/2 - (n'-n)}\right\}, \tag{11}$$

where the dimensionless quantities

$$\Omega = \omega/\omega_{1c}, \qquad q_1 = \alpha q_x,$$

$$q_2 = \alpha q_z, \qquad n_0 = \mu_0/\hbar\omega_{1c} - 1/2$$

have been introduced (μ_0 is the chemical potential of a degenerate Fermi gas in a magnetic field).

For q_1 and q_2, and $\Omega \ll |n'-n|$, and also $n \gg 1$ (large quantum numbers), it follows from (11) that

$$k_0^2 = 3\omega_{01}^2/v_0^2, \tag{12}$$

where v_0 is the Fermi velocity.

It is easy to see that these conditions correspond to the case in which the Larmor radius of the orbit is small in comparison with the sound wavelength. Only in this case can the dispersion of the sound velocity and its anisotropies be neglected for $q^2 \ll k_0^2$.

We now return to a consideration of the dependence of k_0 on the magnetic field and on the angle ϑ between the field and the wave vector of the sound wave. It is shown that k_0 vanishes for $\vartheta = \pi/2$ in the case of very strong magnetic fields. Actually, it follows from (11), for the case $q_z = 0$, that

$$-k_0^2 = \frac{4\sqrt{2}\,m_1e^2}{\pi\hbar^2\alpha}\sum_{nn'} F_{nn'}^2(q_1)\,\frac{(n_0-n)^{1/2} - (n_0-n')^{1/2}}{\Omega - (n'-n)}. \tag{13}$$

The quantity k_0 vanishes for $n_0 < 1$. This means that the screening radius becomes infinite in a direction perpendicular to $\mathbf{H}$. This was also to be expected, since the frequency of plasma oscillations vanishes for $n_0 < 1$ in the direction $\vartheta = \pi/2$ (see reference 6).

The vanishing of the frequency of plasma oscillations for $\vartheta = \pi/2$ can be illustrated by analogy with the oscillations of plasma in a periodic field. In the case $n_0 < 1$, all the electrons are found in the level with $n = 0$, and the nearest level $n = 1$ is separated by an energy $\hbar\omega_{1c} > E_0$ (for metals, $E_0 \sim \hbar\omega_{01}$). In the band model, this is analogous to the situation in which the completely filled band is separated by an energy gap ΔE from an empty band, and plasma oscillations are impossible if $\Delta E > \hbar\omega_{01}$. We note that the screening radius in the direction of the magnetic field does not depend on the field, since $\lim_{q\to 0} F_{nn'} = \delta_{nn'}$, and the magnetic field falls out of the expression for k_0^2.

Thus it is shown that the screening radius, which has a constant value (independent of the magnetic field) along the field, can become infinite for $\vartheta = \pi/2$ in strong fields. It follows from what has been shown that the acoustic vibrations degenerate into ion vibrations with a frequency

$$\omega^2 = \omega_{02}^2 + \omega_{2c}^2 \tag{14}$$

in the case $n_0 < 1$ and $\vartheta = \pi/2$.

The development of acoustic oscillations for $\vartheta = \pi/2$ comes about abruptly when n_0 reaches a value equal to unity. Upon further decrease in H, the dispersion of the sound changes abruptly every time n_0 increases by unity, since new components appear in (13).

5. We now proceed to our fundamental problem — the investigation of the attenuation of ultrasonic vibrations. The damping frequency γ is expressed in accord with (14) by the following formula

$$\gamma = \frac{m_1\omega^3 G(q)}{4\pi\hbar^2\alpha\omega_{02}^2}\sum_{nn'} F_{nn'}^2(q_1)\int d\zeta f_{01}(E_{\zeta,n}) \times \{\delta(E_{\zeta-q_2,n'} - E_{\zeta,n} + \Omega) - \delta(E_{\zeta+q_2,n'} - E_{\zeta,n} - \Omega)\} \tag{15}$$

for $\omega \gg \gamma$ (which is confirmed by the result). For $q_1 = 0$ and $\hbar\omega \ll E_0$, it follows from (15) that

$$\gamma|_{\vartheta=0} = \gamma_0 = \frac{3\pi}{4}\,\frac{\omega_{01}^3}{\omega_{02}^2}\,\frac{\omega}{v_0^3 r^3}. \tag{16}$$

If we neglect the dispersion of the sound velocity in (16), then (16) goes over into the following formula

$$\gamma_0 = \frac{\pi}{12}\,\frac{m_1}{m_2}\,v_0 q, \tag{16a}$$

which was first obtained by Silin,[4] and later by Steinberg.[7] It also follows from (16) that γ does not depend on the magnetic field for a longitudinal ultrasonic wave with $\mathbf{q} \parallel \mathbf{H}$.

The greatest interest attaches to the ultrasonic wave with $\mathbf{q} \perp \mathbf{H}$, since the properties of the electron plasma in a direction perpendicular to $\mathbf{H}$ change appreciably in a magnetic field (in contrast with its properties in a direction parallel to the field). It follows from (15) that in this case γ has the form

$$\gamma|_{\theta=\pi/2} = \gamma_{\perp} = \frac{m_1\omega^3 G(q)}{\sqrt{2}\,\pi\hbar^2\alpha\omega_{02}^2} \sum_{n=0}^{n_0-\Omega} F_{n,\,n+\Omega}^2(q)\,\{(n_0-n)^{1/2} - (n_0-n-\Omega)^{1/2}\}. \quad (17)$$

This expression is very complicated in the general case.

For small values of the quantum number ($n \sim 1$), the value of $\gamma_{\perp}$ undergoes a discontinuous change upon change in the magnetic field strength, since the number of components in (17) changes abruptly upon an increase or decrease of n_0 by unity. For large quantum numbers $n \gg 1$, the asymptotic expansion

$$F_{n,\,n+\Omega}^2(q) = J_{\Omega}^2[q(2n+\Omega+1)^{1/2}], \quad (18)$$

can be used where $J_{\Omega}(x)$ is a Bessel function of the first kind of order Ω. If we take into account the inequalities $n \gg 1$ and $n_0 \gg \Omega$, we can convert in (17) from summation over n to integration and, expanding the term in curly brackets in a series in Ω, we obtain

$$\gamma_{\perp} = \frac{m_1\omega^4 G(q)\sqrt{n_0}}{\sqrt{2}\,\hbar^2\alpha\omega_{02}^2\omega_{1c}} \int_0^{\pi/2} J_{\Omega}^2(r_c q\sin\varphi)\sin\varphi\,d\varphi, \quad (19)$$

where $r_c = v_0/\omega_{1c}$. The last expression can be rewritten in the form

$$\frac{\gamma_{\perp}}{\gamma_0} = 2r_c q \int_0^{\pi/2} J_{\Omega}^2(r_c q\sin\varphi)\sin\varphi\,d\varphi, \quad (20)$$

where γ_0 is determined by Eq. (16).

Upon removal of the magnetic field, Eq. (20) changes to the equality

$$\gamma_{\perp} = \gamma_0, \quad (21)$$

since the anisotropy disappears.

For large values of the argument, or, more precisely, for $r_c q \gg \Omega$, the Bessel function can be replaced by its asymptotic expansion:

$$J_{\Omega}(x) = \sqrt{2/\pi x}\cos(x - \pi\Omega/2 - \pi/4); \quad (22)$$

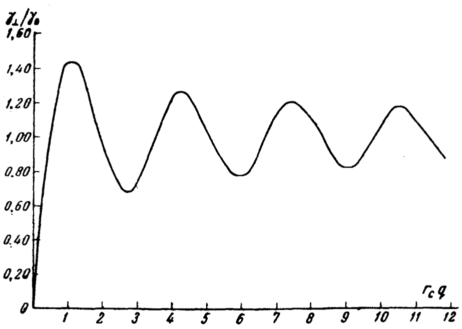

FIG. 2. Dependence of $\gamma_{\perp}/\gamma_0$ on $r_c q$ for $\Omega \ll 1$, according to Eq. (20).

We then get in place of (20)

$$\frac{\gamma_{\perp}}{\gamma_0} = \frac{4}{\pi}\int_0^{\pi/2} \cos^2\left(r_c q\sin\varphi - \frac{\pi}{2}\Omega - \frac{\pi}{4}\right)d\varphi. \quad (23)$$

The latter integral is equal to

$$\gamma_{\perp}/\gamma_0 = 1 + \sin\pi\Omega J_0(2r_c q) + \cos\pi\Omega\,[2\Omega_0(2r_c q) - S_0(2r_c q)], \quad (24)$$

where Ω_0 and S_0 are respectively the Lommel-Weber and Struve functions.

For $H \to 0$ or $r \to \infty$, the ratio $\gamma_{\perp}/\gamma_0 = 1$. This is seen immediately from (23), since $\cos^2(x)$ can be replaced by its average value (equal to $1/2$) for $r_c \to \infty$.

The dependence of $\gamma_{\perp}/\gamma_0$ on $r_c q$, which follows from (20), is shown in Fig. 2.

In conclusion, we note that Eq. (20) is valid under the assumptions $\lambda \ll l$, $E_0 \gg \hbar\omega_{1c}$ and $E_0 \gg \hbar\omega(q)$.

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