# UNSTABLE PARTICLE IN THE LEE MODEL

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We consider an unstable particle which can decay according to $V \to N + \theta$ in the Lee model of a nonrelativistic second-quantized theory. Perturbation theory for an eigenstate expansion of any initial state is generalized to the case of an unstable particle. A quantity playing the role of the norm of the state of such a particle is defined. A new method is given for finding the amount of time a stable $V'$ particle, which can undergo virtual transitions to the $N + \theta$ state, spends in each of its two possible states. This method is then applied to an unstable V particle, and a definite value is obtained for the amount of time it spends in each state; however, this value is found to be complex.

We follow Lee[1] and consider a system consisting of three particles, V, N, and $\theta$, whose interaction leads to the decay $V \to N + \theta$. Unlike Lee, however, we assume that $m_V > m_N + m_\theta$, so that the V particle is unstable and decays spontaneously into an N and a $\theta$. Then the set of stationary states with positive real energy E does not include V states. In the spectrum of this system, the unstable V will appear simply as a resonance in the $\theta$ and N states at a given energy $E_0$ (with $E_0 > m_N + m_\theta$, and some resonance width $\gamma$). It is clear, furthermore, that in the nonstationary problem the wave function of the system will have a term proportional to $\exp(-iE't) = \exp(-iE_0t - \gamma t)$, describing V decay.

In the case of a stable V particle, the interaction gives rise to a term in its wave function which describes an admixture of the $N + \theta$ state. In configuration space and in the simplest case of an S wave, this mixed state is described by a wave function of the form $f(r) = Ce^{-\kappa r}/r$, where r is the distance between the N and the $\theta$.[2] It is then easy to normalize the physical V state and to calculate how much bare V and how much $N + \theta$ it contains.

The wave function of an unstable V particle is a superposition of the bare V wave function and a diverging $N + \theta$ function, which in configuration space (in the case of an S wave) is of the form $f(r) = Ce^{ikr}/r$.

At first sight such a state cannot be normalized, and it would then seem that it is impossible to find the amount of bare V in the unstable physical V state. It will be shown below, however, that if one uses the techniques developed for the description of unstable states in the Schrödinger quantum mechanics of a single particle,[3] one can uniquely answer all questions for unstable particles which are solvable in an elementary way for stable ones.

With this in mind, we start with a new definition of the concept of the fractional amount of a given state, using first the case of stable particles (for which everything is clear).

Consider an eigenstate $\Psi$ of a Hamiltonian H including interaction ($H\Psi = E\Psi$), and let this state be written in the form

$$\Psi = \left| a\psi_V + \int f(r)\, d\tau\, \psi_N\psi_\theta \right| 0 \rangle \tag{1}$$

where $\psi_V$ is the creation operator for a bare V particle, $\psi_N$ and $\psi_\theta$ are creation operators for N and $\theta$ particles, and $|0\rangle$ is the vacuum state.* The normalization condition is

$$|a|^2 + \int |f(r)|^2\, d\tau = 1. \tag{2}$$

The usual procedure is the following. We suppose that the interaction leading to the transformation $V \to N + \theta$ is instantaneously turned off (the Hamiltonian changes from H to $H_0$). In the system described by $H_0$, the number of V particles is conserved, and the number of $N + \theta$ particles is conserved separately, and therefore the fraction of such states at the instant before the interaction was turned off is given, respectively, by $|a|^2$ and $\int |f(r)|^2\, d\tau$.

Our new approach is the following. We apply to the system in state $\Psi$ a small perturbation of the form $\delta H = \delta_V \psi_V^* \psi_V$. In the first approximation the energy is perturbed by an amount

*The wave function could also be written in Fock space as a column, each row of which corresponds to a different number of particles. Then the row with amplitude a would correspond to V, while that with amplitude f(r) would correspond to $N + \theta$.

$$\delta E = K\delta_V. \quad (3)$$

The coefficient K in this formula is then the fraction of bare V in the physical V state. The principle behind this new approach can be explained by analogy: the fraction of iron in a mixture of iron and wood filings can be found by placing the mixture in a weak magnetic field (which will not destroy the mixture), and measuring the force on it; the usual approach corresponds to destroying the mixture (by sorting).

In the same way one can find the fraction of $N + \theta$ by turning on, for instance, a perturbation of the form $\delta_2 H = \delta_N \psi_N^* \psi_N$ and calculating the first-order correction to the energy

$$\delta E = q\delta_N. \quad (4)$$

The fraction of $N + \theta$ is then given by q. The normalization of $\Psi$ leads to the identity $K + q = 1$.

For a stable particle the new approach gives exactly the same result as the usual one.

This new approach, however, can be applied also to the case of an unstable particle. Then for a particle whose wave function is of the form

$$\Psi_V = \Big| a\psi_V + \int f(r)\, d\tau\, \psi_N \psi_\theta \Big| 0 \Big\rangle, \quad (5)$$

we obtain[3]

$$K = a\tilde{a}^* \Big/ \Big[ a\tilde{a}^* + \int f(r) \tilde{f}^*(r)\, d\tau \Big], \quad (6)$$

$$q = \int f(r) \tilde{f}^*(r)\, d\tau \Big/ \Big[ a\tilde{a}^* + \int f(r)\tilde{f}^*(r)\, d\tau \Big], \quad (7)$$

where $\tilde{a}$ and $\tilde{f}$ correspond to the solution with the complex conjugate boundary conditions. In the simplest case of an S wave and a real coupling constant g, we have $f(r) = Ce^{ikr}/r$, $\tilde{f}^* = f$, and $\tilde{a}^* = a$, and then the integral in Eqs. (6) and (7) becomes

$$4\pi C^2 \int e^{2ikr} dr.$$

For an unstable particle the k here is complex, and $e^{2ikr}$ increases exponentially as $r \to \infty$. Now as has been shown elsewhere,[3] the integral can be given a well defined meaning, and it turns out to be equal to $-4\pi C^2/2ik$. Thus for S waves the fraction of bare V in the physical V state is

$$K = \frac{a^2}{a^2 - 4\pi C^2/2ik}, \quad (8a)$$

and the fraction of $N + \theta$ is

$$q = -\frac{4\pi C^2/2ik}{a^2 - 4\pi C^2/2ik}. \quad (8b)$$

These formulas are analytic continuations of the corresponding formulas* for the stable particles, where for S waves one finds

$$f(r) = C\frac{1}{r}e^{-\varkappa r}, \quad K = \frac{|a|^2}{|a|^2 + 4\pi|C|^2/2\varkappa},$$

$$q = \frac{4\pi|C|^2/2\varkappa}{|a|^2 + 4\pi|C|^2/2\varkappa}. \quad (9)$$

In particular, if we let $m_V$ approach $m_N + m_\theta$, i.e., go to the boundary between stable and unstable V, we find that K approaches zero and q approaches 1, independently of whether we approach this boundary from the region of stable or unstable V. Some artificiality in this concept of the "fraction of the state" for the unstable particle is manifested by the fact that K and q are then complex.

For particles with spin, in which case the decay products have orbital angular momentum, the wave function has a nonintegrable singularity at $r = 0$. This difficulty exists both for the stable and unstable cases, and can be removed only by a cutoff for small r (or a form factor in the interaction). On the other hand, it is a simple matter[3] to eliminate difficulties connected with the fact that $f(r)$ increases exponentially for large r when $l \neq 0$ for the unstable-particle case.

Let us now consider the Cauchy problem. Let the initial wave function be of the form

$$\Psi(t=0) = \Big| b\psi_V + \int h(r)\, d\tau\, \psi_N\psi_\theta \Big| 0 \Big\rangle \quad (10)$$

with arbitrary b and $h(r)$. It can be shown that the leading term in the solution is then of the form

$$\Psi(t) = ne^{-iE't}\Psi_V = ne^{-iE't}\Big| a\psi_V + \int f(r)\,\psi_N\psi_\theta d\tau \Big| 0 \Big\rangle, \quad (11)$$

and that the coefficient n in this expression is given by

$$n = \frac{(\tilde{\Psi}_V^* \Psi(t=0))}{(\tilde{\Psi}_V^* \Psi_V)} = \frac{1}{a^2 - 4\pi C^2/2ik}\Big[ab + \int h(r)\, C\frac{e^{ikr}}{r} d\tau\Big]. \quad (12)$$

The specific expression on the right side of our equation is correct for the simplest case of an S wave and a real coupling constant g, for which $f(r) = Ce^{ikr}/r$.

Thus also in the Cauchy problem the denominator of Eq. (8a) represents the norm of the state for the decaying particle. For an S wave and a real coupling constant g in the interaction term $g(\psi_V^*\psi_N\psi_\theta + \psi_N^*\psi_\theta\psi_V)$ we find that the quantities entering into (1) are

$$f(r) = Ce^{-\varkappa r}/r, \quad C = 2agm_r/4\pi, \quad m_{\text{np}} = m_N m_\theta/(m_N + m_\theta),$$

$$\varkappa = \sqrt{2m_r(m_N + m_\theta - m_V)};$$

and that the quantities entering into (5) are

$$f(r) = Ce^{ikr}/r, \qquad C = 2agm_r/4\pi,$$

$$k = \sqrt{2m_r(m_V - m_N - m_\theta - i\gamma)} \approx k_0 - i\gamma m_r/k_0,$$

$$k_0 = \operatorname{Re} k = \sqrt{2m_r(m_V - m_N - m_\theta)},$$

$$\gamma = -\operatorname{Im} E_V = (g^2/2\pi)\, m_r k_0.$$

*Since $C \sim a$ and $C/a$ is real, Eqs. (9) can be written without the absolute value sign on a and C, which emphasizes their resemblance to (8a) and (8b).

Now introducing the decay probability $w = 2\gamma$, writing $\Delta = m_V - m_N - m_\theta$, and expressing $g^2$ and $k_0$ in terms of these quantities, we arrive at the fraction of $N + \theta$ in the unstable V state (for $w \leq k_0$), namely

$$q = iw / 16\pi\Delta. \tag{13}$$

The $m_V$ that enters into all these expressions is the renormalized mass.

We shall not go into the simple operations which lead one to the above results, but shall merely describe the basic assumptions. The Lee model is an example of a theory in which there are particles which can decay into each other. In this particular case it is possible to solve the problem completely because one needs to consider only a finite number of different kinds of states (for instance V and N, $\theta$ in one problem, or V, $\theta$ and N, $2\theta$ in another problem).

In the Lee model this happens because of selection rules of the same kind as charge conservation. In particular, V + N (the baryon number) is conserved, and V + $\theta$ (the electric charge) is conserved. These selection rules, however, would not lead one to the goal if it were not for the fact that at the same time anti-particles and therefore $\theta$, $\overline{\theta}$ pair creation, etc., were not excluded from consideration.

A consistent relativistic theory cannot exist without antiparticles, and therefore it is logical to make still another simplifying assumption by considering the nonrelativistic problem in which the particle energy is $E = E_0 + p^2/2m$, so that one need not introduce the factor $1/2\omega$ into the $\theta$ normalization. At the same time, the concept of wave-function and mass (or energy) renormalization is maintained in the nonrelativistic theory. The description of real and virtual decay in a second-quantized nonrelativistic theory, and in particular the definition of the "fraction of a state" in virtual decay (i.e., for a stable state) has been given elsewhere[2] [in particular, Eqs. (28) and (26) of the cited reference].

As has been mentioned by N. A. Dmitriev, the expression

$$N = \int [\psi^2 (r) - (Ce^{ikr}/r)^2]\, d\tau - 4\pi C^2 / 2ik \tag{14}$$

which plays the role of the norm in an unstable-state solution of a Schrödinger problem,[3] (the analog of this expression is

$$N = a^2 - 4\pi C^2 / 2ik \tag{15}$$

of the present paper), has already essentially appeared in some work of Ning Hu,[4] involving dispersion relations.

Indeed, according to Heisenberg, a normalized stable (bound) state whose asymptotic wave function is of the form $\psi = Ce^{-\kappa r}/r$ leads to a residue in the scattering amplitude S(k) at the pole corresponding to the state, that is at $k = i\kappa$, given by

$$\oint_{k=i\varkappa} S(k)\, dk = 8\pi^2 |C|^2. \tag{16}$$

Using the fact that the bound-state function is real, we write this result for an unnormalized function in the form

$$\oint_{k=i\varkappa} S(k)\, dk = 8\pi^2 C^2 \Big/ \int \psi^2 (r)\, d\tau \tag{17}$$

Ning Hu gives a similar expression [Eq. (28) in his cited work] for the residue at $k = k_1$, with $\operatorname{Im} k_1 < 0$, which corresponds to an unstable state. His expression is

$$2ik_1 \left(\frac{dS}{dk}\right)_{k=-k_1} = 2k_1 \int_0^R (r\psi)^2\, dr + i\, [R\psi(R)]^2, \tag{18}$$

where $\psi(r)$ is the unstable-state solution whose asymptotic form is $\psi = e^{ik_1 r}/r$. We choose R so large that $\psi$ is essentially in its asymptotic form, and introduce C by writing $\psi = Ce^{ik_1 r}/r$ for $r \geq R$. Some elementary operations lead to

$$\begin{aligned}\oint_{k=k_1} S(k)\, dk &= -2\pi i \left(\frac{dS}{dk}\right)^{-1}_{k=-k_1} \\ &= 2\pi C^2 \Big/ \int_0^R \psi^2 r^2\, dr + \frac{i}{2k_1} [R\psi(R)]^2 \\ &= 2\pi C^2 \Big/ \int_0^R \psi^2 r^2 dr + \frac{i}{2k_1} C^2 e^{2ik_1 R} \\ &= 8\pi^2 C^2 \Big/ \int_0^\infty \left[\psi^2 - \left(\frac{Ce^{ik_1 r}}{r}\right)^2\right] d\tau - \frac{4\pi C^2}{2ik_1}. \end{aligned} \tag{19}$$

The resemblance of (17) to (19) is obvious.

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