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STABILITY OF A PLASMA PINCH WITH ANISOTROPIC PARTICLE VELOCITY DISTRIBUTION AND ARBITRARY CURRENT DISTRIBUTION

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The necessary conditions for stability of a plasma with an anisotropic distribution of particle velocities located in a helical magnetic field are derived on the basis of kinetic theory without taking close collisions into account.

In Suydam's work[1], a necessary condition for stability of a plasma pinch with arbitrary electric current distribution is obtained in the magnetohydrodynamic approximation. The magnetohydrodynamic approximation is only valid for perturbations with frequency and growth rate significantly lower than the frequency of close collisions. To investigate perturbations of frequency significantly greater than the frequency of the close collisions, a kinetic analysis is necessary. For perturbations in the region in which the "drift" approximation is applicable, a stability criterion can be obtained from the generalized energy principle.[2]

In this paper, the stability of a cylindrical plasma pinch with an anisotropic particle velocity distribution is investigated with the aid of the generalized energy principle and with account the charge neutrality of the plasma.[3] As in Suydam's work,[1] the magnetic lines of force are assumed to be helical, with pitch depending on the distance from the symmetry axis. We note that the "drift" approximation is not valid for particles in the tail of the equilibrium distribution. The number of these particles is small and therefore, as shown by Sagdeev and Shafranov,[4] the oscillation build-up effects that lead to instability, produced by them, are also small. No such instabilities exist in the case of a "cut-off" equilibrium distribution.

In this paper we obtain the necessary conditions for stability of a plasma pinch with anisotropic particle-velocity distribution. One of these criteria, which places a limit on the magnetic field gradient and on the particle distribution function, is the analog of Suydam's condition. Two other criteria with a local character are related to the possible appearance of "kinetic" instabilities caused by the anisotropy in the distribution function. These conditions have the same form as in the case of an unbounded homogeneous plasma.

A necessary condition for the stability of a plasma is that the energy change, δW, resulting from a small possible perturbation be greater than or equal to zero. Choosing the displacement vector of the plasma, $\boldsymbol{\xi}(\mathbf{r})$, in the form

$$\xi(\mathbf{r}) = [\xi_r(r),\ \xi_\varphi(r),\ \xi_z(r)] \exp(ikz + im\varphi) \tag{1}$$

and minimizing the previously obtained[3] expression for δW with respect to the displacement components $\xi_\varphi(r)$ and $\xi_z(r)$, we find

$$\delta W = \pi \int_0^R r\,dr \left(A\frac{\xi^2}{r^2} + 2B\frac{\xi\xi'}{r} + C\xi'^2\right). \tag{2}$$

Here $\xi = \xi_r$; the prime signifies differentiation with respect to r;

$$A = \gamma n_z^4 + \eta(g^2r^2 - n_\varphi^4) - r\frac{d}{dr}(\eta n_\varphi^2) - \frac{[\gamma n_z^2(mr^{-1}n_z - kn_\varphi) + \eta n_\varphi(gn_z + k)]^2}{\gamma(mr^{-1}n_z - kn_\varphi)^2 + \eta g^2}, \tag{3}$$

$$B = -\eta n_\varphi^2 + k^2\gamma\eta/[\gamma(mr^{-1}n_z - kn_\varphi)^2 + \eta g^2], \tag{4}$$

$$C = g^2\gamma\eta/[\gamma(mr^{-1}n_z - kn_\varphi)^2 + \eta g^2], \tag{5}$$

where

$$\mathbf{n} = \mathbf{H}/H, \quad g = kn_z + mr^{-1}n_\varphi, \quad \eta = H^2/4\pi + p_\perp - p_\parallel,$$
$$\gamma = H^2/4\pi + 2p_\perp + 2q,$$

$p_\parallel$ and $p_\perp$ are the longitudinal and transverse components of the pressure tensor, and R is the radius of the pinch. The quantity q is related to the anisotropy of the particle distribution function $f_0^{(i)}(r, v_\parallel, v_\perp)$:

$$q = \frac{1}{2}\sum_i m_i \iint \frac{H^3}{v_\parallel}\mu^2\frac{\partial f_0^{(i)}}{\partial\varepsilon}d\mu\,d\varepsilon - \Big(\sum_i e_i \iint \frac{H^2}{v_\parallel}\mu\frac{\partial f_0^{(i)}}{\partial\varepsilon}d\mu\,d\varepsilon\Big)^2 \Big/ 2\sum_i \frac{e_i^2}{m_i}\iint \frac{H}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial\varepsilon}d\mu\,d\varepsilon. \tag{6}$$

The variables of integration, μ and ε, are given in terms of the longitudinal and transverse velocity components by $\mu = v_\perp^2/2H$ and $\varepsilon = \frac{1}{2}(v_\parallel^2 + v_\perp^2)$. The summation in Eq. (6) is over the particle types (m_i and e_i are the mass and charge of particles of the i-th type).

In the derivation of (2) it was assumed that the displacement ξ_r is zero on the boundary of the plasma and on the axis of the cylinder. In order that the quadratic form (2) be non-negative, it is necessary that the coefficient of ξ'^2 be positive. We find another necessary condition by minimizing δW with respect to ξ. This minimization leads to the Euler equation

$$\xi'' + \left(\frac{d}{dr}\ln Cr\right)\xi' - \frac{1}{Cr^2}\left(A - r\frac{dB}{dr}\right)\xi = 0. \qquad (7)$$

If ξ satisfies Eq. (7), then the expression under the integral sign in (2) is a perfect differential, so that

$$\int r\,dr\left(A\frac{1}{r^2}\xi^2 + 2B\frac{1}{r}\xi\xi' + C\xi'^2\right) = B\xi^2 + Cr\xi\xi'. \qquad (8)$$

Equation (7) is an equation with a regular singularity at $r = a$, where $g(a) = 0$. The general solution of Eq. (7) has a branch point or pole at $r = a$ and can be represented in the form

$$\xi = (r-a)^{s_1}u_1(r) + (r-a)^{s_2}u_2(r), \qquad (9)$$

where $u_1(r)$ and $u_2(r)$ are functions analytic at $r = a$ and $s_{1,2} = -\frac{1}{2} \pm \frac{1}{2}(1 - 4M^2)^{1/2}$ are the roots of the indicial equation

$$s^2 + s + M^2 = 0, \qquad M^2 = \left[\left(\frac{\nu}{\nu'}\right)^2\nu^2\left(1 + \frac{\eta}{\gamma} + r\frac{\eta'}{\eta n_\varphi^2}\right)\right]_{r=a},$$

$$\nu = \frac{n_\varphi}{rn_z} = \frac{H_\varphi}{rH_z}. \qquad (10)$$

For all values of M^2, the displacement ξ is infinite at $r = a$. Therefore we substitute in Eq. (2) for δW values of ξ which satisfy Eq. (7) for all r except in the interval $a - \varepsilon < r < a + \varepsilon$, in which we choose ξ constant:

$$\xi = \varepsilon^{s_1}u_1(a) + \varepsilon^{s_2}u_2(a). \qquad (11)$$

The width, 2ε, of the interval is chosen to be such that when $u_1(r)$ and $u_2(r)$ are expanded in powers of $r - a \approx \varepsilon$, only the first term need be kept. In the following determination of the sign of δW, the values of $u_1(a)$ and $u_2(a)$ at $r = a$ are not essential and are chosen to be equal to unity.

If the roots of the indicial equation (10) are real ($4M^2 \leq 1$), we find

$$\delta W = 2\pi\varepsilon^{2s_2+1}\left[r\eta n_\varphi^2 n_z^2(\nu'/\nu)^2\right]_{r=a}(|s_2| - M^2) \geqslant 0. \qquad (12)$$

If the roots s_1 and s_2 are complex ($1 - 4M^2 = -\rho < 0$), then the energy change δW has the form

$$\delta W = \frac{1}{2}\pi\left[r\eta n_\varphi^2 n_z^2(\nu'/\nu)^2\right]_{r=a}\left[(1 - 2M^2)(1 + \cos 2\psi) + \rho\sin 2\psi\right], \qquad (13)$$

where $\psi = \frac{1}{2}\rho\ln\varepsilon + \Phi$ with Φ a constant phase.

It is easy to see that the right hand side of (13) can be made negative by a suitable choice of the interval width, 2ε. Thus, for the stability of the plasma it is necessary that the inequalities

$$H^2/4\pi + p_\perp - p_\parallel \geqslant 0, \qquad H^2/4\pi + 2p_\perp + 2q \geqslant 0, \qquad (14)$$

$$4\left(\frac{H_\varphi}{rH_z}\right)^2\left[1 + \frac{H^2/4\pi + p_\perp - p_\parallel}{H^2/4\pi + 2p_\perp + 2q} + r\frac{H^2}{H_\varphi^2}\frac{d}{dr}\ln(H^2/4\pi + p_\perp - p_\parallel)\right]$$

$$\leqslant \left(\frac{d}{dr}\ln\frac{H_\varphi}{rH_z}\right)^2 \qquad (15)$$

hold at all points, with $m \neq 0$.

In the isotropic case, $f_0^{(i)} = f_0^{(i)}(r, v_\parallel^2 + v_\perp^2)$, the inequalities (14) are automatically fulfilled and (15) becomes the stability criterion obtained by Suydam[1] in the magnetohydrodynamic approximation. This shows that collisions do not affect the stability condition (15) in an isotropic plasma. We note that the inequalities (14) are identical with the stability criteria for an unbounded homogeneous plasma.[5]

In the case $m = 0$, the stability conditions can be obtained directly from (2); they are

$$\eta = H^2/4\pi + p_\perp - p_\parallel \geqslant 0, \qquad \gamma = H^2/4\pi + 2p_\perp + 2q \geqslant 0,$$

$$\eta(\gamma - \eta)\left[\gamma(rn_\varphi^2)' + \eta n_z^2\right] - r\gamma'\eta^2 n_z^2 - r\eta'\gamma^2 n_\varphi^2$$

$$+ \eta(\gamma n_\varphi^2 + \eta n_z^2)^2\left[(k^2r^2 + 1)n_z^2 - 2n_\varphi^2\right] \geqslant 0. \qquad (16)$$

We note that the criteria (16) are not valid if the magnetic lines of force are circular ($n_z = 0$). This is due to the fact that when $n_z = 0$, integrals along the lines of force which are zero for $n_z \neq 0$ contribute to the expression for the energy change δW. We have obtained the stability criteria for $n_z = 0$ in a previous paper.[3]

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186