# *ON GAUGE TRANSFORMATIONS OF GREEN'S FUNCTIONS*

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Submitted to JETP editor October 13, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 926-932 (March, 1961)

The law is obtained for the transformation of many-particle Green's functions under changes of gauge. General identities of the type of Ward's identity follow from this law.

## 1. INTRODUCTION

THE paper by Landau and Khalatnikov[1] on the laws of gauge transformation of the Green's functions $\langle T\psi(x)\bar{\varphi}(y)\rangle$ and $\langle T\psi(x)\bar{\varphi}(y)A_\mu(u)\rangle$ has been followed by a number of papers connected with this question.[2–5]

The present paper proves a general theorem which establishes the law of gauge transformation for Green's functions with an arbitrary number of operators of arbitrary charged and neutral fields, without choosing a specific form of the interaction.

The laws of gauge transformations are determined only by the transformation properties of the Heisenberg operators under changes of gauge, independent of special forms of the interactions.

In electrodynamics the law of transformation of somewhat differently defined many-particle Green's functions has been derived recently by Okubo.[4] His proof, however, is valid only in electrodynamics and is connected with the expansion of the Green's functions in perturbation-theory series.

By means of the law for gauge transformations of Green's functions one can give a derivation of identities of the type of the Ward identity which is independent of gauge and is the most general and natural proof of these relations. This has also been done for the connection between the vertex part and the one-particle Green's function (cf. references 4 and 5). This method is used here to obtain general identities connecting many-particle Green's functions, and some concrete examples of identities of the Ward type are presented.

## 2. DEFINITIONS OF THE GREEN'S FUNCTIONS

We present the definitions of the many-particle Green's functions for which we shall consider the laws of gauge transformations. The Green's function without photon ends is defined as the vacuum expectation value of the T product,

$$G(x^1\ldots x^n,\ y^1\ldots y^n,\ z^1\ldots)$$
$$= \langle T\Psi\rangle \equiv \langle T\psi_1(x^1)\ldots\psi_n(x^n)\varphi_1(y^1)\ldots\varphi_n(y^n)\chi_1(z^1)\ldots\rangle, \tag{1}$$

where $\psi(x)$ and $\varphi(y)$ are operators for arbitrary lepton, meson, and baryon charged fields, which transform under gauge transformation by the law

$$\psi'(x) = \exp\ [ie\Lambda(x)]\psi(x), \qquad \varphi'(y) = \varphi(y)\exp\ [-ie\Lambda(y)], \tag{2}$$

and $\chi(z)$ are operators of arbitrary neutral fields other than the electromagnetic field.

We introduce the Green's functions that include electromagnetic-field operators $A_\mu(u)$ in the following way:

$$G_\mu(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u) = \langle T\Psi A_\mu(u)\rangle,$$

$$G_{\mu_1\mu_2}(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u^1u^2) \tag{3}$$
$$= \langle T\Psi A_{\mu_1}(u^1)A_{\mu_2}(u^2)\rangle - \langle T\Psi\rangle\langle TA_{\mu_1}(u^1)A_{\mu_2}(u^2)\rangle \tag{4}$$

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$$G_{\mu_1\mu_2\ldots\mu_m}(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u^1u^2\ldots u^m)$$
$$= \langle T\Psi A_{\mu_1}(u^1)A_{\mu_2}(u^2)\ldots A_{\mu_m}(u^m)\rangle$$
$$- \sum_{k>l} G_{\mu_1\ldots\mu_{l-1}\mu_{l+1}\ldots\mu_{k-1}\mu_{k+1}\ldots\mu_m}\langle TA_{\mu_l}(u^l)A_{\mu_k}(u^k)\rangle$$
$$- \sum_{i>j>k>l} G_{\mu_1\ldots\mu_{l-1}\mu_{l+1}\ldots\mu_{k-1}\mu_{k+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_m}$$
$$\times\langle TA_{\mu_i}(u^i)A_{\mu_j}(u^j)A_{\mu_k}(u^k)A_{\mu_l}(u^l)\rangle - \ldots \tag{5}$$

These expressions can be obtained formally by means of functional differentiation with respect to an external current:

$$G_{\mu_1\mu_2\ldots\mu_m}(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u^1\ldots u^m)$$
$$= \left.\frac{\delta^m\langle T\Psi\rangle}{\delta J_{\mu_1}(u^1)\,\delta J_{\mu_2}(u^2)\ldots\delta J_{\mu_m}(u^m)}\right|_{J=0}, \tag{6}$$

where the functional derivative is defined by the rule (Schwinger[6])

$$\frac{\delta\langle T\Psi\rangle}{\delta J_\mu(u)} = \langle T\Psi A_\mu(u)\rangle - \langle T\Psi\rangle\langle A_\mu(u)\rangle, \tag{7}$$

and for $J = 0$ the vacuum expectation value of an odd number of factors $A_\mu(u)$ goes to zero.

The Green's functions defined by the formulas (3) — (5) transform more simply under changes of gauge than do vacuum expectation values, and for the present functions one gets the natural generalization of Ward's identity. In perturbation theory this definition corresponds to the omission of diagrams that contain unconnected parts in which there are only photon external ends.

## 3. THE TRANSFORMATION OF THE GREEN'S FUNCTIONS UNDER CHANGES OF GAUGE

Let us express the Green's function (5) in an arbitrary gauge in terms of the Green's function in the true gauge of Landau and Khalatnikov (L.K.), in which by definition the photon Green's function has no longitudinal part and the commutator of electromagnetic-field operators at equal times is zero.[5]

The change from the Heisenberg operators $\psi^T$, $\varphi^T$, $\chi^T$, and $A_\mu^T$ in the L.K. gauge to the operators $\psi$, $\varphi$, $\chi$, and $A_\mu$ in an arbitrary gauge is made according to the rules

$$\psi(x) = \exp[ie\Lambda(x)]\,\psi^\tau(x), \quad \varphi(y) = \varphi^\tau(y)\exp[-ie\Lambda(y)],$$
$$\chi(z) = \chi^\tau(z), \qquad A_\mu(u) = A_\mu^\tau(u) + \partial\Lambda(u)/\partial u_\mu, \tag{8}$$

where $\Lambda(x)$ is a suitably chosen Hermitian operator, which can be represented in the form[1]

$$\Lambda(x) = \int d^4k\,\lambda(k^2)\,(a_k e^{ikx} + \overset{+}{a}_k e^{-ikx}) \tag{9}$$

($a_k$ and $a_k^+$ are creation and annihilation operators).

We may assume that the operator $\Lambda$ acts in a different Hilbert space from that in which the operators $\psi^T$, $\varphi^T$, $\chi^T$, and $A_\mu^T$ act,[1,5] and thus commutes with them. We note also that when we change to a gauge in which the equal-time commutators of the electromagnetic fields are zero by definition (we shall call these true gauges) the choice of $\Lambda$ is restricted by the condition[5]

$$[\Lambda(\mathbf{x},0),\, \dot{\Lambda}(0)] = 0. \tag{10}$$

Then the photon Green's function in an arbitrary true gauge can be expressed in the following way:

$$D^c_{\mu\nu}(u^1-u^2) = \langle TA_\mu(u^1)A_\nu(u^2)\rangle = \langle TA_\mu^\tau(u^1)A_\nu^\tau(u^2)\rangle$$
$$+ \frac{\partial^2}{\partial u^1_\mu \partial u^2_\nu}\langle T\Lambda(u^1)\Lambda(u^2)\rangle = \int d^4q\exp\{iq(u^1-u^2)\}$$
$$\times [q^2\delta_{\mu\nu} - q_\mu q_\nu)\, d_t(q^2) + q_\mu q_\nu d_l(q^2)]. \tag{11}$$

Let us now find the transformation law for the Green's function (1). According to what has been said,

$$G(x^1\ldots, y^1\ldots, z^1\ldots)$$
$$= G^\tau(x^1\ldots, y^1\ldots, z^1)\,\langle T\exp(ie\Phi)\rangle, \tag{12}$$

where

$$\Phi = \Lambda(x^1) + \ldots + \Lambda(x^n) - \Lambda(y^1) - \ldots - \Lambda(y^n). \tag{13}$$

Owing to the choice of the operator $\Lambda(x)$ in the form (9) the expression $\langle T\exp(ie\Phi)\rangle$ can be calculated by means of Wick's theorem, and furthermore the theorem can be applied directly to $\Phi$, since $\Phi$ depends linearly on $\Lambda$. Thus

$$\langle T\exp(ie\Phi)\rangle = \sum_{k=0}^{\infty}\frac{(ie)^{2k}}{(2k)!}\langle T\Phi^{2k}\rangle$$
$$= \sum_{k=0}^{\infty}\frac{(ie)^{2k}}{(2k)!}(2k-1)!!\,\langle T\Phi^2\rangle^k = \exp\left(-\frac{e^2}{2}\langle T\Phi^2\rangle\right), \tag{14}$$

and the transformation law for $G$ is of the form

$$G = \exp(e^2\rho)\,G^\tau, \tag{15}$$

$$\rho \equiv \rho(x^1\ldots, y^1\ldots) = -\tfrac{1}{2}\langle T\Phi^2\rangle$$
$$= \frac{1}{2}\sum_{i,j=1}^{n}[\langle T\Lambda(x^i)\Lambda(y^j)\rangle$$
$$-\langle T\Lambda(x^i)\Lambda(x^j)\rangle - \langle T\Lambda(y^i)\Lambda(y^j)\rangle]. \tag{16}$$

The Green's function $G_{\mu_1\mu_2\ldots\mu_m}$ in an arbitrary gauge is expressed in terms of the Green's function in the L.K. gauge according to the law

$$G_{\mu_1\mu_2\ldots\mu_m} = \exp(e^2\rho)\Big\{G^\tau_{\mu_1\ldots\mu_m} + ie\sum_{r=1}^{m}\rho_{\mu_r}G^\tau_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m}$$
$$+ (ie)^2\sum_{r_2>r_1=1}^{m}\rho_{\mu_{r_1}}\rho_{\mu_{r_2}}G^\tau_{\mu_1\ldots\mu_{r_1-1}\mu_{r_1+1}\ldots\mu_{r_2-1}\mu_{r_2+1}\ldots\mu_m} + \ldots$$
$$+ (ie)^k\sum_{r_k>r_{k-1}\ldots>r_1=1}^{m}\rho_{\mu_{r_1}}\rho_{\mu_{r_2}}\ldots\rho_{\mu_{r_k}}$$
$$\times G^\tau_{\mu_1\ldots\mu_{r_1-1}\mu_{r_1+1}\ldots\mu_{r_k-1}\mu_{r_k+1}\ldots\mu_m} + \ldots$$
$$+ (ie)^m\rho_{\mu_1}\rho_{\mu_2}\ldots\rho_{\mu_m}G^\tau\Big\}, \tag{17}$$

where

$$\rho_{\mu_r} \equiv \rho_{\mu_r}(x^1\ldots, y^1\ldots, u^r) = \left\langle T\frac{\partial\Lambda(u^r)}{\partial u^r_{\mu_r}}\Phi\right\rangle$$
$$= \sum_{j=1}^{n}\left\langle T\frac{\partial\Lambda(u^r)}{\partial u^r_{\mu_r}}[\Lambda(x^j) - \Lambda(y^j)]\right\rangle. \tag{18}$$

For the proof we note that we can represent the definition (6) of the many-particle Green's functions in the form

$$G_{\mu_1\mu_2\ldots\mu_m} = \frac{\delta^m\langle T\Psi\rangle}{\delta J_{\mu_1}(u^1)\,\delta J_{\mu_2}(u^2)\ldots\delta J_{\mu_m}(u^m)}\Bigg|_{J=0}$$
$$= \prod_{r=1}^{m}\left(\frac{\delta\cdot}{\delta J^\tau_{\mu_r}} + \frac{1}{i\alpha}\frac{\partial}{\partial u^r_{\mu_r}}\right)$$
$$\times\langle T\Psi^\tau\rangle\frac{\left\langle T\exp\left[ie\Phi + i\alpha\sum_{l=1}^{m}\Lambda(u^l)\right]\right\rangle}{\left\langle T\exp\left[i\alpha\sum_{l=1}^{m}\Lambda(u^l)\right]\right\rangle}\Bigg|_{J^\tau=0,\ \alpha=0}, \tag{19}$$

where the functional derivative $\delta/\delta J^T_\mu$ is defined by the rule (7) with $A_\mu$ replaced by $A^T_\mu$, and acts only on $\langle T\Psi^T\rangle$. We note that as long as $J^T$ and the parameter $\alpha$ are not set equal to zero the operator $\delta/\delta J^T_\mu + (i\alpha)^{-1}\partial/\partial u_\mu$ gives $A_\mu(u)$ in the form of the sum $A^T_\mu + \partial\Lambda/\partial u_\mu$, according to the rule (7), in which we have only to replace the average $\langle T\ldots\rangle$ by the average

$$\langle T\ldots\exp[i\alpha\Sigma\Lambda(u^l)]\rangle/\langle T\exp[i\alpha\Sigma\Lambda(u^l)]\rangle.$$

After calculations analogous to Eq. (14) we get

$$\frac{\langle T\exp[ie\Phi+i\alpha\Sigma\Lambda(u^l)]\rangle}{\langle T\exp[i\alpha\Sigma\Lambda(u^l)]\rangle}=\exp\left[e^2\rho-\alpha e\sum_{l=1}^{m}\langle T\Phi\Lambda(u^l)\rangle\right]. \tag{20}$$

Differentiation of the last expression obviously gives

$$\left(\frac{1}{i\alpha}\right)^k\frac{\partial^k}{\partial u^1_{\mu_1}\ldots\partial u^k_{\mu_k}}\exp\left[e^2\rho-\alpha e\sum_{l=1}^{m}\langle T\Phi\Lambda(u^l)\rangle\right]\Bigg|_{\alpha=0}$$

$$=(ie)^k\rho_{\mu_1}\rho_{\mu_2}\ldots\rho_{\mu_k}\exp(e^2\rho). \tag{21}$$

Since

$$G^\tau_{\mu_1\mu_2\ldots\mu_k}=\frac{\delta^m\langle T\Psi^\tau\rangle}{\delta J^\tau_{\mu_1}(u^1)\,\delta J^\tau_{\mu_2}(u^2)\ldots\delta J^\tau_{\mu_k}(u^k)}\Bigg|_{J^\tau=0} \tag{22}$$

$$\prod_{r=1}^{m}\left(\frac{\delta}{\delta J^\tau_{\mu_r}}+\frac{1}{i\alpha}\frac{\partial}{\partial u^r_{\mu_r}}\right)=\frac{\delta^m}{\delta J^\tau_{\mu_1}\ldots\delta J^\tau_{\mu_m}}$$

$$+\frac{1}{i\alpha}\sum_{r=1}^{m}\frac{\delta^{m-1}}{\delta J^\tau_{\mu_1}\ldots\delta J^\tau_{\mu_{r-1}}\delta J^\tau_{\mu_{r+1}}\ldots\delta J^\tau_{\mu_m}}\frac{\partial}{\partial u^r_{\mu_r}}$$

$$+\left(\frac{1}{i\alpha}\right)^2\sum_{r_2>r_1=1}^{m}\frac{\delta^{m-2}}{\delta J^\tau_{\mu_1}\ldots\delta J^\tau_{\mu_{r_1-1}}\delta J^\tau_{\mu_{r_1+1}}\ldots\delta J^\tau_{\mu_{r_2-1}}\delta J^\tau_{\mu_{r_2+1}}\ldots\delta J^\tau_{\mu_m}}$$

$$\times\frac{\partial^2}{\partial u^{r_1}_{\mu_{r_1}}\partial u^{r_2}_{\mu_{r_2}}}+\ldots+\left(\frac{1}{i\alpha}\right)^m\frac{\partial^m}{\partial u^1_{\mu_1}\ldots\partial u^m_{\mu_m}}, \tag{23}$$

the transformation law (17) follows from the expression (19).

We shall make two comments. First, we note that Eq. (17) can also be proved without the functional differentiation (6), (7) by the method of mathematical induction, if we use the definition (5) of the Green's functions. Second, we emphasize that in the proof of the theorem essential use has been made of the definition (9), which enables us to apply Wick's theorem directly to the operators $\Lambda$. If we renounce this definition, the law in question is not true. For example, if we set*

$$\Lambda(x)=\Lambda_1(x)\Lambda_2(x), \tag{24}$$

where $\Lambda_1(x)$ and $\Lambda_2(x)$ are Hermitian operators of the form (9) that act in different Hilbert spaces, and take for simplicity

$$\langle T\Lambda_1(x)\Lambda_1(y)\rangle_1=\langle T\Lambda_2(x)\Lambda_2(y)\rangle_2=F(x-y), \tag{25}$$

then the transformation law of the one-particle Green's function is

$$\langle T(\psi x)\overline{\psi}(y)\rangle=\langle T\psi^\tau(x)\overline{\psi}^\tau(y)\rangle\{1+e^2[F^2(0)-F^2(x-y)]\}^{-1}. \tag{26}$$

In this connection we note that the derivation of the law (15) for $\langle T\psi(x)\overline{\psi}(y)\rangle$ from group-theory arguments in the paper of Evans, Feldman. and Matthews[5] is erroneous, since the infinitesimal transition operator has been found only from a special (L.K.) gauge, and not from an arbitrary gauge. Therefore the passage from Eq. (3.15) to Eq. (3.16) in reference 5 is illegitimate, in spite of the fact that gauge transformations actually do form a Lie group. Thus in the example just given Eq. (3.15) holds and Eq. (3.16) does not. Equation (3.16) holds only in the case of operators $\Lambda(x)$ of the form (9), but to establish it one actually needs to know the final result.

## 4. GENERAL IDENTITIES OF THE WARD TYPE

In the L.K. gauge

$$\frac{\partial}{\partial u^r_{\mu_r}}G^\tau_{\mu_1\ldots\mu_r\ldots\mu_k}=0 \tag{27}$$

in consequence of the Lorentz condition[5]

$$\frac{\partial A^\tau_\mu(u)}{\partial u_\mu}=0 \tag{28}$$

and the fact that for $t_x = t_y = t_z = t_{u'} = t_u$

$$[\psi^\tau(x),A^\tau_\mu(u)]=[\varphi^\tau(y),A^\tau_\mu(u)]=[\chi^\tau(z),A^\tau_\mu(u)]$$

$$=[A^\tau_\nu(u'),A^\tau_\mu(u)]=0. \tag{29}$$

Calculating the divergence of both sides of Eq. (17) and using Eq. (27), we get general identities in an arbitrary gauge:

$$\frac{\partial}{\partial u^r_{\mu_r}}G_{\mu_1\ldots\mu_r\ldots\mu_m}=ie\frac{\partial\rho_{\mu_r}}{\partial u^r_{\mu_r}}G_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m}. \tag{30}$$

In the special case of the Feynman gauge[5] they take the form

$$\frac{\partial}{\partial u^r_{\mu_r}}G^F_{\mu_1\ldots\mu_r\ldots\mu_m}=-e\sum_{l=1}^{n}[D_c(u^r-x^l)$$

$$-D_c(u^r-y^l)]\,G^F_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m},$$

$$D_c(x)=\frac{1}{(2\pi)^4}\int e^{ikx}\frac{dk}{k^2-i\varepsilon}, \tag{31}$$

since the change from the L.K. gauge to the Feyn-

*This is a purely illustrative example, since after such a transformation the commutator of the electromagnetic-field operators is no longer a c number.

man gauge is accomplished by means of a $\Lambda(x)$ for which

$$\langle T\Lambda(x)\Lambda(y)\rangle = \frac{-i}{(2\pi)^4}\int e^{ik(x-y)}\frac{dk}{(k^2)^2}. \quad (32)$$

If we now apply to Eq. (31) the d'Alembertian operator $\Box_{u^r}$, we get

$$\Box_{u^r}\frac{\partial}{\partial u^r_{\mu_r}} G^F_{\mu_1\ldots\mu_r\ldots\mu_m} = e\sum_{l=1}^{n}[\delta(u^r - x^l) - \delta(u^r - y^l)]\, G^F_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m}. \quad (33)$$

With suitable definitions of the higher vertex parts the identities (30) lead to gauge-independent general Ward identities for processes with arbitrary numbers of charged and neutral particles and of photons. This derivation of the Ward identities is based only on the laws of gauge transformation of the fields, and assumes absolutely no knowledge of the concrete forms of the interactions or of their renormalizability.

As an example let us consider the case of a single charged particle $[\Psi = \psi(x)\overline{\psi}(y)]$ and m photons. Taking as the definition of the vertex part $\Gamma_{\mu_1\mu_2\ldots\mu_m}$ the formula

$$G_{\mu_1\mu_2\ldots\mu_m}(x, y, u^1\ldots u^m) = (2\pi)^{-4(m+1)}e^m \times\int dp^1dp^2dq^1\ldots dq^m \exp(ip^1x - ip^2y + i\Sigma q^lu^l)\,\delta\Big(p^1 - p^2 + \sum_{l=1}^{m} q^l\Big) \times\prod_{j=1}^{m} D^c_{\mu_j\nu_j}(q^j)\, S^c(p^1)\,\Gamma_{\nu_1\ldots\nu_m}(p^1, p^2, q^1\ldots q^m)\, S^c(p^2) \quad (34)$$

($S^c$ is the exact one-particle Green's function of the charged particle), we get from Eq. (30) after dividing through by $q_r^2 d_l(q_r^2)$ and by all of the $D^c_{\mu_j\nu_j}(q^j)$:

$$S^c(p^1)\, q^r_{\nu_r}\Gamma_{\nu_1\ldots\nu_r\ldots\nu_m}(p^1, p^2, q^1\ldots q^m)\, S^c(p^2) = S^c(p^1)\,\Gamma_{\nu_1\ldots\nu_{r-1}\nu_{r+1}\ldots\nu_m} \times(p^1, p^2 - q^r, q^1\ldots q^{r-1}q^{r+1}\ldots q^m)\, S^c(p^2 - q^r) - S^c(p^1 + q^r)\,\Gamma_{\nu_1\ldots\nu_{r-1}\nu_{r+1}\ldots\nu_m} \times(p^1 + q^r, p^2, q^1\ldots q^{r-1}q^{r+1}\ldots q^m)\, S^c(p^2), \quad (35)$$

where

$$p^1 - p^2 + \sum_{l=1}^{m} q^l = 0.$$

For m = 1 we can write this identity in the form

$$S^c(p^1)\, q_\nu\Gamma_\nu(p^1, p^1 + q, q)\, S^c(p^1 + q) = S^c(p^1) - S^c(p^1 + q). \quad (35a)$$

Let us also consider the case of two charged particles and examine the connections between the processes

$$\pi^+ + p \to \pi^+ + p, \quad \pi^+ + p \to \pi^+ + p + \gamma.$$

Introducing the standard definitions

$$G(x^1x^2, y^1y^2) = (2\pi)^{-12}\int dp^1\, dp^2\, dp^3\, dp^4 \times\exp(ip^1x^1 + ip^2x^2 - ip^3y^1 - ip^4y^2) \times\delta(p^1 + p^2 - p^3 - p^4)\, G(p^1, p^2, p^3, p^4),$$

$$G_\mu(x^1x^2, y^1y^2, u) = (2\pi)^{-16}e\int dp^1\, dp^2\, dp^3\, dp^4\, dq \times\exp(ip^1x^1 + ip^2x^2 - ip^3y^1 - ip^4y^2 + iqu)\,\delta(p^1 + p^2 + q - p^3 - p^4)\, D^c_{\mu\nu}(q)\, S^c(p^1)\,\Delta^c(p^2) \times\Gamma_\nu(p^1p^2, p^3p^4q)\, S^c(p^3)\,\Delta^c(p^4), \quad (36)$$

where $S^c(p)$ and $\Delta^c(p)$ are the exact one-particle Green's functions of the proton and the $\pi$ meson, we arrive at the identity

$$S^c(p^1)\,\Delta^c(p^2)\, q_\mu\Gamma_\mu(p^1, p^2, p^3, p^4, q)\, S^c(p^3)\,\Delta^c(p^4) = G(p^1, p^2, p^3, p^4 - q) + G(p^1, p^2, p^3 - q, p^4) - G(p^1, p^2 + q, p^3, p^4) - G(p^1 + q, p^2, p^3, p^4), \quad (37)$$

where $p^1 + p^2 + q = p^3 + p^4$.

The identities (35), which are a direct generalization of Ward's identity, were found by Fradkin,[2] who obtained them as a consequence of the Schwinger system of equations for the Green's functions in electrodynamics, i.e., by the use of a concrete form of renormalizable interaction. Quite recently identities of the form (35) have been derived by Kazes[7] by means of perturbation theory.

The identity (33) in the Feynman gauge is directly related to an identity for the coefficient functions of perturbation theory obtained by Bogolyubov and Shirkov[8] in quantum electrodynamics.

The generalized Ward identity (35a), also formulated by Green,[9] has also been proved without perturbation theory by Takahashi[10] by means of the formula (33) in the Feynman gauge, which he derived for m = 1. A gauge-independent derivation of this identity in electrodynamics has been given by Okubo[4] by the method of Caianello, which is closely connected with perturbation theory. Finally, Evans, Feldman, and Matthews[5] have obtained this same identity as a consequence of a law for gauge transformations of the Green's functions[1]

$$\langle T\psi(x)\overline{\psi}(y)\rangle \text{ and } \langle T\psi(x)\overline{\Psi}(y)A_\mu(u)\rangle.$$

which does not require any reference to the concrete form of the interaction.

In conclusion we express our gratitude to M. A. Markov, Ya. A. Smorodinskii, and Chou Kuang-Chao for helpful discussions.

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Translated by W. H. Furry
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