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DISPERSION FORMULAS WHICH TAKE INTO ACCOUNT THE OPTICAL INTERACTION

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A formal theory of nuclear reactions is developed which is more adequate for treating nuclear problems than the Wigner R-matrix theory. The approximate orthogonality between the wave functions of the compound nucleus and the unexcited ground state functions is used to separate the compound nucleus resonances. The theoretical treatment leads to specific dispersion formulas which take into account the overlapping levels and the interaction between the particles and the optical potential of the nucleus.

1. A large number of articles have appeared recently in which it is attempted to describe, within the framework of a single mathematical scheme, nuclear reactions involving the production of a compound nucleus, direct nuclear reactions, and optical scattering. Previously,[1,2] we developed a variant of a formal theory in which the compound nucleus resonances were isolated by separation of the polar part of the wave function and the S matrix for complex energies.

It turned out that such a method separates correctly only those levels whose width is considerably less than the distance to the nearest threshold of a nuclear reaction. Undoubtedly, the approach suggested by Feshbach[3] is more successful. Feshbach's formalism, however, does not describe nuclear collisions accompanied by a redistribution of the particles (nuclear reactions in a narrow sense) Moreover, the dispersion formulas obtained by Feshbach were left in very general form, and their application was not discussed. Feshbach's theory is improved and developed in the present article as regards these two aspects.

It is shown in Sec. 2 that the smallness of the parameter (1) (see below) suggested by us[1] makes it possible to establish the property of approximate orthogonality of the wave functions of the compound nucleus to the functions of the reaction channels. This property permits the extension of the procedure suggested by Feshbach for the separation of compound nucleus resonances to nuclear reactions accompanied by a redistribution of particles (Secs. 3 and 4).

It is suggested in Sec. 5 that direct nuclear interactions play a minor role, and that the optical potential is energy independent. Under these assumptions, the resonance half-widths split into the products of the penetration factors, which take into account the optical interaction and the amplitudes depending only on the level. The dispersion formula (22) describing the compound nucleus resonances is found for the energy region in which the influences of the optical interaction and nuclear reaction thresholds are important. In Sec. 6, the general formula is applied to the special case in which only one of the compound nucleus levels makes an essential contribution.

2. It is known that optical interactions and direct nuclear reactions take place without a rearrangement of the internal structure of the nucleus. In contrast to these processes, the production of a compound nucleus is accompanied by the transfer of excitation to all or, at least, several nucleons. The physical difference between these states requires that the wave functions of the compound nucleus be (at least, approximately) orthogonal to the wave functions of the channels open at medium energies. It is convenient to introduce numerical criteria which make it possible to consider the accuracy to which this orthogonality is fulfilled.

Let χ_λ denote the wave function of a compound nucleus and $c_1, \ldots, c_P$ the indices for the channels open at medium energies. We shall consider the ratio η of the mean square of the wave function of the compound nucleus on parts of the hypersurface corresponding to channels $c_1, \ldots, c_P$ which can be open at medium energies to the mean over the volume. This ratio can be written in the form

$$\eta = \frac{\sum\limits_{c}^{P} |\langle c | \chi_\lambda \rangle_R|^2}{\frac{1}{\tau}\int |\chi_\lambda|^2 \, d\tau = \frac{1}{3} \sum\limits_{c}^{P} \frac{|\gamma_{\lambda c}|^2}{E_R}}. \qquad (1)$$

Here τ is that part of configuration space in which the nucleus is situated, R is the radius of the nucleus in channel c. The integrals over the internal variable $\langle c \mid \chi_\lambda \rangle$ taken for $r = R$ are related to the amplitudes of the reduced half-widths $\gamma_{\lambda c}$. The energy E_R, which is equal to $\hbar^2/2mR$, varies from 0.1 to 1 Mev for different nuclei.

It was shown by the present author[1] that the parameter η is the ratio of the mean time of free flight of the particles from the space in which the unstable compound nucleus is situated to the lifetime of the compound nucleus. We now note that at the same time, relation (1) indicates what part of the wave function χ_λ lies in the functional subspace which is filled by the wave functions of the channels. The smallness of η means that, in a first approximation, the subspace P of the wave functions $c_1, \ldots, c_P$ is orthogonal to the subspace $\{\chi_2\}$ of the wave functions describing the states of the compound nucleus. The property of approximate orthogonality of the functions $\mid c \rangle$ and χ_λ permits us to exclude formally the intermediate nucleus states and introduce an effective interaction whose smooth part describes the direct and optical processes.

3. We denote by P and $P_\perp$ the projection operators in subspace P and in subspace $P_\perp$ complementary to it; $P_\perp$ is equal to $1 - P$, where by the symbol 1 we have in mind the entire configuration space of the coordinates of the nucleons of the system. Schrödinger's equation with infinitesimal imaginary additional terms which automatically separate the outgoing wave solution at infinity has the form

$$(E - H + i\varepsilon)\,\psi^{(+)} = i\varepsilon\varphi. \tag{2}$$

Here φ is the incident plane wave, $\psi^{(+)}$ is the solution, and $\epsilon \to +0$.

For brevity, we introduce the notation:

$$\Gamma = P_\perp H P, \qquad \Gamma^+ = PHP_\perp, \qquad H_\perp = PHP_\perp. \tag{3}$$

The projections of Eq. (2) in the subspaces P and $P_\perp$ have the form

$$\begin{aligned}(E - PHP + i\varepsilon)\,P\psi^{(+)} &= i\varepsilon\varphi + \Gamma^+\psi^{(+)},\\ (E - H_\perp + i\varepsilon)\,P_\perp\psi^{(+)} &= \Gamma\psi^{(+)},\end{aligned} \tag{4}$$

from which we obtain the equation in subspace P equivalent to (2):

$$(E - H_P + i\varepsilon)\,P\psi^{(+)} = i\varepsilon\varphi, \tag{5}$$

where the "effective Hamiltonian" is

$$H_P = PHP + \Gamma^+ (E - H_\perp)^{-1}_\perp \Gamma. \tag{6}$$

The subscript ⊥ after the parenthesis in (6) denotes that the inverse operator acts in the complementary space. The quantity $i\epsilon$ is omitted, since the operator $H_\perp$ has only discrete, real eigenvalues in the energy region under consideration.

The eigenfunctions of $H_\perp$ describe stationary states of the compound nucleus whose decay is artificially forbidden. The part PHP of the total Hamiltonian H_P corresponds to processes not connected with the rearrangement of the internal structure of the nucleus and which occur within a relatively small time interval. The second term of the right-hand part of (6) describes the production and decay of the compound system. We denote by χ_λ the eigenfunctions of the Hamiltonian $H_\perp$ corresponding to the eigenvalue E_λ. In the medium energy region, the operator $H_\perp$ does not have a continuous spectrum; therefore the energy E_λ is real and the wave functions χ_λ satisfy the generalized condition of realness:

$$\chi^*_\lambda = (-)^{J-M} O \chi_\lambda(-M), \tag{7}$$

where O is the time-reversal operator (spin matrix) and M is the quantum number characterizing the projection of the total angular momentum J. The minus sign in the parenthesis following χ_λ denotes that the projection of the total angular momentum in the state $\chi_\lambda(-M)$ has the opposite sign.

We write the Green's function in relation (6) in the form of a spectral expansion

$$(E - H_\perp)^{-1}_\perp = \sum_\lambda \frac{\mid \chi_\lambda\rangle\langle\chi_\lambda \mid}{E - E_\lambda}, \tag{8}$$

in which case the effective Hamiltonian takes the form

$$H_P = H^0_P + \sum_\lambda \frac{\Gamma^+ \mid \chi_\lambda\rangle\langle\chi_\lambda \mid \Gamma}{E - E_\lambda}. \tag{9}$$

It is clear that all difficulties connected with the singularities in the behavior close to the thresholds do not involve H_P. The expansion (9) is fundamental to the present theory. Further transformations are of a formal character, and, in many ways, are analogous to those presented by Lane and Thomas.[5]

4. Let $\psi_0^{(+)}$ denote the solution of Schrödinger's equation (5) in which the Hamiltonian H_P is replaced by the resonanceless operator H^0_P. Let $H = H_0 + V$, $H^0_P = H_0 + V^0$, where H_0 is the Hamiltonian of free motion described by the wave functions φ, while V and V^0 are interaction operators in the corresponding channels. It is known[4] that the amplitude for transition from the state φ into some other state φ' is expressed by a matrix element of the form $\langle \varphi' \mid V' \mid \psi^{(+)} \rangle$. Using Eq. (2) for the functions $\psi^{(+)}$ and $\psi_0^{(+)}$, we find, after simple algebraic transformations (similar to those given by Feshbach[3]):

$$\langle\varphi'|V'|\psi^{(+)}\rangle = \langle\varphi'|V^{0'}|\psi_0^{(+)}\rangle,$$
$$+\sum_\lambda \frac{\langle\psi_0^{(-)'}|\Gamma^+|\chi_\lambda\rangle\langle\chi_\lambda|\Gamma|\psi^{(+)}\rangle}{E-E_\lambda}. \qquad (10)$$

The first term of the right-hand part of (10) gives the amplitude of optical scattering and of the direct processes.

The matrix element for transition into the state $\psi_0^{(-)'}$ is a smooth function of the energy, the matrix element for transition into the state $\psi^{(+)}$ can be expressed in terms of quantities that are smooth functions of the energy by means of the following system of equations:

$$\sum_\lambda [\delta_{\mu\lambda}(E-E_\lambda) - K_{\mu\lambda}] \frac{\langle\chi_\lambda|\Gamma|\psi^{(+)}\rangle}{E-E_\lambda} = \langle\chi_\mu|\Gamma|\psi_0^{(+}\rangle, \qquad (11)$$

where

$$K_{\mu\lambda} = \langle\chi_\mu|\Gamma(E-H_P^0+i\varepsilon)^{-1}\Gamma^+|\chi_\lambda\rangle. \qquad (12)$$

Feshbach assumed that, in first approximation, the matrix element for transition to the state $\psi_0^{(-)'}$ and the quantity $K_{\mu\lambda}$ do not depend on the energy. This enables one to construct readily the solution (11) by means of an expansion in eigenvectors of a homogeneous equation. Of course, it is assume here that $\psi_0^{(-)}$ and $K_{\mu\lambda}$ vary little over energy intervals of the order of the width of the compound nucleus levels. If the levels overlap with the nuclear reaction thresholds, this method of treatment is known to be inapplicable.

In Eq. (11), it is easy to go over to the matrix elements of the S matrix. To do this, it is sufficient to use the formulas given in the appendix to reference 1. In place of the functions $\psi^{(+)}$ and $\psi_0^{(+)}$, we use the functions $u_c^{(\pm)}$ and $u_c^{0(\pm)}$ describing the motion with the initial quantum numbers αslJM. Schrödinger's equation for the functions $u_c^{0(\pm)}$ has the form

$$(E-H_P^0)\,u_c^{0(\pm)} = 0, \qquad (13)$$

where in the external region, in which there is no interaction,

$$|u_c^{0(+)}\rangle = \sum_{c'} \frac{\delta_{c'c}I_{c'} - S_{c'c}^{0J}O_{c'}}{v'^{1/2}r'}|c'\rangle,$$
$$\langle u_c^{0(-)}| = \sum_{c'} \frac{\delta_{c'c}I_{c'} - S_{c'c}^{0J}O_{c'}}{v'^{1/2}r'}\langle c'|. \qquad (14)$$

Here, the functions I_c and O_c are the same as those given by Lane and Thomas.[5] The matrix element $S_{c'c}^{0J}$ describes processes occurring without the formation of a compound nucleus.

5. We shall consider in detail the low-energy case, where the kinetic energy of the incident (or emitted) particles does not exceed 5 — 10 Mev. At these energies, the optical potential has only a real part, and the direct nuclear processes do not play an important role. It will be assumed that the Hamiltonian H_P^0 and the wave functions $\psi_0^{(\pm)}$ describe scattering on an optical potential.

In the approximation $E_{kin} \leq 40$ Mev, the behavior of the optical wave functions inside the potential well does not depend on the energy, and the functions $u_c^{0(\pm)}$ split into the product of two factors, one of which depends only on the energy (normalizing factor) and the other, only on the coordinates. We can then write

$$u_c^{0(+)} = k_c(E)\,g_c(r)\,|c\rangle, \qquad u_c^{0(-)*} = k_c(E)\,g_c(r)\,\langle c|. \qquad (15)$$

These relations do not hold close to the boundary of the potential well, and therefore the coefficients $k_c(E)$ cannot be found directly from the boundary conditions. We differentiate Schrodinger's equation for the function $u_c^{0(+)}$ with respect to the energy and write the conjugate equation for $u_c^{0(-)*}$:

$$(E-H_P^0)\,\partial u^{0(\pm)}_c/\partial E + (1-\partial V^0/\partial E)\,u_c^{0(+)} = 0,$$
$$(E-H_P^{0*})\,u_c^{0(-)*} = 0. \qquad (16)$$

Here V^0 is the optical potential in the channel c. We multiply these equations by $u_c^{0(-)*}$ and $u_c^{0(+)}$ respectively, subtract one from the other, and integrate over the internal part of the potential well. We now apply Green's theorem and, using the boundary conditions (14) for the functions $u_c^{0(\pm)}$, we obtain

$$\int_0^R \left\langle u_c^{0\,(-)} \left|\left(1-\frac{\partial V^0}{\partial E}\right)\right| u_c^{0\,(+)}\right\rangle = 2\hbar P_c\Omega_c^2\,\frac{\partial L_c^0/\partial E}{(L_c^0-L_c)^2}. \qquad (17)$$

Here L_c denotes, as usual, the logarithmic derivative of the function O taken at the point R and multiplied by R; similarly, L_c^0 denotes the logarithmic derivative of the optical wave functions $u_c^{0(+)}$ for r = R, multiplied by R. The quantities P_c and Ω_c are defined in the review article of Lane and Thomas.[5]

At small energies, it can be assumed that the optical potential does not depend on the energy, and the derivative in the left-hand part of (17) can be dropped. We normalize the real functions $g_c(r)$ to unity, so that the right-hand part of (17) is equal to $[k_c(E)]^2$. For convenience of notation, we introduce the energy-independent amplitudes $b_{\lambda c}$:

$$b_{\lambda c} = E_R^{-1/2}\langle\chi_\lambda|\Gamma|cg_c\rangle = E_R^{-1/2}\langle cg_c|\Gamma^+|\chi_\lambda\rangle. \qquad (18)$$

From relation (7) for χ_λ^* and from similar relations for the functions $|c\rangle$ resulting from the properties of the time-reversal operator, it follows that $b_{\lambda c}$ is real.

We note that the logarithmic derivative L_c^0 in the minus first degree is an R function (for the definition and properties of R functions, see Lane and Thomas,[5] p. 272). This means, in particular, that the quantity $\partial L_c^0/\partial E$ is always less than zero.

We introduce the auxiliary quantity γ_c^0

$$\gamma_c^0 = +\sqrt{-(\partial L_c^0/\partial E)^{-1}}, \tag{19}$$

whose physical sense will be explained below. It can be shown that the quantity γ_c^0 weakly depends on the energy; the characteristic interval over which γ_c^0 changes is equal in order of magnitude to E_R.

We define the positive penetration factors

$$\Pi_c = E_R P_c / (\gamma_c^0)^2 \, |L_c^0 - L_c|^2 \tag{20}$$

and the matrix Ω_c^0 whose square gives the diagonal of the S matrix for optical scattering. From the adjointness conditions at the boundary we readily find

$$(\Omega_c^0)^2 = S_c^{\text{opt}} = \Omega_c^2 (L_c^0 - L_c^*) / (L_c^0 - L_c). \tag{21}$$

The dispersion expansion (10) taking into account resonances of the compound nucleus and optical processes takes the form

$$S_{c'c}^J = \delta_{c'c} S_c^{\text{opt}} - 2i \sum_\lambda \Omega_{c'}^0 \Pi_{c'}^{1/2} b_{\lambda c'} \Omega_c^0 \Pi_c^{1/2} x_{\lambda c}. \tag{22}$$

The energy dependence of the resonance is contained in the quantities $x_{\lambda c}$ for which the equation

$$\sum_\lambda [\delta_{\mu\lambda} (E - E_\lambda) - K_{\mu\lambda}] \, x_{\lambda c} = b_{\mu c} \tag{23}$$

is valid.

It is seen that the contribution of one resonance in (22) does not satisfy the reciprocity theorem. The coefficients $K_{\mu\lambda}$ in (23) are determined by (12). The imaginary part of $K_{\mu\lambda}$ contains a δ function of the energy and is readily expressed by the quantities Π_c and $b_{\lambda c}$. From the completeness relation for the functions $u_c^{0(+)}$, we find

$$\delta(E - H_P^0) = -\frac{1}{2\pi\hbar} \sum_c | u_c^{0(+)} \rangle \langle u_c^{0(+)} |,$$

$$\operatorname{Im} K_{\mu\lambda} = -\sum_c \Pi_c b_{\mu c} b_{\lambda c}. \tag{24}$$

The summation in (24) includes only channels open at a given energy.

In order to find also the real part of $K_{\mu\lambda}$, we employ the analytical character of the dependence of the quantity Λ_c on the energy. Let

$$\Lambda_c = (E_R / (\gamma_c^0)^2) (L_c^0 - L_c)^{-1}. \tag{25}$$

It is readily shown that

$$K_{\mu\lambda} = -\sum_c^P \Lambda_c b_{\mu c} b_{\lambda c}. \tag{26}$$

As a matter of fact, both parts of relation (26) represent functions which are analytical in the principal plane. The imaginary parts of (26) coincide, since Im Λ_e is equal to Π_c. As $E \to \infty$, the expression for $K_{\mu\lambda}$ in (12) tends to zero as does the quantity Λ_c. At points for which S_c^{opt} tends to infinity, both parts have poles, but the values at the poles coincide.

Far away from the optical resonances $|L_c^0| \gg 1$, Ω_c^0 goes over into Ω_c, the product $\Pi_c^{1/2} b_{\lambda c}$ goes over into $P_c^{1/2} \gamma_{\lambda c}$, where $\gamma_{\lambda c} = E_R^{-1/2} b_{\lambda c} / \gamma_c^0 \, |L_c^0 - L_c|$, and we obtain the same results as in R-matrix theory.

The optical resonances usually lie sufficiently far away from one another and it is necessary to take into account only one resonance. For the resonance energy E_r, the logarithmic derivative L_c^0 vanishes. Close to the resonance, we expand L_c^0 in powers of the difference $E - E_r$. Substitution of the expansion into (21) leads to the usual formula for the optical resonance with the width Γ_c^0 equal to $2P_c(\gamma_c^0)^2$. It is seen that the square of γ_c^0 has the sense of a reduced width of optical resonance. The quantity Λ_c and the penetration factor Π_c take the form

$$\Lambda_c = -\frac{E_R}{E - E_r + L_c (\gamma_c^0)^2},$$

$$\Pi_c = \frac{E_R \Gamma_c^0 / 2}{(E - E_r + S_c (\gamma_c^0)^2)^2 + (\Gamma_c^0)^2 / 4}. \tag{27}$$

6. We consider the special case in which one of the compound nucleus levels lies close to the threshold in channel c′. Let the width Γ'_λ and the shift Δ'_λ contain no contribution from channel c′:

$$\Gamma'_\lambda = 2 \sum_c{}' \Pi_c b_{\lambda c}^2, \qquad \Delta'_\lambda = -\sum_c^P{}' \operatorname{Re} \Lambda_c b_{\lambda c}^2, \tag{28}$$

where the summation in the expression for Γ'_λ is carried out only over the open channels. It is clear that these quantities vary little close to the threshold. The dispersion formula takes on the form

$$S_{c''c}^J = \delta_{c''c} S_c^{\text{opt}} - 2i \frac{\Omega_{c''}^0 \Pi_{c''}^{1/2} b_{\lambda c''} \Omega_c^0 \Pi_c^{1/2} b_{\lambda c}}{E - E_\lambda - \Delta'_\lambda + i\Gamma'_\lambda/2 + \Lambda_{c'} b_{\lambda c'}^2}. \tag{29}$$

The reaction cross section in the region below the threshold of channel c′ displays an interesting behavior. If the optical interaction is of such a type that capture on a single-particle level is possible, then, at an energy corresponding to the binding energy, the quantity $\Lambda_{c'}$ in the denominator has a pole and the reaction cross section $\sigma_{c''c}$ vanishes.

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