# ON THE PHYSICAL MEANING OF NEGATIVE PROBABILITIES

J. -P. VIGIER and Ya. P. TERLETSKII

Institute Henri Poincaré, Paris, France

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It is shown that the calculation of the statistical averages of a series of physical quantities can be carried out with the help of the distribution function instead of the probability density. The distribution function is the mathematical expectation of the particle density and is proportional to the probability density for a single particle only if all particles are absolutely identical. For a set of particles and antiparticles (with respect to some single property) the distribution function can also assume negative values. In this case it is no longer proportional to the probability, but it can be used to compute the averages of a number of physical quantities. It is shown that in the field theory of elementary particles the average values of some quantities characterizing the entire field (energy, momentum, charge, etc.) can be computed with the help of the corresponding distribution function.

## 1. INTRODUCTION

IT is known that many attempts at a generalization or a new interpretation of the apparatus of quantum mechanics have led to negative or even complex "probabilities." Thus, for example, the density matrix in the mixed (coordinate-momentum) representation, which plays the role of the quantum mechanical probability density in the phase space, must either be complex[1,2] or a real quantity which can assume negative values.[3,4] Negative "probability" densities appeared to be inevitable in relativistic quantum mechanics which includes states with negative energies.[5,6] Finally, quantum mechanics in Feynman's functional representation[7] involves complex (or real, but sometimes negative) "probability" densities in the functional space of the particle paths.[8]

The problem of the physical meaning of complex or negative "probabilities" also turns up in the new interpretation of quantum theory as the classical statistical theory of systems which interact with an "imaginary" thermostatic oven, i.e., with a thermostatic oven that has an imaginary temperature, as has been proposed by one of us.[9] Since it is possible, in all cases known to us, to avoid the appearance of imaginary parts in the "probabilities" by an appropriate reformulation, the question of principal interest is that of the meaning of "probabilities" which can have negative as well as positive values. The discussion of this question forms the subject of the present paper.

If "probability" is understood as a quantitative measure of the possibility of the presence of some object or of the occurrence of some process, where probability one implies certainty and probability zero corresponds to impossibility, negative values of the probability have no real meaning whatsoever. Thus, if the quantity which plays the role of a probability in a physical theory is not positive definite, it cannot be an actual "probability"; it only fulfills the function of the genuine probability in the computation of certain average values. This quantity is more accurately called the formal probability or quasiprobability.

The quasiprobability (more precisely, the density of the quasiprobability) is, in general, a distribution function, i.e., the mathematical expectation of the particle density. If the particles are completely identical and indistinguishable, the probability density function for a single particle differs from the distribution function only by normalization factor. However, for a set of positive and negative (with respect to some property, e.g., the charge) particles which otherwise have identical properties, the distribution function can take positive as well as negative values, since it represents the average "charge" density. This distribution function does not coincide with the probability density; however, it can be used in place of the probability density for the computation of certain average values.

The appearance of negative "probabilities" in a physical theory may therefore mean that we are actually not dealing with a probability density but with a distribution function which plays the role of a probability density in the computation of average values of certain quantities. Let us now consider these questions in more detail.

## 2. PROBABILITY DENSITY AS MATHEMATICAL EXPECTATION

In every physical theory one always ignores some physical quantities and only focusses attention on certain chosen ones, namely, the observables for which the theory has been formulated. Let X denote the set of observable quantities and Y, the set of ignored quantities. We can introduce a probability density W(X, Y) for the whole set of physical quantities (X, Y) in such a way that WdXdY has all the properties of a probability. With the help of this probability density we can also compute the probability density for the observed quantities

$$W(X) = \int_{(Y)} W(X, Y)\, dY. \tag{1}$$

On the other hand, expression (1) can also be written in the form

$$W(X) = \int_{(X', Y')} \delta(X - X')\, W(X', Y')\, dX' dY', \tag{2}$$

where $\delta(X - X')$ stands for the corresponding product of $\delta$ functions if X denotes more than one variable. Now the last integral is the mathematical expectation of the quantity

$$F = \delta(X - X'). \tag{3}$$

The probability density of the observed quantities can therefore be regarded as the average value of a quantity of the type (3).

In the above-mentioned example W(X) cannot take negative values, since $W(X, Y) \geq 0$ and $F \geq 0$; W, therefore, has all the properties of a probability density.

## 3. DISTRIBUTION FUNCTION AND PROBABILITY DENSITY

If the physical system under consideration consists of a set of N identical particles (i.e., the matter density is distributed over the three-dimensional space in the form of a set of N $\delta$-function-like maxima), its statistical behavior is sufficiently, completely described by a distribution function defined as the average of the particle density.

In the classical statistical theory of gases the distribution function is defined as the average of

$$\sigma = \sum_{k=1}^{N} \delta(\mathbf{r}_k - \mathbf{r})\, \delta(\mathbf{p}_k - \mathbf{p}), \tag{4}$$

in the six-dimensional phase space of coordinates and momenta; $\mathbf{r}_k$ and $\mathbf{p}_k$ are the coordinates and momenta of the individual mass points. This means we have

$$f(\mathbf{r}, \mathbf{p}) = \bar{\sigma} = \int \sigma W(\mathbf{r}_1, \mathbf{p}_1; \mathbf{r}_2, \mathbf{p}_2; \ldots; \mathbf{r}_N, \mathbf{p}_N)\, d\mathbf{r}_1 \ldots d\mathbf{p}_N, \tag{5}$$

where $W(\mathbf{r}_1, \ldots, \mathbf{p}_N)$ is the probability density in the (6N)-dimensional phase space. If all particles are identical, the function W must be symmetric under the interchange of the particles, and therefore

$$f(\mathbf{r}, \mathbf{p}) = N \int \delta(\mathbf{r} - \mathbf{r}_1)\, \delta(\mathbf{p} - \mathbf{p}_1)\, W(\mathbf{r}_1, \ldots, \mathbf{p}_N)\, d\mathbf{r}_1 \ldots d\mathbf{p}_N, \tag{6}$$

i.e., the distribution function is, according to (2), equal to N times the probability density that a single particle has the given coordinates $\mathbf{r}$ and momenta $\mathbf{p}$. Thus the average value of any physical quantity which is the sum of identical functions of the coordinates and momenta of the individual particles can with equal success be computed either with the help of f or with the help of W, for obviously

$$\overline{\sum_{k=1}^{N} \varphi(\mathbf{r}_k, \mathbf{p}_k)} = N \int \varphi(\mathbf{r}_1, \mathbf{p}_1)\, W(\mathbf{r}_1, \ldots, \mathbf{p}_N)\, d\mathbf{r}_1 \ldots d\mathbf{p}_N$$
$$= \int \varphi(\mathbf{r}, \mathbf{p})\, f(\mathbf{r}, \mathbf{p})\, d\mathbf{r} d\mathbf{p}. \tag{7}$$

In the computation of average values of physical quantities of a certain form, the distribution function can therefore fulfill the same function as the probability density. It is clear that this result holds not only for the distribution function in the theory of gases, but also in all other cases (for example, for the distribution function in configuration space).

## 4. DISTRIBUTION FUNCTION FOR A SYSTEM OF OPPOSITELY CHARGED IDENTICAL PARTICLES

In the preceding example the distribution function, just like the probability density, cannot take negative values, since always $\sigma \geq 0$. It can therefore replace the probability density in all respects, as it differs from the latter only by a normalization factor. The situation is different for a set of identical but oppositely charged particles. Here "charge" must be understood in the widest sense, i.e., as a certain property which distinguishes between particles and antiparticles which are identical to each other in all other respects. The "charge" can thus be interpreted as the electric charge in the case of electrons and positrons, as the baryon number in the case of nucleons and antinucleons, and, finally, as the rest mass in the case of Dirac particles with positive and negative masses.

Let us consider the case of a system of oppositely charged particles. Let it contain $N^+$ positive and $N^-$ negative particles. Together there are $N = N^+ + N^-$ particles described by a set of generalized coordinates $x_1, x_2, \ldots x_{N^+}, x_{N^++1}, \ldots, x_N$. The first $N^+$ coordinates, which can also be denoted by $x_k = x_k^+$, are the coordinates of the positive particles, and the remaining $N^-$ coordinates $x_{N^++i} = x_i^-$ are the coordinates of the negative particles. The total probability density describing the statistical behavior of the total ensemble of particles must be a function of all coordinates:

$$W(x_1, \ldots, x_N)$$
$$= W(x_1^+, x_2^+, \ldots, x_{N+}^+; x_1^-, x_2^-, \ldots, x_{N-}^-). \qquad (8)$$

Since the particles are indistinguishable, the function (8) is evidently symmetric with respect to interchanges of particles of the same sign, but may not, in general, be symmetric with respect to interchanges of a positive and a negative particle. Hence

$$W_+(x) = \int \delta(x - x_k^+)\, W(x_1^+, \ldots, x_{N-}^-)\, dx_1^+ \ldots dx_{N-}^-$$
$$\neq W_-(x) = \int \delta(x - x_i^-)$$
$$\times W(x_1^+, \ldots, x_{N-}^-)\, dx_1^+ \ldots dx_{N-}^-. \qquad (9)$$

i.e., the probability density for a positive particle is not equal to the probability density for a negative particle.

The role of the distribution function for the considered system of oppositely charged particles will, obviously, be played by the average "charge" density, i.e., by the average value of the quantity

$$\rho = \sum_{k=1}^{N} \varepsilon_k \delta(x - x_k). \qquad (10)$$

If we set $|\epsilon_k| = 1$, this expression can also be written in the form

$$\rho = \sum_{k=1}^{N+} \delta(x - x_k^+) - \sum_{i=1}^{N-} \delta(x - x_i^-). \qquad (11)$$

The average value of $\rho$ is, according to (9) — (11), equal to

$$\bar{\rho} = N_+ W_+(x) - N_- W_-(x). \qquad (12)$$

Because of (9), this quantity can take both positive and negative values.

It is easy to see that we can use the average density to compute the average values of quantities of the form

$$F = \sum_{k=1}^{N} \varepsilon_k \varphi(x_k) \qquad (13)$$

with the help of the usual formula for the mathematical expectation. Indeed,

$$\bar{F} = \int \sum_{k=1}^{N} \varepsilon_k \varphi(x_k)\, W(x_1, \ldots, x_N)\, dx_1 \ldots dx_N$$
$$= \int \varphi(x)\, [N_+ W_+(x) - N_- W_-(x)]\, dx = \int \varphi(x)\, \bar{\rho}(x)\, dx \qquad (14)$$

hence, if we substitute the quantity $N\varphi(x)$ instead of $\bar{F}$, the quantity $\bar{\rho}(x)/N$ will play the role of the probability density in x space.

One might think that the situation is analogous in those cases where negative "probabilities" appear. If the formal apparatus of the theory leads to a negative probability for some particle coordinate, this may indicate that we are actually dealing with an ensemble of particles and antiparticles (with respect to some property) and that we are actually computing the average value of a quantity of the type (13), and not a function of the coordinate of a single particle.

## 5. AVERAGE VALUES IN THE FIELD THEORY OF PARTICLES

If the elementary particles are regarded as particular solutions of nonlinear field equations (for example, as the particle-like solutions of references 9 — 11), only quantities characterizing the whole field, not those referring to the individual particles, have a physical meaning.

In the example considered above, such a quantity is the "charge" of a given volume. A theory which uses the distribution function instead of the probability allows us to calculate the average "charge" of the unit volume, but says nothing about the average "charge" of a given particle. In the case of a set of Dirac particles with positive and negative masses, we can compute the average energy and momentum in a given volume with the help of the distribution function. Many basic average values in the field theory of elementary particles can thus be computed with the help of the corresponding distribution function, using the same rules that apply if the probability density is used. The distribution function may take negative values; but it is not correct to interpret this as the appearance so to speak of negative probabilities.

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