ON THE MANDELSTAM REPRESENTATION IN PERTURBATION THEORY FOR AN ANOMALOUS MASS RELATION

V. N. GRIBOV, M. V. TERENT'EV, and K. A. TER-MARTIROSYAN

Submitted to JETP editor August 8, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 337-340 (January, 1961)

The analytic properties of the box diagram are investigated in perturbation theory. With this diagram as an example, it is demonstrated how the Mandelstam representation must be modified in the case of an anomalous relation between the particle masses.

As is well known, the Mandelstam integral representation for scattering amplitudes[1] is valid only if the process does not contain "anomalous" diagrams. In this paper we discuss the modification of the Mandelstam representation in the anomalous case by considering the simplest diagram, namely the box diagram shown in Fig. 1. The notation is as follows: p_i, k_i are particle four-momenta; m, μ are the masses of the "internal" particles; $p_1 + p_2 + p_3 + p_4 = 0$. We consider the case when $p_1^2 = p_2^2 = p_3^2 = p_4^2 = M^2$. The diagram is a function of the invariants $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$:

$$A(s,t)= \frac{1}{i\pi^3}$$

$$\times\int \frac{\delta(p_1+k_4-k_1)\,\delta(p_2+k_1-k_2)\,\delta(p_3+k_2-k_3)\,d^4k_1\,d^4k_2\,d^4k_3\,d^4k_4}{(k_1^2-\mu^2+i\varepsilon_1)(k_2^2-m^2+i\varepsilon_2)(k_3^2-\mu^2+i\varepsilon_3)(k_4^2-m^2+i\varepsilon_4)}.$$

For the "normal" case when $M^2 < m^2 + \mu^2$ we can write

$$A(s,t) = \frac{1}{\pi^2}\int_{\Omega_N} \frac{\rho(s',t')\,ds'\,dt'}{(s'-s-i\delta)(t'-t-i\sigma)};$$

$$\rho(s,t) = \pi/\sqrt{-stf(s,t)},$$

$$f(s,t) = 4(M^2-m^2-\mu^2)^2 + (t-4\mu^2)(4m^2-s). \qquad (1)$$

The region of integration Ω_N is bounded by the curve L_N in the st-plane (see Fig. 2); and $f(s,t) = 0$ on the curves L_N and L_A.

The quantity $A(s,t)$ may also be represented in the form

$$A(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{A_1(s',t)}{s'-s-i\delta}\,ds',$$

$$A_1(s,t) = \frac{1}{\sqrt{-stf(s,t)}}\ln\frac{\sqrt{f(s,t)}+\sqrt{t(4m^2-s)}}{\sqrt{f(s,t)}-\sqrt{t(4m^2-s)}},$$

$$t<0, \qquad (2)$$

where the integral is to be calculated along the contour C_1 in the complex s-plane shown in Fig. 3. The function $A_1(s,t)$, which is for $t < 0$ formally continued into the region $s < 4m^2$, has two branch points:

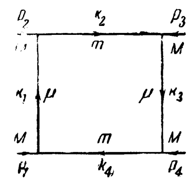

FIG. 1

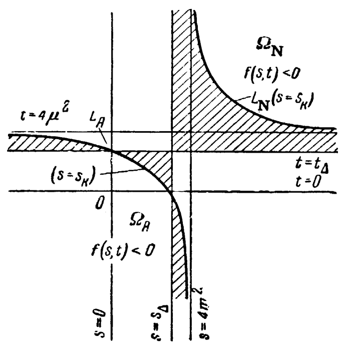

FIG. 2

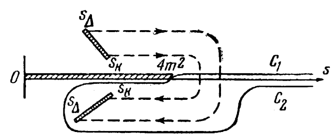

FIG. 3

$$\text{at } s = s_\Delta = 4m^2 - (M^2-m^2-\mu^2)^2/\mu^2$$

a logarithmic type branch point, and at

$$s = s_k = 4m^2 + 4(M^2-m^2-\mu^2)^2/(t-4\mu^2)$$

a root type branch point. The function $A_1(s,t)$ may be made single-valued in the complex s-plane by introducing a cut from s_Δ to s and from 0 to $4m^2$ (see Fig. 3).

Let us investigate in more detail the motion of the singular points s_Δ and s_k as the mass M changes, making use of the method proposed by Mandelstam.[2] Let at first $M^2 < m^2 + \mu^2$. We make the substitution $M^2 \to M^2 + i\epsilon$. Then the points

s_Δ and s_k move off the real axis. As M increases these points move in the manner shown by the dashed lines in Fig. 3; when $M^2 = m^2 + \mu^2$ they cut the contour C_1 to the right of the point $s = 4m^2$, and then, for $M^2 > m^2 + \mu^2$, take up positions to the left of the point $s = 4m^2$, as shown in Fig. 3. In order to avoid cutting the contour of integration when passing to the anomalous case it is necessary to replace in the expression (1) from the very beginning the contour C_1 by the contour C_2. In the anomalous case this leads to the appearance of the integral over the discontinuity of the function $A_1(s, t)$ in the interval $s_\Delta < s < s_k$. Passing to the limit $\epsilon \to 0$ and evaluating the jump in A_1 in this interval, one can obtain for $M^2 > m^2 + \mu^2$

$$A(s, t) = a(s, t) + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{A_1(s', t)}{s' - s - i\delta} ds', \tag{3}$$

$$a(s, t) = \frac{1}{\pi}\int_{s_\Delta}^{s_k} \frac{a_1(s', t)}{s' - s - i\delta} ds', \quad a_1(s, t) = \frac{2\pi}{\sqrt{stf(s, t)}}. \tag{4}$$

The integral in Eq. (4) can be evaluated directly. As a result it turns out that for $t < 0$ and $s_\Delta < s < s_k$ the function a is of the form

$$a(s, t) = \frac{2}{\sqrt{stf(s, t)}}\left[\ln\frac{\sqrt{st(s_\Delta - 4m^2)} + \sqrt{-s_\Delta f(s, t)}}{\sqrt{st(s_\Delta - 4m^2)} - \sqrt{-s_\Delta f(s, t)}} + i\pi\right] \tag{5}$$

(it is obvious that a is symmetric under the exchange $s \to t$, $\mu \to m$).

Now the function $a(s, t)$ may be analytically continued into the region $s > 4m^2$, and then in the variable t into the region $t > 0$. As a result the following representation is obtained for $a(s, t)$ for t in the various regions as indicated:

$$a(s, t) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_{s_k}^{s_\Delta} \frac{a_1(s', t)}{s' - s - i\delta} ds', & 0 < t < t_\Delta \\ -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{s_k} \frac{a_1(s', t)}{s' - s - i\delta} ds' + \frac{1}{\pi}\int_{\infty}^{s_\Delta} \frac{a_1(s', t)}{s' - s - i\delta} ds', & t_\Delta < t < 4m^2; \\ \dfrac{1}{\pi}\displaystyle\int_{s_k}^{s_\Delta} \frac{a_1(s', t)}{s' - s - i\delta} ds', & t > 4m^2 \end{cases}$$

$$t_\Delta = 4\mu^2 - (M^2 - m^2 - \mu^2)^2 / m^2. \tag{6}$$

Thus the diagram of Fig. 1 may be represented in the anomalous case in the form

$$A(s, t) = a(s, t) + \frac{1}{\pi^2}\int_{\Omega_N} \frac{\rho(s', t')\, ds'\, dt'}{(s' - s - i\delta)(t' - t - i\sigma)} = a(s, t) + A_0(s, t), \tag{7}$$

where the function a is determined by the relations (6) and (4) and has discontinuities in the variable s for fixed real t in the dashed-in region in Fig. 2 (it is obvious that in the same region $a(s, t)$ has analogous discontinuities in the variable t for fixed s).

It is interesting to note that as $t \to 0$ the function $a(s, t)$ has as $s \to s_k \to s_\Delta$ a pole type singularity, namely

$$a(s, 0) \underset{s \to s_\Delta}{=} \frac{\xi}{s - s_\Delta}, \qquad \xi = \frac{1}{\mu^2}\sqrt{(4m^2 - s_\Delta)/s_\Delta}.$$

We are most grateful to L. D. Landau who called to our attention the fact that in the anomalous case there must necessarily exist the singular region shown shaded in Fig. 2.

One can verify that for the function $A(s, t)$ the curve L_N is not singular in the anomalous case, since along this curve the singularities of the functions a and A_0 compensate each other. Indeed, for $t > 4\mu^2$ the function $A_0(s, t)$ has the following representation:

$$A_0(s, t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{1}{i\sqrt{s'tf(s', t)}} \times \ln\left(\frac{\sqrt{f(s', t)} + i\sqrt{t(s' - 4m^2)}}{\sqrt{f(s', t)} - i\sqrt{t(s' - 4m^2)}}\right)\frac{ds'}{s' - s - i\delta}.$$

Let us separate out of A_0 the part that is singular at $s = s_k$ (for $t > 4\mu^2$); then $A(s, t) = A_0' + a'$, where

$$A_0' = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{1}{i\sqrt{s'tf}} \ln\left(\frac{i\sqrt{t(s' - 4m^2)} + \sqrt{f}}{i\sqrt{t(s' - 4m^2)} - \sqrt{f}}\right)\frac{ds'}{s' - s - i\delta},$$

$$a'(s, t) = \int_{s_k}^{s_\Delta} \frac{2\, ds'}{\sqrt{s'tf}(s' - s - i\delta)} + \int_{4m^2}^{\infty} \frac{ds'}{\sqrt{s'tf}(s' - s - i\delta)}.$$

The function A_0' has no branch points at $s = s_k$. After performing the integration a' may be written in the form (for $0 < s < s_\Delta$)

$$a'(s,t)=\frac{1}{\sqrt{stf}}\left\{2\ln\frac{\sqrt{s_\Delta(s_k-s)}-\sqrt{s(s_k-s_\Delta)}}{\sqrt{s(s_k-s_\Delta)}+\sqrt{s_\Delta(s_k-s)}}\right.$$
$$+\ln\frac{\sqrt{4m^2(s_k-s)}+\sqrt{s(s_k-4m^2)}}{\sqrt{4m^2(s_k-s)}-\sqrt{s(s_k-4m^2)}}$$
$$\left.+\ln\frac{i\sqrt{s_k-s}-\sqrt{s}}{i\sqrt{s_k-s}+\sqrt{s}}\right\}.$$

It is easy to show after analytic continuation into the region $s \sim s_k$ that the function a' also has no singularities along the line $s = s_k$.

If one considers the case when the anomaly appears only in the simplest diagrams shown in Fig. 1, 4 and 5 (it is possible that this is the case

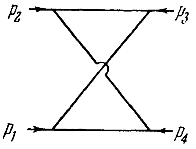

FIG. 4

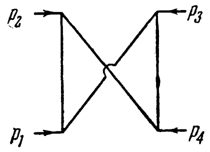

FIG. 5

for the real scattering processes of Σ and Λ hyperons[3]) then the exact scattering amplitude F(s, t) may be represented by

$$F(s,t)=a(s,t)+a(s_c,t)+a(s,s_c)+F_0(s,t),$$

where $s_c = 4M^2 - s - t$, and the function $F_0(s, t)$ has the usual Mandelstam representation.

In conclusion it is important to note that the diagram under consideration, symmetric in the masses of the particles, constitutes a very special case. For arbitrary masses of the particles with the structure of the integrand preserved the region of integration involved in the determination of the function a(s, t) is substantially modified. In addition, if the particle stability condition is violated the diagrams develop complex singularities, with the result that the contour of integration in a(s, t) is no longer entirely along the real axis.

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[2] S. Mandelstam, Phys. Rev. Lett. **4**, 84 (1960).

[3] Patashinskiĭ, Rudik, and Sudakov, JETP, in press.

Translated by A. M. Bincer
50