HIGHER MOMENTS OF THE CHARGE AND MAGNETIC MOMENT DISTRIBUTIONS OF NUCLEONS

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The unitarity condition with only two-pion intermediate states included is used to calculate the moments of the charge and magnetic moment distributions of nucleons. Asymptotic expressions for these moments are obtained. We calculate how the phase shifts in electron scattering from such nucleons differ from the phase shifts that would be obtained for point nucleons.

1. Chew et al.[1] and Federbush, Goldberger, and Treiman[2] have investigated the spectral distributions g_1^V and g_2^V of the isovector electromagnetic nucleon form factors G_1^V and G_2^V which result from two-pion states. The calculation requires an expression for the pion-nucleon scattering amplitude in the nonphysical region of momentum and energy transfer. If one uses the contribution to this amplitude from the pole term and the experimental data on pion-nucleon scattering, one can obtain an expression for the scattering amplitude only for momentum transfer t close to $t = 4\mu^2$. This is no help whatsoever if one wants to perform a reliable calculation of the anomalous magnetic moment and mean square radius r^2 of the nucleon without additional assumptions on the nature of pion-pion scattering[3] or the contribution from states with large numbers of pions and nucleons.

The problem we have set for ourselves is to calculate the quantities that depend specifically on the spectral distributions for t in the neighborhood of $4\mu^2$. Such quantities are the higher moments of the charge and magnetic moment distribution

$$(\overline{r^{2n}})_{1,2}^V = \frac{(-1)^n(2n+1)!}{n!} G_{1,2}^{V(n)}(0) = \frac{(2n+1)!}{\pi}\int^{\infty} \frac{g_{1,2}^V(t)\,dt}{t^{n+1}}$$

or the higher multipole potentials of nucleon transitions, which are related to the nucleon structure and determine, in particular, how the phase shifts in electron-nucleon scattering differ from the values obtained with point nucleons, e.g., expressions of the form

$$a_l = \int \frac{G^V(t)-G(0)}{t} P_l(z)\,dz$$

$$= -\frac{1}{\pi p^2}\int_{4\mu^2}^{\infty} \frac{g^V(t)}{t} Q_l\left(1+\frac{t}{2p^2}\right)dt, \qquad (2)$$

where P_l and Q_l are Legendre polynomials of the first and second kind, $t = -2p^2(1-z)$, and p is the momentum in the electron-nucleon center-of-mass system.

For sufficiently large n or l, the integrals in (1) and (2) depend only on a region close to the lower limit, where the g^V functions can be found up to a factor which represents the value of the pion form factor $\Pi(t)$ for $t \sim 4\mu^2$. This quantity can then be measured by comparing the results of calculations according to (1) and (2) with experiment. Unfortunately the asymptotic nature of Eqs. (1) and (2) makes such a comparison difficult.

2. The spectral distributions obtained from the unitarity condition including only the two-pion states are of the form[1]

$$g_1^V\gamma_\mu + \frac{g_2^V}{4M}(\gamma_\mu\hat{q} - \hat{q}\gamma_\mu)$$

$$= \frac{\Pi(t)}{32\pi}\frac{1}{\sqrt{t(t-4M^2)}}\int_{-\sqrt{(t-4M^2)(t-4\mu^2)}}^{\sqrt{(t-4M^2)(t-4\mu^2)}} k_\mu A(t,\nu)\,d\nu, \qquad (3)$$

where $k = k_1 + k_2$, $q = k_1 - k_2 = p_1 - p_2$, p_i and k_i are the nucleon and meson momenta, $\nu = |\mathbf{k}|\,|\mathbf{p}|$, M is the nucleon mass, μ is the meson mass, and A is the isovector amplitude for two-pion nucleon annihilation.

This amplitude A can be divided into a pole term and a regular part, $A = A^p + A'$. Those terms in the g^V obtained by putting A^p in (3) were found by Chew et al.[1] For small $\xi = \sqrt{t/4\mu^2 - 1}$, they may be written (here $\epsilon = \mu/M$)

$$g_{1,p}^V = \tfrac{8}{3}f^2\xi^3/\varepsilon^2, \qquad g_{2,p}^V = 4f^2\xi^5/\varepsilon^4, \qquad \xi \ll \varepsilon/2, \qquad (4)$$

$$g_{1,p}^V = f^2[2\xi - \pi\varepsilon/2 + \varepsilon^2/2\xi],$$

$$g_{2,p}^V = -2f^2[2\xi - \pi\xi^2/2\varepsilon], \quad \varepsilon/2 < \xi \ll 1. \qquad (5)$$

The terms obtained from the regular part of A can be found by inserting the first terms of the expansion of A′ about $t = 4\mu^2$ into (3). These were obtained by Galanin et al.[4] by analytic continuation of the experimental pion-nucleon scattering amplitudes:

$$A' = (g^2/\mu^2)(a_1\nu/M - a_2\hat{k}), \quad a_1 = 0.0065, \quad a_2 = 0.0125. \tag{6}$$

Inserting (6) into (3), we obtain the corresponding terms in the g^V, namely

$$g_1^{V'} = \tfrac{4}{3} f^2\xi^3 (a_1 - a_2)/\varepsilon^2, \quad g_2^{V'} = -\tfrac{2}{3} f^2 \xi^3 a_1/\varepsilon^2$$
$$(f^2 = (\varepsilon/2)^2 (g^2/4\pi) = 0.08). \tag{7}$$

3. Comparison of (7) with (4) and (5) shows that the regular part $g^{V'}$ of the spectral distribution is significant only for very small values of ξ. Bosco and Alfaro[5] have used expressions such as (7) to calculate the asymptotic behavior of the isovector part of the charge and magnetic moment distributions, obtaining

$$\rho^V(r) = \frac{1}{(2\pi)^2} \int_{4\mu^2}^{\infty} g^V(t) \frac{1}{r} \exp(-\sqrt{t}r)\, dr. \tag{8}$$

This expression, however, is good only asymptotically for $r \gg 1/\mu\epsilon$, and can therefore not be used to construct the entire $\rho^V(r)$ curve by interpolation, as was attempted by Alfaro and Predazi.[6] This is because in the region $1/2\mu \ll r < 1/\mu\epsilon$ the fundamental role is played by values of ξ in which the g^V are given by the pole term (5).

Inserting (5) into (1), we obtain

$$(\overline{r_1^{2n}})^V = \frac{4f^2(2n+1)!}{(4\mu^2)^n n!} \left[-\frac{(n-1)!\varepsilon}{8} + \frac{(2n-3)!!}{2^{n+1}} + \frac{\varepsilon^2}{4} \frac{(2n-1)!!}{2^{n+1}} \right],$$
$$(\overline{r_2^{2n}})^V = \frac{f^2(2n+1)!}{\varepsilon(4\mu^2)^n n!} \left[\frac{(n-2)!}{4} - 2\varepsilon \frac{(2n-3)!!}{2^{n+1}} \right]. \tag{9}$$

Inserting (5) into (8), we obtain

$$\rho_1^V(r) = \frac{f^2}{(2\pi)^2} e^{-2\mu r} \left[-\frac{\pi\varepsilon}{r^3} + 3\sqrt{\frac{\pi}{\mu}} \frac{1}{r^{7/2}} \right],$$
$$\rho_2^V(r) = \frac{f^2}{(2\pi)^2} e^{-2\mu r} \left[-6\sqrt{\frac{\pi}{\mu}} \frac{1}{r^{7/2}} + \frac{4\pi}{\mu\varepsilon} \frac{1}{r^4} \right]. \tag{10}$$

4. Expressions can be given for the difference between the electron-nucleon phase shift and their values for a point nucleon. It is more convenient, however, to give the a_i^k and b_i^k, which are related to these differences in the phase shifts by the expressions

$$[\sqrt{l(l-1)}/(2l-1)](a_l^{l-1} - b_l^{l-1}) = 2i\gamma^{l-1},$$
$$[\sqrt{(l+1)(l+2)}/(2l+3)](a_l^{l+1} - b_l^{l+1}) = 2i\gamma^{l+1},$$
$$[(l+2)/(2l+3)]\, a_l^{l+1} + [(l+1)/(2l+3)]\, b_l^{l+1} = -2i\,{}^3\delta_l^{l+1},$$
$$[(l-1)/(2l-1)]\, a_l^{l-1} + [l/(2l-1)]\, b_l^{l-1} = -2i\,{}^3\delta_l^{l-1},$$
$$a_l^l = -2i\,{}^3\delta_l^l, \quad a_{0,l}^l = -2i\,{}^1\delta_l^l, \quad b_{0,l}^l = 2i\gamma_{0,1}^l. \tag{11}$$

Here ${}^{3,1}\delta_l^J$ are the triplet and singlet phase shifts for total angular momentum J, and $\gamma_{0,1}^J$ and γ^J are parameters which describe the amount of mixing between states with total spin 0 and 1 and states with orbital angular momenta $J - 1$ and $J + 1$. Then the a_i^k and b_i^k are given by the following expressions, where we use Eqs. (5) for the spectral distributions:

$$a_l^l = NQ_l(1+2\eta^2) \left\{ -\left[\frac{A}{p} + \frac{\omega B}{lM}\right] J_1 + \left[\frac{\omega(1+2\eta^2)}{lM} - 4\frac{l+1}{l}\frac{\eta^4(1-\eta^4)}{M}\right] J_2 \right\},$$
$$a_l^{l-1} = NQ_l(1+2\eta^2) \left\{ -\left[\frac{A}{p} + \frac{l+1}{l}\frac{\omega B}{M}\right] J_1 + \left[\frac{l+1}{l}\frac{\omega(1+2\eta^2)}{M} + \frac{4(l+1)^2\eta^4(1-\eta^4)}{l(l-1)M}\right] J_2 \right\},$$
$$a_l^{l+1} = NQ_l(1+2\eta^2) \left\{ -\left[\frac{A}{p} - \frac{\omega B}{M}\right] J_1 + \left[-\frac{\omega(1+2\eta^2)}{M} + \frac{4(l+1)\eta^4(1-\eta^4)}{(l+2)M}\right] J_2 \right\},$$
$$b_l^{l-1} = NQ_l(1+2\eta^2) \left\{ -\left[\frac{A}{2p} + \frac{l+1}{l}\frac{\omega B}{M}\right] J_1 + \left[\frac{l+1}{l}\frac{\omega(1-2\eta^2)}{M} - \frac{2\eta^2}{M}\right] J_2 \right\},$$
$$b_l^{l+1} = NQ_l(1+2\eta^2) \left\{ -\left[\frac{A}{2p} - \frac{\omega B}{M}\right] J_1 + \left[-\frac{\omega}{M}(1-2\eta^2) - \frac{2\eta^2}{M}\right] J_2 \right\},$$
$$a_{0,l}^l = NQ_l(1+2\eta^2) \left\{ -\frac{A}{2p} J_1 - \frac{2\eta^2}{M} J_2 \right\},$$
$$b_{0,l}^l = NQ_l(1+2\eta^2) \sqrt{\frac{l+1}{l}} \frac{\eta^2}{\omega} \left[-\frac{\omega^2 B}{\eta M} J_1 - \frac{\omega^2 J_2}{\eta M} \right];$$
$$A = \frac{Mm_e + Mp + p^2}{Mp} + (1+2\eta^2)\frac{p(M+p)}{M(m_e+p)}, \quad B = 1 + \frac{M}{m_e+p};$$
$$J_1 = \frac{f^2\sqrt{\pi}}{2} \frac{1}{L^{3/2}} \left[1 - \frac{\sqrt{\pi}}{8\zeta}\left(1 - \frac{1}{L}\right) \right],$$
$$J_2 = -2f^2\sqrt{\pi} \frac{1}{L^{3/2}} \left[1 - \frac{\sqrt{\pi}}{8\zeta}\left(1 - \frac{1}{L}\right) \right];$$
$$\zeta = \varepsilon\sqrt{L}/2, \quad \eta = \mu/p, \quad \omega = \eta\sqrt{1+\eta^2},$$
$$N = ie^2p/2\pi, \quad L = \eta(l+1)/\sqrt{1+\eta^2}. \tag{12}$$

Calculation for the case $l = 4$ and $p = \mu$ leads to very small values (of the order of 10^{-4}) for the differences in the phase shifts and the mixing parameters. This is a reflection of the following situation, which makes it difficult to apply to the present case the methods of calculation suggested by Galinin et al.[4] Equations (12) are the leading terms in a power series expansion in $1/L$, where

$$L = l + 1, \quad \eta \gg 1; \tag{13a}$$
$$L = \eta(l+1), \quad \eta \ll 1. \tag{13b}$$

On the other hand, angular momenta $l \lesssim l_{\text{eff}}$

$\sim r_0 p$ play a role in the scattering, where r_0 may be called the radius of interaction (for our case it is of the order of the nucleon radius). In case (13a) a satisfactory expansion is one in powers of $1/L$ for $l \gtrsim 4-5$; but then $l_{eff} \ll 1$, and the phases obtained for $l = 4$, of course, turn out to be very small. For $p \gg \mu$, the value of l_{eff} increases, but the values of l for which the phases can be calculated also increase linearly with p. It is most convenient to investigate the region where $p \sim \mu$, but our calculations have shown that in this case also the differences in the phases and mixing parameters are extremely small.

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APPENDIX

We give here the formulas relating the phases and mixing parameters with the matrix elements $S^J_{S', l'; S, l}$ of the scattering matrix in the representation in which the total angular momentum J is diagonal (S, l, S′, and l' are the spin and orbital angular momenta before and after scattering):

$$S^J_{1, J+1; 1, J+1} = \cos 2\gamma^J \exp(2i^3\delta^J_{J+1}),$$
$$S^J_{1, J-1; 1, J-1} = \cos 2\gamma^J \exp(2i^3\delta^J_{J-1}),$$
$$S^J_{1, J+1; 1, J-1} = i \sin 2\gamma^J \exp(i^3\delta^J_{J-1} + i^3\delta^J_{J+1}),$$
$$S^J_{0, J; 0, J} = \cos 2\gamma^J_{0,1} \exp(2i^1\delta^J_J),$$
$$S^J_{1, J; 1, J} = \cos 2\gamma^J_{0,1} \exp(2i^3\delta^J_J),$$
$$S^J_{1, J; 0, J} = i \sin 2\gamma^J_{0,1} \exp(i^1\delta^J_J + i^3\delta^J_J).$$

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