ON THE APPLICATION OF THE METHODS OF SUPERCONDUCTIVITY THEORY TO THE PROBLEM OF THE MASSES OF ELEMENTARY PARTICLES

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The methods of superconductivity theory are applied to a Heisenberg type four-fermion two-component Lagrangian with cut-off. Owing to the rearrangement of the vacuum state, the two-component nature of the initial field does not hinder the appearance of a fermion mass. Boson excitations are found which are analogous to acoustic excitations in a superconductor. Interaction between the excitations is discussed.

WE shall consider the simplest Lagrangian,[1] containing only a two-component spinor field u(x):

$$L = -u^+\sigma pu - \tfrac{1}{4}\lambda(ugu)(u^+gu^+)$$
$$= -u^+\sigma pu - \tfrac{1}{8}\lambda(u^+\sigma_r u)(u^+\sigma_r u). \qquad (1)$$

Here $\sigma_r = (\sigma, 1)$, $\sigma p = \sigma_r p_r = \boldsymbol{\sigma}\cdot\mathbf{p} - \sigma_0 p_0$, σ is the Pauli spin matrix and $g = i\sigma_y$. The Heisenberg operators $u(x)$ and $u^+(y)$ obey the usual anti-commutation relations $\{u(\mathbf{x}, t), u^+(\mathbf{y}, t)\} = \delta(\mathbf{x} - \mathbf{y})$. We assume a cut-off at a large momentum Λ, the physical reason for which is not considered. The Lagrangian L is not invariant with respect to the operations $P: \mathbf{x} \to -\mathbf{x}$ and $C: u \to gu^+$, but is invariant under CP and T.

From (1) we have

$$\sigma pu + \tfrac{1}{2}\lambda(ugu)gu^+ = 0. \qquad (2)$$

We use the methods of the theory of superconductivity in order to solve (2). Following the work of Gor'kov[2] we introduce the quantities

$$G_1 = \langle Tu(x)u^+(y)\rangle,$$
$$F = \langle u(x)u(y)\rangle, \qquad \Delta = \tfrac{1}{2}\lambda\langle u(x)gu(x)\rangle; \qquad (3)$$

where the symbol $\langle\ldots\rangle$ denotes an expectation value for the physical vacuum. Taking into account, as in the theory of superconductivity, only qualitative effects connected with rearrangement of the vacuum, we obtain from (2) and (3)

$$\sigma pG_1(p) - \Delta gF^+(p) = -i, \qquad \tilde{\sigma}pF^+(p) + \Delta^* gG_1(p) = 0; \qquad (4)$$

$$\Delta^* = -\frac{\lambda}{2}\int d^4p\,\mathrm{Sp}\,(F^+(p)g) = -i\lambda\Delta^*\int\frac{d^4p}{p^2 + |\Delta|^2 - i\delta}, \qquad (5)$$

where $d^4p = (2\pi)^{-4}d\mathbf{p}dp_0$, $p^2 < \Lambda^2$, $\delta \to +0$.

Besides the trivial solution $\Delta^* = 0$, Eq. (5) can have a non-zero solution for the condition $(4\pi)^{-2}\lambda\Lambda^2 > 1$. $|\Delta|^2$ then satisfies the relation

$$1 = -i\lambda\int\frac{d^4p}{p^2 + |\Delta|^2 - i\delta} = \frac{\lambda}{(4\pi)^2}\left(\Lambda^2 - |\Delta|^2\ln\frac{\Lambda^2}{|\Delta|^2}\right). \qquad (6)$$

Assuming that the mass $m = |\Delta| \ll \Lambda$, we find that for (6) to be fulfilled it is essential that $(4\pi)^{-2}\lambda\Lambda^2$ should be very close to 1, i.e. $\lambda m^2 \ll 1$. The hypothesis of such a connection between the interaction constant and the cut-off was put forward in Zel'dovich's work.[3] A more rigorous discussion leads to an integral equation instead of (5), with the constant λ replaced by an irreducible four-pole. We shall find that this only changes the relation between the mass and the parameters λ and Λ.

It is convenient to go over to a four-component description. We introduce

$$\psi = \begin{pmatrix} u \\ (\Delta/m)\,gu^+ \end{pmatrix}, \quad G = \langle\psi(x)\psi(y)\rangle. \qquad (7)$$

Equation (4) then takes the form

$$(\hat{p} + m)G(p) = -i, \qquad (8)$$

where $\hat{p} = \gamma_r p_r = \boldsymbol{\gamma}\cdot\mathbf{p} - \gamma_0 p_0$, γ_r being the usual γ matrix in the Weyl representation. Equations (7) and (8) describe a Majorana particle of mass m: if we introduce the charge-conjugation matrix

$$C = \frac{\Delta}{m}\begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix},$$

then

$$\psi_c(x) \equiv \bar{\psi}(x)C = \psi(x).$$

With the notation of (7), the Lagrangian (1) takes the form

$$L = -\tfrac{1}{2}\bar{\psi}\hat{p}\psi + \tfrac{1}{16}\lambda\left[(i\bar{\psi}\gamma_5\psi)^2 + (\bar{\psi}\psi)^2\right]$$
$$= -\tfrac{1}{2}\bar{\psi}\hat{p}\psi + \tfrac{1}{32}\lambda(i\bar{\psi}\gamma_r\gamma_5\psi)^2, \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \qquad (9)$$

We shall find collective excitations analogous to excitations in a superconductor.[4] For this purpose let us consider the two-particle Green's functions. There are six types of such functions corresponding to the number of possible antisymmetrical ψ combinations:

$$K_+ = \tfrac{1}{8}\langle i\bar{\psi}(x)\gamma_5\psi(x),\ i\bar{\psi}(y)\gamma_5\psi(y)\rangle,$$

$$K_- = \tfrac{1}{8}\langle\bar{\psi}(x)\psi(x),\ \bar{\psi}(y)\psi(y)\rangle,$$

$$K_{r+} = -K_{+r} = \tfrac{1}{8}\langle i\bar{\psi}(x)\gamma_5\gamma_r\psi(x),\ i\bar{\psi}(y)\gamma_5\psi(y)\rangle.$$

As above, if we replace the irreducible four-pole by the constant λ, we obtain the following equations:

$$K_-(q) = \Pi_-(q) + i\lambda\Pi_-(q)K_-(q),$$
$$K_+(q) = \Pi_+(q) + i\lambda\Pi_+(q)K_+(q) + i\lambda\Pi_{+r}(q)K_{r+}(q),$$
$$K_{r+}(q) = \Pi_{r+}(q) + i\lambda\Pi_{r+}(q)K_+(q) + i\lambda\Pi_{rn}(q)K_{n+}(q);$$
$$\Pi_\pm = -\int\frac{d^4p}{s_1s_2}(p_1p_2 \pm m^2),\qquad \Pi_{r+} = -mq_r\int\frac{d^4p}{s_1s_2} = -\Pi_{+r}, \tag{10}$$
$$\Pi_{rn} = \int\frac{d^4p}{s_1s_2}[\delta_{rn}(p_1p_2 - m^2) - 2p_{1r}p_{2n}],$$
$$p_1 = p + q/2,\qquad p_2 = p - q/2,\qquad s_{1,2} = p^2_{1,2} + m^2. \tag{11}$$

The excitation spectrum is determined by the poles of the functions K. From (11) and (10) taking account of (6), we find that in the region $q^2 \ll \Lambda^2$ the function K_{r+} does not have a pole, K_+ has a pole $q^2 = 0$ and K has a pole at the point $-q^2 \approx 4m^2 - m^2/\ln(\Lambda^2/m^2)$. Thus there exist odd-CP excitations of zero mass and even CP related to a two-particle state.

Let us examine the interaction between the particles obtained. As an example we shall find the interaction between two fermions through exchange of the massless "K_+ particle". The scattering amplitude of particles with momenta p_1 and p_2 into the state with momenta p_3 and p_4 is given by the expression

$$\langle p_3, p_4\left|T\exp\left\{\frac{i\lambda}{16}\int d^4x\,[(\bar{\psi}i\gamma_5\psi)^2 + (\bar{\psi}\psi)^2]\right\}\right|p_1p_2\rangle.$$

Let us look, for example, at the non-exchange term. The term of interest to us is of the form

$$(i\gamma_5)_{3,1}(i\gamma_5)_{4,2}\Big[\frac{i\lambda}{2} - \frac{\lambda^2}{16}\int\Big\langle i\bar{\psi}\Big(\frac{x}{2}\Big)\gamma_5\psi\Big(\frac{x}{2}\Big),$$
$$i\bar{\psi}\Big(-\frac{x}{2}\Big)\gamma_5\psi\Big(-\frac{x}{2}\Big)\Big\rangle e^{-iqx}d^4x\Big]. \tag{12}$$

Using equation (10) for the expression in square brackets in (12) to the accuracy of terms $\sim \lambda m^2$, we obtain

$$\frac{i\lambda'(q)}{2} = \frac{i\lambda}{2}\Big(1 + \frac{i\lambda\Pi_+}{1 - i\lambda\Pi_+}\Big) = \frac{i\lambda}{2}\,\frac{1}{1 - i\lambda\Pi_+(q)}. \tag{13}$$

According to (6) and (11) we have

$$\frac{i\lambda'(q)}{2} = \frac{i\lambda}{2i\lambda[\Pi_+(0) - \Pi_+(q)]} = \frac{i(4\pi)^2}{q^2[\ln(\Lambda^2/m^2) - 3/4 + 2(1-\theta\cot\theta)]}, \tag{14}$$

where $\sin^2\theta = -q^2/4m^2$. In spite of the smallness of the primary constant λ, the effective interaction $\lambda'(q)$ is therefore not very small. As was pointed out by Zel'dovich,[3] a similar situation occurs in the problem of resonance scattering, which has some features in common with the model considered.

We shall explain the relation between the results obtained and the usual description of the interaction in the form

$$L_{int} = ei\bar{\psi}(x)\gamma_5\psi(x)\varphi(x). \tag{15}$$

The magnitude of e is determined by comparing (14) with the scattering amplitude for small values of q^2, obtained by perturbation theory from (15). We obtain

$$e^2/4\pi^2 = \Big[\ln\frac{\Lambda^2}{m^2} - \frac{3}{4}\Big]^{-1}. \tag{16}$$

Let us consider the question of renormalizations for the interaction (5). If we do not taken into account, as in deriving (14), the correction to the function G and the vertex part, we obtain for the effective interaction, analogously to (14)

$$-4e_0^2D(q) = 4ie_0^2/[q^2 + ie_0^2 8\Pi_+(q)]. \tag{17}$$

Here the boson Green's function $D(x-y) = \langle\varphi(x), \varphi(y)\rangle$ and Π_+ is given by Eq. (11). According to the rules of renormalization, we must subtract $ie_0^2 8\Pi_+(0)$ from the demoninator of expressions (7), after which (7) takes the form

$$-4e_0^2D = \frac{4ie_0^2}{q^2}\Big[\Big(1 + \frac{e_0^2}{(2\pi)^2}\Big(\ln\frac{\Lambda_0^2}{m^2} - \frac{3}{4}\Big)\Big) + \frac{e_0^2}{(2\pi)^2}2(1-\theta\cot\theta)\Big]^{-1}. \tag{18}$$

By comparing (18) with (14) we can verify that expression (16) does in fact determine the renormalized physical charge of the usual theory:

$$e^2 = e_0^2\Big[1 + \frac{e_0^2}{4\pi^2}\Big(\ln\frac{\Lambda_0^2}{m^2} - \frac{3}{4}\Big)\Big]^{-1}. \tag{19}$$

It can be seen from (19) and (16) that (18) agrees with (14). In this way we can understand in the model discussed the calculated quadratically divergent "self mass" of a boson in the usual renormalization.

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Note added in proof (15 December, 1960): After our work was sent to be printed we became aware of a work by Nambu in which analogous results were obtained (Y. Nambu, Report to Midwest Conference on Theoretical Physics, March 1960, preprint).

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