INTEGRAL EQUATION FOR PION-NUCLEON SCATTERING AT LOW ENERGIES

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A set of coupled integral equations for the S and P pion-nucleon scattering wave amplitudes at small energies is deduced on basis of the dispersion relations for forward and backward scattering and the unitarity condition. The contribution of the cut in the nonphysical region is taken into account without applying the analytic continuation by the Legendre expansion. The integral equations include $N\bar{N}$-annihilation amplitudes, which are explicitly connected with the presence of the $\pi\pi$-interaction.

1. INTRODUCTION

SEVERAL authors[1-3] have recently attempted to solve the problem of strong interaction at low energies on the basis of the two-dimensional dispersion relation proposed by Mandelstam.[4-6] They used from the very outset the dispersion relations for the partial amplitudes. The scattering amplitudes in the nonphysical region were obtained by means of an analytic continuation of the Legendre expansion from the physical region, which becomes invalid on the boundary of the region, where the spectral functions do not vanish.

Efremov, Meshcheryakov, Shirkov, and Chou[7] have shown, however, that the higher partial waves in the Legendre expansion cannot be neglected. In particular, if the contribution from the nonphysical region is not cut off at high energies, then the coefficients of the higher partial amplitudes even diverge.

In the present paper we derive a system of coupled integral equations for the amplitudes of the S and P waves of pion-nucleon scattering at low energies, using the dispersion relations for the forward and backward scattering and the unitarity conditions. The dispersion relations for the forward and backward scattering have the advantage that the scattering amplitudes in the nonphysical region can be expressed directly in terms of the amplitudes of the crossing reactions without the use of analytic continuation. The problem of the violation of the Legendre expansion does not arise here. It is very possible that these integral equations will permit a more accurate evaluation of the contribution due to cutoff in the nonphysical region. The dispersion relation for the backward scattering, together with the integral equations, contains the amplitudes of the annihilation reaction $N + \bar{N} \rightarrow \pi + \pi$, so that the influence of the $\pi\pi$ interaction can be taken into account.

The integral equations obtained should be solved simultaneously with the integral equations for the annihilation $N + \bar{N} \rightarrow \pi + \pi$. Quite recently, Efremov, Meshcheryakov and Shirkov[8] obtained a system of integral equations for the scattering, under the assumption that the $N\bar{N}$ annihilation proceeds in the low-energy region predominantly via the S and P states. They have also used the advantage of the dispersion relation for the backward scattering.[9] Their system of integral equations has an interesting feature in that it can be solved without knowledge of the $N\bar{N}$ annihilation amplitudes, provided the $\pi\pi$ scattering phases are known.

The integral equation derived in the present paper is suitable for the case when the real part of the annihilation amplitude in the states with higher moments cannot be neglected compared with the annihilation amplitudes in the S and P states.

In the second section of the present paper we investigate the locations of the singularities of the forward and backward scattering amplitudes.

In Sec. 3 we write out the dispersion relations for the forward and backward scattering, and give the connection between the amplitudes of the S and P waves, on the one hand, and the forward and backward scattering amplitudes on the other. In Sec. 4 we derive integral equations for the amplitudes of the S and P waves. The results obtained are compared with those of Chew, Goldberger, Low, and Nambu.[10]

2. REGION OF VALIDITY OF THE LEGENDRE EXPANSION

The notation used in the present paper is standard. For convenience, however, we shall define it in this section.

As is well known, processes of the form

$$\begin{aligned} &\text{I.} \quad \pi(p_1, \alpha) + N(p_3) \to \pi(-p_2, \beta) + N(-p_4),\\ &\text{II.} \quad \pi(p_2, \beta) + N(p_3) \to \pi(-p_1, \alpha) + N(-p_4),\\ &\text{III.} \quad N(p_3) + \overline{N}(p_4) \to \pi(-p_1, \alpha) + \pi(-p_2, \beta) \end{aligned} \tag{1}$$

are described by a single Green's function. Here π, N, and $\overline{\mathrm{N}}$ are respectively the pion, nucleon, and antinucleon. The p's inside the parentheses denote the corresponding four-momentum vectors, directed inward; α and β are the isotopic spin indices.

In the momentum representation, the Green's function has the form

$$T = \delta_{\beta\alpha}\{-A^+(s, \bar{s}, t) + \tfrac{1}{2} i (\hat{p}_1 - \hat{p}_2) B^+(s, \bar{s}, t)\} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha]\{-A^-(s, \bar{s}, t) + \tfrac{1}{2} i (\hat{p}_1 - \hat{p}_2) B^-(s, \bar{s}, t)\}, \tag{2}$$

where

$$s = -(p_1 + p_3)^2 = -(p_2 + p_4)^2,$$
$$\bar{s} = -(p_1 + p_4)^2 = -(p_2 + p_3)^2,$$
$$t = -(p_1 + p_2)^2 = -(p_3 + p_4)^2, \qquad s + \bar{s} + t = 2m^2 + 2 \tag{3}$$

(m is the nucleon mass, and the pion mass is set equal to unity). Only two of the three variables s, $\bar{s}$, and t are independent here.

The invariant functions $A^\pm$ and $B^\pm$ satisfy the following relations:

$$A^\pm(s, \bar{s}, t) = \pm A^\pm(\bar{s}, s, t), \qquad B^\pm(s, \bar{s}, t) = \mp B^\pm(\bar{s}, s, t). \tag{4}$$

According to Mandelstam,[4] they also satisfy the following two-dimensional dispersion relations

$$\begin{aligned} A^\pm(s, \bar{s}, t) = {} & \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{(m+1)^2}^{\infty} d\bar{s}' \frac{a_{12}^\pm(s', \bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_4^{\infty} dt' \frac{a_{13}^\pm(s', t')}{(s'-s)(t'-t)}\\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} d\bar{s}' \int_4^{\infty} dt' \frac{a_{23}^\pm(\bar{s}', t')}{(\bar{s}'-\bar{s})(t'-t)},\\ B^\pm(s, \bar{s}, t) = {} & \frac{g^2}{m^2 - s} \mp \frac{g^2}{m^2 - \bar{s}}\\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{(m+1)^2}^{\infty} d\bar{s}' \frac{b_{12}^\pm(s', \bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_4^{\infty} dt' \frac{b_{13}^\pm(s', t')}{(s'-s)(t'-t)}\\ & + \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} d\bar{s}' \int_4^{\infty} dt' \frac{b_{23}^\pm(\bar{s}', t')}{(\bar{s}'-\bar{s})(t'-t)}, \end{aligned} \tag{5}$$

where $a^\pm$ and $b^\pm$ are spectral functions, and g is the renormalized rationalized πN-coupling constant.

To facilitate the analysis we introduce the following variables in the center-of-mass system: k and Φ — momentum and scattering angle for reaction I; $\bar{k}$ and $\bar{\Phi}$ — momentum and scattering angle for reaction II; p, q, and θ — respectively the nucleon and antinucleon momentum, the pion momentum, and the scattering angle for reaction III. We then have the following relations for reaction I:

$$\begin{aligned} s &= m^2 + 1 + 2k^2 + 2\sqrt{(m^2 + k^2)(1 + k^2)},\\ \bar{s} &= m^2 + 1 - 2k^2x - 2\sqrt{(m^2 + k^2)(1 + k^2)},\\ t &= -2k^2(1 - x), \qquad x \equiv \cos\Phi. \end{aligned} \tag{6}$$

The corresponding relations for reaction II are

$$\begin{aligned} s &= m^2 + 1 - 2\bar{k}^2\bar{x} - 2\sqrt{(m^2 + \bar{k}^2)(1 + \bar{k}^2)},\\ \bar{s} &= m^2 + 1 + 2\bar{k}^2 + 2\sqrt{(m^2 + \bar{k}^2)(1 + \bar{k}^2)},\\ t &= -2\bar{k}^2(1 - \bar{x}), \qquad \bar{x} \equiv \cos\bar{\Phi}. \end{aligned} \tag{7}$$

For reaction III we have

$$\begin{aligned} &s = -p^2 - q^2 + 2pqz, && \bar{s} = -p^2 - q^2 - 2pqz,\\ &t = 4(m^2 + p^2) = 4(1 + q^2), && z = \cos\theta, \end{aligned} \tag{8}$$

It is known from experiment that only small values of the momentum are significant for πN scattering at low energies, and the other states can be neglected. As indicated in reference 7, to determine a small number of partial amplitudes it is sufficient to have dispersion relations for only several fixed values of the angle. Thus, in the energy region where all the states except S and P can be neglected, we need dispersion relations for two angles only. The most suitable angles are $\Phi = 0$ and $\Phi = \pi$.

The forward scattering for reaction I is determined by the condition

$$t = 0, \tag{9}$$

which determines also the forward scattering for reaction II. The backward scattering in reaction I is determined by the condition

$$\bar{s}s = (m^2 - 1)^2. \tag{10}$$

Simple calculations show that (10) determines also the backward scattering in reaction II and the forward and backward scattering in reaction III. In Fig. 1, condition (9) is represented by a straight line while condition (2) corresponds to the hyperbola and to the ellipse.

The branch of the hyperbola in region $t \leq 4$ is connected with the backward scattering in reactions I and II. The second branch, in the region $t \geq 4m^2$ is connected with the backward and forward scattering in reaction III. The ellipse, on which s and $\bar{s}$ are complex, corresponds to the nonphysical region of reaction III, where $4 \leq t \leq 4m^2$.

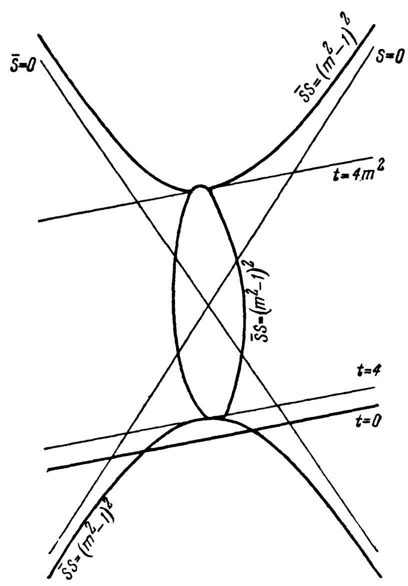

FIG. 1

The dispersion integral for the forward scattering is taken along the line t = 0. The scattering amplitudes for reaction I in the nonphysical region are directly connected with the amplitudes for reaction II in the physical region when cos Φ = 1. Thus, no analytic continuation is necessary.

For the backward scattering the dispersion integral is taken along the curve given by Eq. (10). The scattering amplitudes of reaction I in the nonphysical region are connected not only with the backward scattering in reaction II, but also with the scattering amplitude of reaction III when cos Φ = ± 1. Inasmuch as the unitarity condition for reaction III, as was shown by Mandelstam,[11] is analytically continued in the region $4 \leq t \leq 4m^2$, no further analytic continuation is necessary in this case, too.

The singularities in the cuts of the forward and backward scattering amplitudes for reaction I in the complex s plane can be obtained by projecting the curves of Fig. 1 on this plane. The singularities of the forward scattering amplitudes are well known; the cuts of the backward scattering amplitudes are shown in Fig. 2. The position of the poles is also well known and is therefore not shown in Fig. 2.

3. DISPERSION RELATIONS FOR FORWARD AND BACKWARD SCATTERING

The dispersion relations for the forward scattering have the form

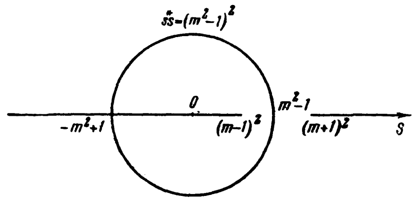

FIG. 2

$$A^{\pm}(s, \cos\Phi = 1) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \left(\frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}_+}\right) A_1^{\pm}(s', 1),$$

$$B^{\pm}(s, 1) = \frac{g^2}{m^2-s} \mp \frac{g^2}{m^2-\bar{s}_+} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \left(\frac{1}{s'-s} \mp \frac{1}{s'-\bar{s}_+}\right) B_1^{\pm}(s', 1). \tag{11}$$

$A_1^{\pm}$ and $B_1^{\pm}$ have been determined by Mandelstam[4] and coincide with the imaginary parts of $A^{\pm}$ and $B^{\pm}$ in the physical region of reaction I; $\bar{s}_+$ is defined as

$$\bar{s}_+ \equiv 2m^2 + 2 - s. \tag{12}$$

We can also calculate directly the dispersion relations for the backward scattering. The positions of the singularities and of the cuts for this case are shown in Fig. 2. From this follow directly the dispersion relations (the pole terms are separated, the integration contour is chosen as shown in Fig. 3):

$$\begin{aligned} A^{\pm}(s, \cos\Phi = -1) &= \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{A_1^{\pm}(s', -1)}{s'-s} - \frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \frac{A_2^{\pm}(s', -1)}{s'-s} \\ &\quad - \frac{1}{\pi}\left\{\int_{g_+} + \int_{g_-} - \int_{-m^2+1}^{0} + \int_{-\infty}^{-m^2+1}\right\} ds' \frac{A_3^{\pm}(s', -1)}{s'-s}, \\ B^{\pm}(s, -1) &= \frac{g^2}{m^2-s} \mp \frac{g^2}{m^2-\bar{s}_-} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{B_1^{\pm}(s', -1)}{s'-s} \\ &\quad - \frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \frac{B_2^{\pm}(s', -1)}{s'-s} - \frac{1}{\pi}\left\{\int_{g_+} + \int_{g_-} - \int_{-m^2+1}^{0} \right. \\ &\quad \left. + \int_{-\infty}^{-m^2+1}\right\} ds' \frac{B_3^{\mp}(s', -1)}{s'-s}. \end{aligned} \tag{13}$$

$A_2^{\pm}$, $B_2^{\pm}$ and $A_3^{\pm}$, $B_3^{\pm}$ are also given in Mandelstam's paper. They coincide with the imaginary parts of

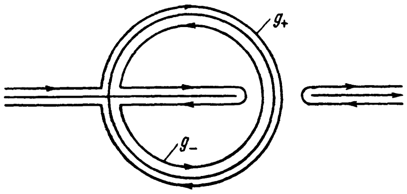

FIG. 3

$A^{\pm}$ and $B^{\pm}$ in the physical region of reaction II and III respectively. The contour g_+ is a semicircle in the upper half plane and is followed clockwise, while the contour g_- is a semicircle in the lower half plane and is followed counterclockwise (Fig. 3). $\bar{s}_-$ is defined as

$$\bar{s}_- \equiv (m^2-1)^2/s. \tag{14}$$

The signs preceding $A_2^{\pm}$, $B_2^{\pm}$ and $A_3^{\pm}$, $B_3^{\pm}$, which appear in the integrals, must be determined with allowance for the signs of the small imaginary parts of $\bar{s}$ and t in the dispersion relations (5). Let us illustrate the determination of the sign proceeding $A_3^{\mp}$ in the integral over the contour g_+. For backward scattering $t = -4k^2$. On the semicircle in the upper half plane, s has the form

$$s = \rho e^{i\varphi}, \quad 0 \leqslant \varphi \leqslant \pi. \tag{15}$$

It is readily shown that

$$\operatorname{Im} t = \frac{1}{\rho}\{(m^2-1)^2-\rho^2\}\sin\varphi. \tag{16}$$

Thus, t has a negative imaginary part outside the semicircle in the upper half plane and a positive imaginary part inside the semicircle of the upper half plane. The sign in front of $A_3^{\mp}$ in the integral over g_+ should be negative, in contradiction to the result obtained by MacDowell.[12]

The dispersion relations (13) can be recast in a form similar to (11). The integrals in (13) are actually taken along the curve $\bar{s}s = (m^2-1)^2$. In some of these it is convenient to make a change of variables

$$\bar{s} = (m^2-1)^2/s. \tag{17}$$

In particular

$$\begin{aligned} -\frac{1}{\pi}\int_0^{(m-1)^2} ds' \frac{A_2^{\pm}(s',-1)}{s'-s} &= \frac{1}{\pi}\int_{(m+1)^2}^{\infty} d\bar{s}' \frac{\bar{s}_-}{\bar{s}'}\frac{A_2^{\pm}(\bar{s}',-1)}{\bar{s}'-\bar{s}_-}, \\ \frac{1}{\pi}\int_{-m^2+1}^{0} ds' \frac{A_3^{\pm}(s',-1)}{s'-s} &= -\frac{1}{\pi}\int_{-\infty}^{-m^2+1} d\bar{s}' \frac{\bar{s}_-}{\bar{s}'}\frac{A_3^{\pm}(\bar{s}',-1)}{\bar{s}'-\bar{s}_-}. \end{aligned} \tag{18}$$

Applying relations (4) to (18), we obtain

$$\begin{aligned} -\frac{1}{\pi}\int_0^{(m-1)^2} ds' \frac{A_2^{\pm}(s',-1)}{s'-s} &= \pm\frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{\bar{s}_-}{s'}\frac{A_1^{\pm}(s',-1)}{s'-\bar{s}_-}, \\ \frac{1}{\pi}\int_{-m^2+1}^{0} ds' \frac{A_3^{\pm}(s',-1)}{s'-s} &= \mp\frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds' \frac{\bar{s}_-}{s'}\frac{A_3^{\mp}(s',-1)}{s'-\bar{s}_-}. \end{aligned} \tag{19}$$

In a similar manner we obtain

$$\begin{aligned} -\frac{1}{\pi}\int_0^{(m-1)^2} ds' \frac{B_2^{\pm}(s',-1)}{s'-s} &= \mp\frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{\bar{s}_-}{s'}\frac{B_1^{\pm}(s',-1)}{s'-\bar{s}_-}, \\ \frac{1}{\pi}\int_{-m^2+1}^{0} ds' \frac{B_3^{\pm}(s',-1)}{s'-s} &= \pm\frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds' \frac{\bar{s}_-}{s'}\frac{B_3^{\mp}(s',-1)}{s'-\bar{s}_-}. \end{aligned} \tag{20}$$

We recognize further that the integral over g_+ is the complex conjugate of the integral over g_- for physical values of s. Therefore (13) can be reduced to the following form:

$$\begin{aligned} A^{\pm}(s,-1) &= \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds'\left(\frac{1}{s'-s} \pm \frac{\bar{s}_-}{s'}\frac{1}{s'-\bar{s}_-}\right)A_1^{\pm}(s',-1) \\ &- \frac{2}{\pi}\operatorname{Re}\int_{g_-} ds' \frac{A_3^{\mp}(s',-1)}{s'-s} - \frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds'\left(\frac{1}{s'-s}\right. \\ &\left.\pm\frac{\bar{s}_-}{s'}\frac{1}{s'-\bar{s}_-}\right)A_3^{\mp}(s',-1), \\ B^{\pm}(s,-1) &= \frac{g^2}{m^2-s} \mp \frac{g^2}{m^2-\bar{s}_-} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds'\left(\frac{1}{s'-s}\right. \\ &\left.\mp\frac{\bar{s}_-}{s'}\frac{1}{s'-\bar{s}_-}\right)B_1^{\pm}(s',-1) - \frac{2}{\pi}\operatorname{Re}\int_{g_-} ds' \frac{B_3^{\mp}(s',-1)}{s'-s} \\ &- \frac{1}{\pi}\int_{-\infty}^{-m^2+1} ds'\left(\frac{1}{s'-s} \mp \frac{\bar{s}_-}{s'}\frac{1}{s'-\bar{s}_-}\right)B_3^{\mp}(s',-1). \end{aligned} \tag{21}$$

If we can neglect the D wave and the higher waves, then the amplitudes of the S and P waves, $f^{\pm}_{S_{1/2}}(s)$, $f^{\pm}_{P_{1/2}}(s)$, $f^{\pm}_{P_{3/2}}(s)$ can be readily expressed in terms of the forward and backward scattering amplitudes

$$\begin{aligned} f^{\pm}_{S_{1/2}}(s) &\approx \frac{1}{2}\{f_1^{\pm}(s,1)+f_1^{\pm}(s,-1)\}, \\ f^{\pm}_{P_{1/2}}(s) - f^{\pm}_{P_{3/2}}(s) &\approx \frac{1}{2}\{f_2^{\pm}(s,1)+f_2^{\pm}(s,-1)\}, \\ f^{\pm}_{P_{3/2}}(s) &\approx \frac{1}{6}\{f_1^{\pm}(s,1)-f_1^{\pm}(s,-1)\}. \end{aligned} \tag{22}$$

The expressions for $f_1^{\pm}$ and $f_2^{\pm}$ have been obtained by Chew, Goldberger, Low, and Nambu (CGLN)[10] and have the following form

$$\begin{aligned} f_1^{\pm}(s,x) &= \frac{E+m}{8\pi\sqrt{s}}\{A^{\pm}(s,x)+(\sqrt{s}-m)B^{\pm}(s,x)\}, \\ f_2^{\pm}(s,x) &= \frac{E-m}{8\pi\sqrt{s}}\{-A^{\pm}(s,x)+(\sqrt{s}+m)B^{\pm}(s,x)\}. \end{aligned} \tag{23}$$

Here E is the nucleon energy in the c.m.s.

4. INTEGRAL EQUATIONS

The problem now consists of expressing $A_1^{\pm}$ and $B_1^{\pm}$ in terms of the imaginary parts of the partial amplitudes of the πN scattering, and expressing $A_3^{\mp}$ and $B_3^{\mp}$ in terms of the imaginary parts of the $N\bar{N}$-annihilation amplitudes. The expressions for $A_1^{\pm}$ and $B_1^{\pm}$ can be obtained directly from (22) and (23):

$$\frac{1}{4\pi}A_1^{\pm}(s,\ 1)\approx\frac{\sqrt{s}+m}{E+m}\,\mathrm{Im}\left\{f^{\pm}_{s_{1/2}}(s)+3f^{\pm}_{p_{3/2}}(s)\right\}$$
$$-\frac{\sqrt{s}-m}{E-m}\,\mathrm{Im}\left\{f^{\pm}_{p_{1/2}}(s)-f^{\pm}_{p_{3/2}}(s)\right\},$$
$$\frac{1}{4\pi}A_1^{\pm}(s,\ -1)\approx\frac{\sqrt{s}+m}{E+m}\,\mathrm{Im}\left\{f^{\pm}_{s_{1/2}}(s)-3f^{\pm}_{p_{3/2}}(s)\right\}$$
$$-\frac{\sqrt{s}-m}{E-m}\,\mathrm{Im}\left\{f^{\pm}_{p_{1/2}}(s)-f^{\pm}_{p_{3/2}}(s)\right\},$$
$$\frac{1}{4\pi}B_1^{\pm}(s,\ 1)\approx\frac{1}{E+m}\,\mathrm{Im}\left\{f^{\pm}_{s_{1/2}}(s)+3f^{\pm}_{p_{3/2}}(s)\right\}$$
$$+\frac{1}{E-m}\,\mathrm{Im}\left\{f^{\pm}_{p_{1/2}}(s)-f^{\pm}_{p_{3/2}}(s)\right\},$$
$$\frac{1}{4\pi}B_1^{\pm}(s,\ -1)\approx\frac{1}{E+m}\,\mathrm{Im}\,\{f^{\pm}_{s_{1/2}}(s)-3f^{\pm}_{p_{3/2}}(s)\}$$
$$+\frac{1}{E-m}\,\mathrm{Im}\,\{f^{\pm}_{p_{1/2}}(s)-f^{\pm}_{p_{3/2}}(s)\}. \tag{24}$$

In order to express $A_3^{\pm}$ and $B_3^{\pm}$ in terms of the scattering amplitudes of reaction III, it is first necessary to obtain a relation between the values of s and $x \equiv \cos\Phi$ for reaction I, and between t and $z \equiv \cos\theta$ for reaction III. It follows from (6) that for backward scattering

$$t=2\,(m^2+1)-s-(m^2-1)^2\,s^{-1}. \tag{25}$$

It can be shown that on the integration contours in (21)

$$z=(s-\bar{s})/4pq=-1. \tag{26}$$

The $N\bar{N}$ annihilation reaction was investigated by Fraser and Fulco.[2] It is easy to express $A_3^{\pm}$ and $B_3^{\pm}$ in terms of the partial wave amplitudes $f_{\pm}^{I,J}$ of $N\bar{N}$ annihilation with a definite helicity, where I and J denote respectively the isotopic spin and the total angular momentum of the NN system. The signs + and − pertain to two helicity states. Taking into account the unitarity condition for reaction III, we obtain after simple calculations

$$A_3^{+}(s,\ -1)=\frac{8\pi}{\sqrt{6}}\sum_{J-\mathrm{even}}\left(J+\frac{1}{2}\right)\frac{(pq)^J}{p^2}\left\{\frac{m}{2}\sqrt{J(J+1)}\right.$$
$$\left.\times\mathrm{Im}\,f_-^{0,J}(t)-\mathrm{Im}\,f_+^{0,J}(t)\right\},$$
$$A_3^{-}(s,\ -1)=4\pi\sum_{J-\mathrm{odd}}\left(J+\frac{1}{2}\right)\frac{(pq)^J}{p^2}$$
$$\times\left\{-\frac{m}{2}\sqrt{J(J+1)}\,\mathrm{Im}\,f_-^{1,J}(t)+\mathrm{Im}\,f_+^{1,J}(t)\right\},$$
$$B_3^{+}(s,\ -1)=-\frac{4\pi}{\sqrt{6}}\sum_{J-\mathrm{even}}\left(J+\frac{1}{2}\right)\sqrt{J(J+1)}$$
$$\times(pq)^{J-1}\mathrm{Im}\,f_-^{0,J}(t),$$
$$B_3^{-}(s,\ -1)=2\pi\sum_{J-\mathrm{odd}}\left(J+\frac{1}{2}\right)\sqrt{J(J+1)}\,(pq)^{J-1}\mathrm{Im}f_-^{1,J}(t). \tag{27}$$

We have here on the contour $g_-\ (0\le\varphi\le\pi)$

$$s=(m^2-1)\,e^{-i\varphi},\quad t=2\,(m^2+1)-2\,(m^2-1)\cos\varphi,$$
$$pq=\tfrac{1}{2}i(m^2-1)\sin\varphi, \tag{28}$$

and for $-\infty<s\le-m^2+1$

$$t=2\,(m^2+1)-s-(m^2-1)^2s^{-1},$$
$$pq=\tfrac{1}{4}\{(m^2-1)^2s^{-1}-s\}. \tag{29}$$

It is interesting to note that on the contour g_- the quantities A_3^{+} and B_3^{-} are real, and A_3^{-} and B_3^{+} are pure imaginary.

The first few terms of the expansion (27) may be sufficient to represent $A_3^{\pm}$ and $B_3^{\pm}$ adequately for values of t not much greater than 4. This region makes an appreciable contribution to the dispersion relations. It pertains, however, to the nonphysical region of reaction III, which is not amenable to experiment. The actual number of terms that should be retained should be established during the process of solving the integral equation.

Introducing the notation

$$\omega\equiv\sqrt{s}-m,\quad K_{\pm}(s',\ s)=\frac{1}{s'-s}\pm\frac{\bar{s}_-}{s'}\,\frac{1}{s'-\bar{s}_-}, \tag{30}$$

we obtain from (11) and (21) — (24) equations which, in conjunction with the unitarity condition, yield integral equations for πN scattering. For the amplitude $f^{\pm}_{s_{1/2}}(s)$ we have

$$f^{\pm}_{s_{1/2}}(s)=P^{\pm}_{s_{1/2}}(s)+\mathrm{I}^{\pm}_{s_{1/2}}(s)+\mathrm{II}^{\pm}_{s_{1/2}}(s)+\mathrm{III}^{\pm}_{s_{1/2}}(s);$$

$$P^{\pm}_{s_{1/2}}(s)=\frac{\omega\,(E+m)\,g^2}{16\pi\sqrt{s}}\left\{\frac{2}{m^2-s}\mp\left(\frac{1}{m^2-\bar{s}_+}+\frac{1}{m^2-\bar{s}_-}\right)\right\},$$

$$\mathrm{I}^{\pm}_{s_{1/2}}(s)=\frac{1}{\pi}\int_{(m+1)^2}^{\infty}\frac{ds'}{s'-s}\left\{\frac{E+m}{E'+m}\,\frac{\sqrt{s'}+\sqrt{s}}{2\sqrt{s}}\,\mathrm{Im}\,f^{\pm}_{s_{1/2}}(s')\right.$$
$$\left.+\frac{E+m}{2\sqrt{s}}\,\frac{\sqrt{s}-\sqrt{s'}}{E'-m}\,[\mathrm{Im}\,f^{\pm}_{p_{1/2}}(s')-\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s')]\right\},$$

$$\mathrm{II}^{\pm}_{s_{1/2}}(s)=\pm\frac{E+m}{4\pi\sqrt{s}}\int_{(m+1)^2}^{\infty}ds'\left\{3K_-(s',\ s)\,\frac{2m+\omega'-\omega}{E'+m}\,\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s')\right.$$
$$+K_+(s',\ s)\left[\frac{2m+\omega'-\omega}{E'+m}\,\mathrm{Im}\,f^{\pm}_{s_{1/2}}(s')-\frac{\omega'+\omega}{E'-m}\mathrm{Im}\,(f^{\pm}_{p_{1/2}}(s')\right.$$
$$\left.\left.-f^{\pm}_{p_{3/2}}(s'))\right]\right\},$$

$$\mathrm{III}^{\pm}_{s_{1/2}}(s)=-\frac{E+m}{16\pi\sqrt{s}}\,\frac{1}{\pi}\left\{\int_{-\infty}^{-m^2+1}ds'\,[K_{\pm}(s',\ s)\,A_3^{\pm}(s',\ -1)+\right.$$
$$+\,\omega K_{\mp}(s',\ s)\,B_3^{\pm}(s,\ -1)]\ +2\,\mathrm{Re}\int_{g_-}\frac{ds'}{s'-s}\,[A_3^{\pm}(s',\ -1)$$
$$\left.+\,\omega B_3^{\pm}(s',\ -1)]\right\}. \tag{31}$$

The quantities $A_3^{\pm}$ and $B_3^{\pm}$ in the equation for $\mathrm{III}^{\pm}_{s_{1/2}}(s)$ should be replaced by the expressions in (27). The corresponding equations for $f^{\pm}_{p_{1/2}}(s)-f^{\pm}_{p_{3/2}}(s)$ have the form

$$f^{\pm}_{p_{1/2}}(s) - f^{\pm}_{p_{3/2}}(s) = P^{\pm}_{p_{1/2},-3/2}(s) + \mathrm{I}^{\pm}_{p_{1/2},-3/2}(s) + \mathrm{II}^{\pm}_{p_{1/2},-3/2}(s) + \mathrm{III}^{\pm}_{p_{1/2},-3/2}(s);$$

$$P^{\pm}_{p_{1/2},-3/2}(s) = \frac{(\sqrt{s}+m)(E-m)g^2}{16\pi\sqrt{s}}\left\{\frac{2}{m^2-s} \mp \left(\frac{1}{m^2-\bar{s}_+} + \frac{1}{m^2-\bar{s}_-}\right)\right\},$$

$$\mathrm{I}^{\pm}_{p_{1/2},-3/2}(s) = \frac{1}{\pi}\int\limits_{(m+1)^2}^{\infty}\frac{ds'}{s'-s}\left\{\frac{E-m}{E'-m}\frac{\sqrt{s'}+\sqrt{s}}{2\sqrt{s}}\,\mathrm{Im}\,[f^{\pm}_{p_{1/2}}(s') - f^{\pm}_{p_{3/2}}(s')] + \frac{E-m}{E'+m}\frac{\sqrt{s}-\sqrt{s'}}{2\sqrt{s}}\,\mathrm{Im}\,f^{\pm}_{s_{1/2}}(s')\right\},$$

$$\mathrm{II}^{\pm}_{p_{1/2},-3/2}(s) = \mp\frac{E-m}{4\pi\sqrt{s}}\int\limits_{(m+1)^2}^{\infty}ds'\left\{3K_-(s',s)\frac{2m+\sqrt{s'}+\sqrt{s}}{E'+m} \times \mathrm{Im}\,f^{\pm}_{p_{3/2}}(s') + K_+(s',s)\left[\frac{2m+\sqrt{s'}+\sqrt{s}}{E'+m}\,\mathrm{Im}\,f^{\pm}_{s_{1/2}}(s') + \frac{2m+\omega-\omega'}{E'-m}\,\mathrm{Im}\,(f^{\pm}_{p_{1/2}}(s') - f^{\pm}_{p_{3/2}}(s'))\right]\right\},$$

$$\mathrm{III}^{\pm}_{p_{1/2},-3/2}(s) = -\frac{E-m}{16\pi\sqrt{s}}\frac{1}{\pi}\left\{\int\limits_{-\infty}^{-m^2+1}ds'\,[-K_{\pm}(s',s)A_3^{\pm}(s',-1) + (\sqrt{s}+m)K_{\mp}(s',s)B_3^{\pm}(s',-1)] + 2\mathrm{Re}\int\limits_{g_-}\frac{ds'}{s'-s}[-A_3^{\pm}(s',-1) + (\sqrt{s}+m)B_3^{\pm}(s',-1)]\right\}. \tag{32}$$

The equations for the amplitude $f^{\pm}_{p_{3/2}}(s)$ have the form

$$f^{\pm}_{p_{3/2}}(s) = P^{\pm}_{p_{3/2}}(s) + \mathrm{I}^{\pm}_{p_{3/2}}(s) + \mathrm{II}^{\pm}_{p_{3/2}}(s) + \mathrm{III}^{\pm}_{p_{3/2}}(s),$$

$$P^{\pm}_{p_{3/2}}(s) = \mp\frac{\omega(E+m)}{48\pi\sqrt{s}}g^2\left(\frac{1}{m^2-\bar{s}_+} - \frac{1}{m^2-\bar{s}_-}\right),$$

$$\mathrm{I}^{\pm}_{p_{3/2}}(s) = \frac{1}{\pi}\int\limits_{(m+1)^2}^{\infty}\frac{ds'}{s'-s}\frac{E+m}{E'+m}\frac{\sqrt{s'}+\sqrt{s}}{2\sqrt{s}}\,\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s'),$$

$$\mathrm{II}^{\pm}_{p_{3/2}}(s) = \pm\frac{E+m}{12\pi\sqrt{s}}\int\limits_{(m+1)^2}^{\infty}ds'\left\{3K_+(s',s)\frac{2m+\omega'-\omega}{E'+m}\,\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s') + K_-(s',s)\left[\frac{2m+\omega'-\omega}{E'+m}\,\mathrm{Im}\,f^{\pm}_{s_{1/2}}(s') - \frac{\omega'+\omega}{E'-m}\,\mathrm{Im}\,(f^{\pm}_{p_{1/2}}(s') - f^{\pm}_{p_{3/2}}(s'))\right]\right\},$$

$$\mathrm{III}^{\pm}_{p_{3/2}}(s) = -\tfrac{1}{3}\,\mathrm{III}^{\pm}_{s_{1/2}}(s). \tag{33}$$

Equations (31) — (33), in spite of their unwieldiness, are rather simple. Each of the amplitudes is the superposition of four terms: $\mathrm{P}^{\pm}$, $\mathrm{I}^{\pm}$, $\mathrm{II}^{\pm}$, and $\mathrm{III}^{\pm}$. The term $\mathrm{P}^{\pm}$ is the contribution from the poles, while $\mathrm{I}^{\pm}$, $\mathrm{II}^{\pm}$, and $\mathrm{III}^{\pm}$ are the contributions from reactions I, II, and III, respectively.

Since these equations contain the amplitudes of reaction III, they should be solved simultaneously with the integral equations for the $N\bar{N}$ annihilation. It is interesting to compare Eqs. (31) — (33) with the corresponding CGLN result.[10] The main difference lies undoubtedly in the fact that our equations contain the terms $\mathrm{III}^{\pm}_{s_{1/2}}$, $\mathrm{III}^{\pm}_{p_{1/2},-3/2}$, and $\mathrm{III}^{\pm}_{p_{3/2}}$, which are completely missing from their equations. These terms clearly represent the effects of the $\pi\pi$ interaction, since the Im $f^{I,J}_{\pm}$ vanish within the limits of the two-meson approximation for the unitarity condition if the $\pi\pi$ interaction vanishes.

To compare the pole contribution and the contributions from the regions of reaction I and II, we can neglect Im $f^{\pm}_{s_{1/2}}$ and Im $f^{\pm}_{p_{1/2}}$ compared with Im $f^{\pm}_{p_{3/2}}$ and discard terms of order $(\omega/m)^2$ compared with the principal term, as was done in the CGLN paper. We then have the following approximate expressions for $\mathrm{P}^{\pm}_{s_{1/2}}$, $\mathrm{I}^{\pm}_{s_{1/2}}$, and $\mathrm{II}^{\pm}_{s_{1/2}}$:

$$P^{\pm}_{s_{1/2}}(s) \approx -\frac{g^2}{4\pi}\frac{1}{2\sqrt{s}}\left\{\left(1-\frac{\omega}{2m}\right) \pm \left(1+\frac{\omega}{2m}\right)\right\},$$

$$\mathrm{I}^{\pm}_{s_{1/2}}(s) \approx \frac{2m^2}{\pi\sqrt{s}}\int_1^{\infty}\frac{d\omega'}{k'^2}\left(1+\frac{\omega'}{2m}-\frac{\omega}{2m}\right)\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s'), \tag{34}$$

$$\mathrm{II}^{\pm}_{s_{1/2}}(s) \approx \pm\frac{2m^2}{\pi\sqrt{s}}\int_1^{\infty}\frac{d\omega'}{k'^2}\left(1+\frac{\omega'}{2m}-\frac{\omega}{2m}\right)\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s').$$

The approximate equations for $\mathrm{P}^{\pm}_{p_{1/2},-3/2}$, $\mathrm{I}^{\pm}_{p_{1/2},-3/2}$, and $\mathrm{II}^{\pm}_{p_{1/2},-3/2}$ are

$$P^{\pm}_{p_{1/2},-3/2}(s) \approx -\frac{f^2k^2}{\omega}\left\{\left(1-\frac{\omega}{2m}\right) \pm \left(1+\frac{\omega}{2m}\right)\right\}\left(1-\frac{\omega}{2m}\right),$$

$$\mathrm{I}^{\pm}_{p_{1/2},-3/2}(s) \approx -\frac{1}{\pi}\int_1^{\infty}d\omega'\frac{k^2}{k'^2}\left(\frac{1}{\omega'-\omega}+\frac{1}{m}\right)\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s'), \tag{35}$$

$$\mathrm{II}^{\pm}_{p_{1/2},-3/2}(s) \approx \pm\frac{1}{\pi}\int_1^{\infty}d\omega'\frac{k^2}{k'^2}\left(\frac{1}{\omega'+\omega}-\frac{1}{m}\right)\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s'),$$

where f is the renormalized pseudo-vector coupling constant, defined as

$$f^2 \equiv g^2/16\pi m^2. \tag{36}$$

The approximate expressions for $\mathrm{P}^{\pm}_{p_{3/2}}$, $\mathrm{I}^{\pm}_{p_{3/2}}$, and $\mathrm{II}^{\pm}_{p_{3/2}}$ have the form

$$P^{\pm}_{p_{3/2}}(s) \approx \pm 2f^2k^2/3\omega,$$

$$\mathrm{I}^{\pm}_{p_{3/2}}(s) \approx \frac{1}{\pi}\int_1^{\infty}d\omega'\left(\frac{1}{\omega'-\omega}+\frac{1}{m}\right)\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s'),$$

$$\mathrm{II}^{\pm}_{p_{3/2}}(s) \approx \pm\frac{1}{\pi}\int_1^{\infty}\frac{d\omega'}{\omega+\omega'}\left\{1+\frac{2(\omega+\omega')^2}{3k'^2}\left[1-\frac{k^2}{(\omega+\omega')^2}\right] - \frac{2\omega}{m} - \frac{2(\omega+\omega')^3}{3mk'^2}\right\}\mathrm{Im}\,f^{\pm}_{p_{3/2}}(s'). \tag{37}$$

Thus, the pole contribution and $\mathrm{I}^{\pm}_{p_{1/2},-3/2}$ are exactly the same as given by CGLN, within the limits of the approximation made.

$\mathrm{I}^{\pm}_{s_{1/2}}$, $\mathrm{II}^{\pm}_{s_{1/2}}$, and $\mathrm{II}^{\pm}_{p_{1/2},-3/2}$ differ from the corresponding CGLN expression in terms of order ω/m. But $\mathrm{I}^{\pm}_{p_{3/2}}$ and $\mathrm{II}^{\pm}_{p_{3/2}}$ are different from each other. In fact, no subtraction was made in our Eq. (33). If the subtraction is made at $\omega = 1$, then the term $\mathrm{I}^{\pm}_{p_{3/2}}$ becomes analogous to their results, but $\mathrm{II}^{\pm}_{p_{3/2}}$ remains different. This probably will lead to some changes in the behavior of the (3, 3) resonance.

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