EXACT SOLUTION OF THE BASIC CASCADE THEORY EQUATIONS

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Exact numerical solutions of the basic cascade theory equations, obtained by the functional transformation method, are given for small depths (up to two radiation lengths).

It is well known (see reference 1) that the initial equations of the cascade theory can be written in the form

$$\frac{\partial P(t,E)}{\partial t} = 2\int_E^\infty \Gamma(t,E')\,W_p(E',E)\,dE' + \int_E^\infty P(t,E')\,W_e(E',E'-E)\,dE' - \int_0^E P(t,E)\,W_e(E,E')\,dE', \qquad (1a)$$

$$\frac{\partial \Gamma(t,E)}{\partial t} = \int_E^\infty P(t,E')\,W_e(E',E)\,dE' - \int_0^E \Gamma(t,E)\,W_p(E,E')\,dE'. \qquad (1b)$$

Here t is the depth in radiation lengths, $P(t, E)\,dE$ is the total number of electrons and positrons in the energy interval $(E, E + dE)$ at a depth t; $\Gamma(t, E)$ is the analogous distribution function for the photons, $W_p(E'\ E)\,dE$ is the probability per radiation length of creation of a pair with a positron energy in the interval $(E, E + dE)$ from a photon of energy E' and $W_e(E, E')\,dE'$ is the probability of an electron or positron of energy E emitting a bremsstrahlung quantum in the interval $(E', E' + dE')$, also per radiation length.

The functions W_e and W_p have in the Born approximation, with allowance for the nuclear field only in the ultrarelativistic limit for large collision parameters which are of importance in this case, the following form[2]

$$W_e(E,E')\,dE' = [1 + (1 - E'/E)^2 - \tfrac{2}{3}(1 - E'/E)]\,dE', \qquad (2a)$$

$$W_p(E',E)\,dE = [(E/E')^2 + (1 - E/E')^2 + \tfrac{2}{3}(E/E')(1 - E/E')]\,dE/E'. \qquad (2b)$$

In writing down the system (1) we have disregarded the Compton effect and the photoeffect for photons, the Rutherford scattering for positrons and electrons, pair annihilation, and other processes which are not significant at high energies. The pair production and bremsstrahlung in the field of the atomic electrons is disregarded in (2). The contribution of the atomic electrons does not exceed 1% of all the effects in the case of heavy elements.

To solve the system of integro-differential equations (1), it is customary to use either the method of functional transformations or the method of moments (see reference 3). The latter is approximate in nature and has limited application. As regard the method of functional transformations, it does permit, in principle, to obtain all the quantities of interest, but the resultant expressions can be calculated analytically only when $t > 1$. In the present paper we obtain exact numerical values for the system (1) in the range of thicknesses up to $t = 2$.

As was shown, for example, in the review by Belen'kiĭ and Ivanenko,[3] the solution of system (1) with the aid of the Laplace-Mellin transformation, subject to the boundary conditions

$$P(t = 0, E) = \delta(E - E_0), \qquad \Gamma(t = 0, E) = 0$$

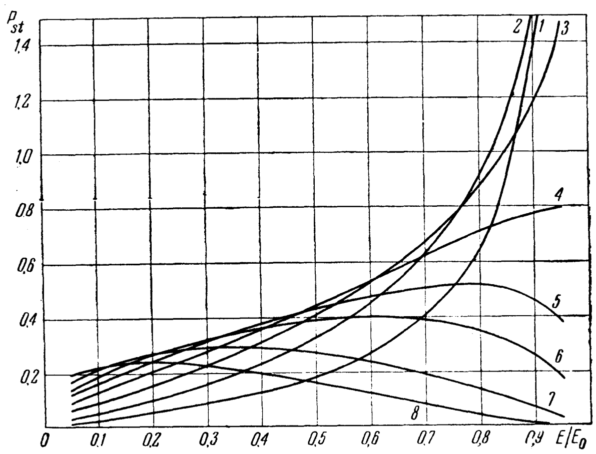

FIG. 1. The P'_{st} curves correspond to the following values of t: 1 – 0.2, 2 – 0.4, 3 – 0.6, 4 – 0.8, 5 – 1.0, 6 – 1.2, 7 – 1.6, 8 – 2.0.

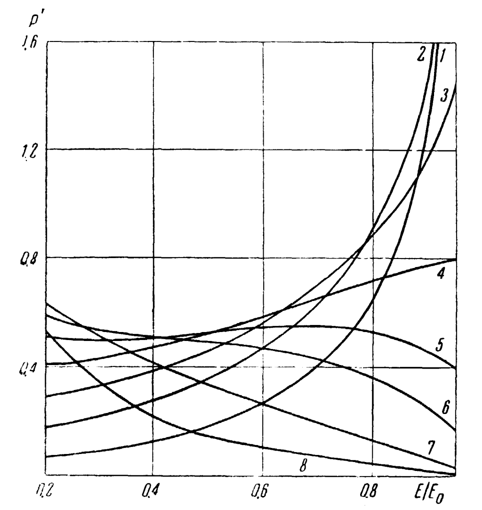

FIG. 2. The P′ curves correspond to the same values of t as in Fig. 1.

has the form

$$P(t, E) = \frac{1}{2\pi i E}\int_{\delta - i\infty}^{\delta + i\infty}\left(\frac{E_0}{E}\right)^s\left[\frac{\sigma+\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_1 t} - \frac{\sigma+\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_2 t}\right]ds,$$

$$\Gamma(t, E) = \frac{1}{2\pi i E}\int_{\delta - i\infty}^{\delta + i\infty}\left(\frac{E_0}{E}\right)^s\frac{C(s)}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)ds \qquad (3)$$

and explicit expressions for the functions $\lambda_1(s)$, $\lambda_2(s)$, and $C(s)$ are given, for example, in the book by Belen'kiĭ.[1]

Here σ is the coefficient of γ-quantum absorption. As usual, it is assumed that σ is equal to its ultrarelativistic limit, $\sigma = 0.773$. The integration in (3) is in the complex plane along a straight line parallel to the imaginary axis, subject to the condition $\delta > 0$.

Cascade theory deals also with the quantity $N(t, E)$, i.e., the total number of electrons and positrons at a depth t with energy greater than E:

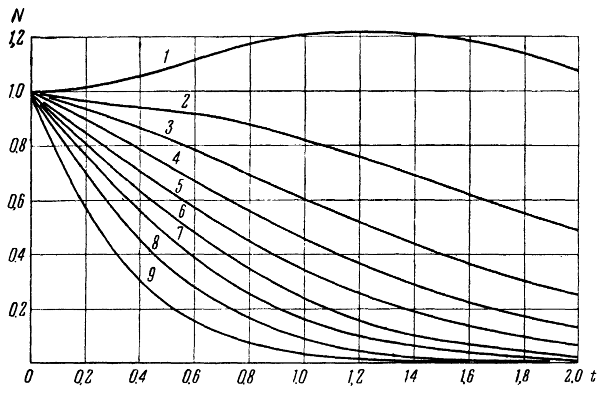

FIG. 3. The N curves correspond to the following values of E/E_0: 1 – 0.1, 2 – 0.2, 3 – 0.3, 4 – 0.4, 5 – 0.5, 6 – 0.6, 7 – 0.7, 8 – 0.8, 9 – 0.9.

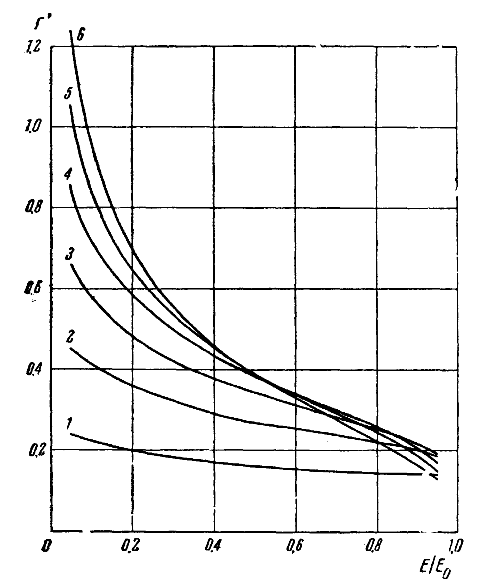

FIG. 4. The Γ′ curves correspond to the following values of t: 1 – 0.2, 2 – 0.4, 3 – 0.6, 4 – 0.8, 5 – 1.0, 6 – 1.2.

$$N(t, E) = \int_E^{\infty} P(t, E)\,dE = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty}\left(\frac{E_0}{E}\right)^s\left[\frac{\sigma+\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_1 t} - \frac{\sigma+\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_2 t}\right]\frac{ds}{s}. \qquad (4)$$

The energy distribution of the initial electrons after passing through a layer of thickness t is also of interest. It is found that the well known Bethe-Heitler expression[2] for the probability that an electron with initial energy E_0 will have an energy between E and E + dE after passing through a thickness t

$$P_{st}^{B-\Gamma}(t, E)\,dE = \frac{dE}{E_0}\left(\ln\frac{E_0}{E}\right)^{t/\ln 2 - 1} \Big/ \Gamma\left(\frac{t}{\ln 2}\right), \qquad (5)$$

is a rather rough approximation.

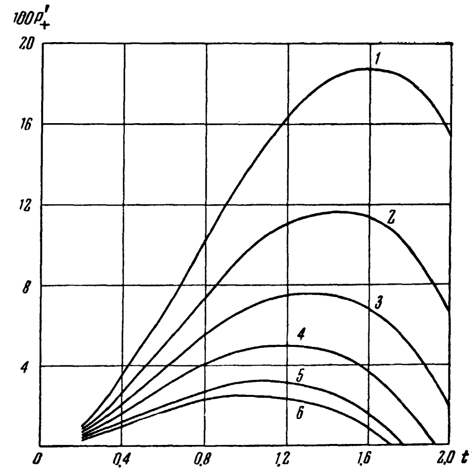

FIG. 5. The P'_+ curves correspond to the following values of E/E_0: 1 – 0.2, 2 – 0.3, 3 – 0.4, 4 – 0.5, 5 – 0.6, 6 – 0.7.

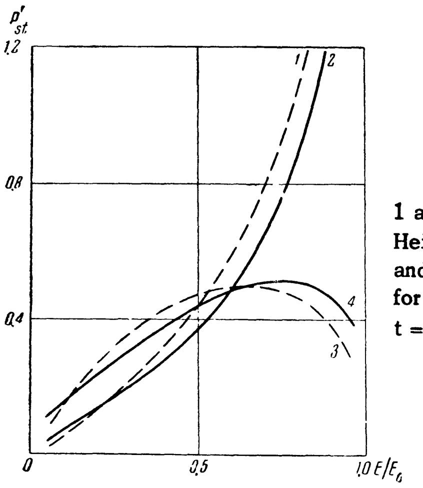

FIG. 6. Curves P'_{st}: 1 and 3 – from the Bethe-Heitler formula (5), 2 and 4 – from (7). t = 0.5 for curves 1 and 2, and t = 1 for curves 3 and 4.

In order to obtain an equation for P_{st} within the framework of the cascade theory, it is evidently sufficient to retain in the right half of (1a) only the terms that describe the radiation retardation

$$\frac{\partial P_{st}(t, E)}{\partial t} = \int_E^\infty P_{st}(t, E) W_e(E', E' - E)\, dE' - \int_0^E P_{st}(t, E) W_e(E, E')\, dE', \qquad (6)$$

from which we readily obtain with the aid of the Laplace-Mellin transformation

$$P_{st}(t, E) = \frac{1}{2\pi i E} \int_{\delta - i\infty}^{\delta + i\infty} \left(\frac{E_0}{E}\right)^s e^{-A(s)t}\, ds. \qquad (7)$$

(for an explicit expression for A(s) see reference 1.)

From (3) and (7) we readily obtain the energy distribution function P_+ of the secondary electrons (positrons) at a depth t:

$$P_+(t, E_+) = \tfrac{1}{2}[P(t, E_+) - P_{st}(t, E_+)]. \qquad (8)$$

The expressions given for P, P_{st}, P_+, N, and Γ have been calculated with an electronic computer accurate to ~1%. The results are shown in Figs. 1 — 6. The plots are given for the "primed" quantities, which are more convenient for calculation. The latter are defined, for example, as $P(t, E) = P'(t, E)\,dE/E$, etc. The energy is measured everywhere in units of the initial energy E_0.

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[1] S. Z. Belen'kiĭ, Лавинные процессы в космических лучах (Cascade Processes in Cosmic Rays), Gostekhizdat, 1948.

[2] W. Heitler, The Quantum Theory of Radiation, Oxford, 1954.

[3] S. Z. Belen'kiĭ and I. P. Ivanenko, Usp. Fiz. Nauk **69**, 591 (1959), Soviet Phys. Uspekhi **2**, 912 (1960).

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