ON THE AZIMUTHAL ANGULAR DISTRIBUTION OF SHOWER PARTICLES

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It is shown that the χ^2 criterion can be employed to investigate the azimuthal angular distribution of a small number of shower particles. The azimuthal angular distribution of secondary particles in jets produced in photographic emulsions by cosmic-ray particles is analyzed by the proposed method. It is concluded that peripheral collisions of high-energy nucleons play an important role.

A possible consequence of the two-center model of multiple particle production in collisions of fast nucleons is the azimuthal anisotropy of secondary particles. If the excited center has a large intrinsic moment of momentum, then the emitted mesons have a tendency to be coplanar.[1,2] In addition, because of the deflection of the emitting centers from the direction of the primary particles,[3,4] an asymmetry can arise in the azimuthal angular distribution of the particles of the diffuse and narrow cones observed in the laboratory frame of reference (l system). Koba and Takagi[2] believe that the detection of an azimuthal asymmetry of shower particles exceeding the limits of simple statistical fluctuations would make the two-center model a very plausible one. The problem is all the more important since the distribution over the angle θ (the angle with the direction of the primary particle) characteristic of the two-center model is often of the order of natural statistical fluctuations.[5]

The azimuthal angular distribution of fast particles produced in the collisions of nucleons with complex nuclei was studied using the Pearson criterion.[6] However, the existence of azimuthal anisotropy has not been proved. It should be noted that the use of the Pearson criterion is mathematically sound only for a sufficiently large number of shower particles n_s,[7] not less than 10 m, where m is the number of equal intervals into which we divide the azimuthal angle φ. It is shown below that it is possible to use the χ^2 criterion to detect the azimuthal anisotropy of secondary particles in the case of small n_s if the small number of particles in each shower can be compensated for by a large number of showers.

In the study of the particle angular distribution, the total azimuthal angle 2π is divided into m equal intervals $\Delta\varphi$, and the number of particles n_k is calculated in each interval $(\sum_{k=1}^{m} n_k = n_s)$.[6] As is well known, the quantity

$$\chi_m^2 = \frac{m}{n_s}\sum_{k=1}^{m}\left(n_k - \frac{n_s}{m}\right)^2 = \frac{m}{n_s}\sum_{k=1}^{m} n_k^2 - n_s \tag{1}$$

has a χ^2 distribution with m − 1 degrees of freedom, if n_s is large, under the assumptions of azimuthal isotropy of shower particles and the statistical independence of their angles of emission φ_i in separate showers with a given n_s. Let us estimate the mathematical expectation and the dispersion of this quantity for an arbitrary n_s. We ascribe to each azimuthal angle of the i-th particle φ_i m random values φ_{ik} (i = 1, 2, ..., n_s; k = 1, 2, ..., m) satisfying the conditions

$$\varphi_{ik} = 1, \tag{2}$$

if φ_i lies within the k-th among the m equal intervals $\Delta\varphi$, and

$$\varphi_{ik} = 0, \tag{3}$$

if φ_i does not lie inside this interval. For such a definition of φ_{ik}, we have

$$n_k = \sum_{i=1}^{n_s} \varphi_{ik}. \tag{4}$$

Taking the statistical independence of the angles φ_i into account, it is easy to find the mathematical expectation of the normalized value $\alpha_m = \chi_m^2/(m-1)$:

$$M(\alpha_m) = 1 + \frac{m}{m-1}(n_s - 1)\sum_{k=1}^{m}\left(p_k - \frac{1}{m}\right)^2, \tag{5}$$

where p_k is the probability that a particle falls within the k-th interval $\Delta\varphi$, which fully determines the distribution of φ_{ik}. If the azimuthal angular distribution is isotropic, $p_k = 1/m$ and

$$M(\alpha_m) = 1. \tag{6}$$

The calculation of the dispersion α_m is rather tedious. Omitting the calculation, we shall present the final result corresponding to the assumption of azimuthal isotropy of shower particles and statistical independence of their angles φ_i:

$$D(\alpha_m) = \frac{2}{m-1} - \frac{1}{n_s}\frac{2}{m-1}. \tag{7}$$

The fact that the value of the dispersion is finite makes it possible to find the critical limit for the quantity α_m averaged over n showers

$$\bar{\alpha}_m = \frac{1}{n}\sum_{i=1}^{n}\alpha_{mi}, \tag{8}$$

The probability that this limit is exceeded is very small. Denoting the probability of the corresponding event by P, we can write the Chebyshev inequality from Eqs. (6) and (7) as

$$P(|\bar{\alpha}_m - 1| \geqslant t\sqrt{\bar{D}_m}/\sqrt{n}) \leqslant 1/t^2,$$
$$\bar{D}_m = \frac{2}{m-1} - \left(\frac{1}{n_s}\right)_{av}\frac{2}{m-1}, \tag{9}$$

where t is an arbitrary positive parameter. The more accurate Chebyshev inequality[8] makes it possible to state that*

$$P(\bar{\alpha}_m - 1 \geqslant t\sqrt{\bar{D}_m}/\sqrt{n}) < \exp(-t^2/4), \tag{10}$$

if

$$0 < t \leqslant 3\sqrt{\bar{D}_m}\sqrt{n}/[(n_s)_{max} - 1]. \tag{11}$$

The Chebyshev inequality clearly results in a considerably greater value than the critical limit $\bar{\alpha}_m$. In order to lower the estimate, we can use the Lyapunov theorem, whose limits of applicability are satisfied for a set of showers with a finite number of particles n_s, e.g., for showers with energy smaller than a certain maximum value. Since, for a sufficiently large number of such showers, the distribution of the quantity $\bar{\alpha}_m$ can be regarded as normal, we can, for instance, claim from Eqs. (6) and (7) that

$$P(\bar{\alpha}_m - 1 > t\sqrt{\bar{D}_m}/\sqrt{n}) \approx (2\pi)^{-1/2}\int_t^\infty \exp(-x^2/2)\,dx. \tag{12}$$

For t ~ 2.5, the right-hand side of Eq. (12) is already less than 0.01.

By means of the χ^2 criterion, we can study the azimuthal angular distribution of shower particles on the whole, independent of the angle θ with the direction of the primary particle. In order to obtain information about the azimuthal asymmetry of the particles travelling at different angles θ, we can use the quantity introduced in reference 9

$$\alpha = n_s \tan^2\theta_0\,(\overline{\cos\theta})^2/\overline{\sin^2\theta}$$
$$= 1 + \sum_{i\neq j}\sin\theta_i\sin\theta_j\cos(\varphi_i - \varphi_j)\Big/\sum_{i=1}^{n_s}\sin^2\theta_i;$$
$$\overline{\cos\theta} = \sum_{i=1}^{n_s}\cos\theta_i/n_s, \qquad \overline{\sin^2\theta} = \sum_{i=1}^{n_s}\sin^2\theta_i/n_s, \tag{13}$$

where θ_i is the angle of emission of the i-th particle with respect to the direction of the primary particle in the l system, φ_i is its azimuthal angle, and θ_0 is the angle between the direction of the primary particle and the shower axis going through the center of gravity of the points of intersection of the secondary charged-particle tracks with a sphere of unit radius whose center coincides with the shower vertex.* By means of this quantity, we can solve the problem concerning the deflection of the excited centers from the direction of the primary particles in peripheral interactions of fast nucleons, since such a deflection leads to an increase in θ_0.

Let $f(\theta_1, \varphi_1, \theta_2, \varphi_2, \ldots, \theta_{n_s}, \varphi_{n_s})$ be the probability density of a multi-dimensional distribution. Assuming the azimuthal isotropy of the angular distribution of each particle for an arbitrary θ and for a fixed direction of the remaining $n_s - 1$ particles, the mathematical expectations of the quantity α and its dispersion are, respectively, given by

$$M(\alpha) = \int \alpha f(\theta_1, \varphi_1, \theta_2, \varphi_2, \ldots, \theta_{n_s}, \varphi_{n_s})\prod_{i=1}^{n_s} d\theta_i\,d\varphi_i$$
$$= \int \alpha f_1(\theta_1, \theta_2, \ldots, \theta_{n_s})\prod_{i=1}^{n_s} d\theta_i\,d\varphi_i/2\pi = 1, \tag{14}$$

$$D(\alpha) = \int\Big[\sum_{i\neq j}\sin\theta_i\sin\theta_j\cos(\varphi_i - \varphi_j)\Big/\sum_{i=1}^{n_s}\sin^2\theta_i\Big]^2$$
$$\times f_1(\theta_1, \ldots, \theta_{n_s})\prod_{i=1}^{n_s} d\theta_i d\varphi_i/2\pi$$
$$= M[1 - \overline{\sin^4\theta}/n_s(\overline{\sin^2\theta})^2] < 1. \tag{15}$$

Hence, it follows that the substitution

$$\bar{D}_m = 1 \tag{16}$$

leaves the inequality (9) valid for the quantity

$$\bar{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\alpha_i, \tag{17}$$

while the equality (12) becomes an inequality.

*Since the quantities α_m, similar to the quantities α and α', considered below, are finite, we have $0 \leqslant \alpha_m \leqslant n_s$.

*The factor $n_s(\overline{\cos\theta})^2/\overline{\sin^2\theta}$ in Eq. (13) is introduced for normalization [see Eq. (14) below].

In a general case, where

$$f(\theta_1, \varphi_1, \theta_2, \varphi_2, \ldots, \theta_{n_s}, \varphi_{n_s}) = f_1(\theta_1, \theta_2, \ldots, \theta_{n_s}) f_2(\varphi_1) f_2(\varphi_2) \ldots f_2(\varphi_{n_s}),$$

$$\int f_2(\varphi)\, d\varphi = 1, \tag{18}$$

the mathematical expectation α differs from unity by the quantity

$$M(\alpha) - 1 = aM\,[n_s\,(\overline{\sin\theta})^2/\overline{\sin^2\theta} - 1] \geqslant 0,$$
$$a = [M(\cos\varphi)]^2 + [M(\sin\varphi)]^2. \tag{19}$$

Since the values of α are finite, we have for a sufficiently large number of showers with a number of particles less than a certain maximum value,

$$\bar{\alpha} \approx M(\bar{\alpha}). \tag{20}$$

The same properties as α are possessed by the quantity*

$$\alpha' = n_s \tan\theta_0' / \overline{\tan^2\theta}$$
$$= 1 + \sum_{i \neq j} \tan\theta_i \tan\theta_j \cos(\varphi_i - \varphi_j) \Big/ \sum_{i=1}^{n_s} \tan^2\theta_i, \tag{21}$$

where θ_0' is the angle between the direction of the primary particle and the straight line going through the vertex of the shower and the center of gravity of the points of intersection of the secondary charged-particle tracks (or their continuations) with the plane tangential to the unit radius sphere at the point of intersection of this sphere with the continuation of the primary-particle track. The experimental values α' can easily be calculated using so-called target diagrams of shower particles.

Before we analyze the experimental data by the above method, let us consider several factors which cause the mathematical expectations α_m, α, and α' to differ from unity.

1. The energy-momentum conservation law subjects the momenta of the particles and their directions to certain constraints which may affect the assumed independence of the φ_i, e.g., in the case of the existence of a particle with a much higher energy and with a very large transverse momentum.† If the transverse momenta of secondary particles are of the same order of magnitude, then, if their number is large and if neutral particles are present, the influence of the energy-momentum conservation law is evidently small. However, if the transverse momenta of nucleons are much greater than those of π mesons,[10] and two nucleons do not have the tendency to be coplanar after their collision, then, for a small (~ 5) number of charged mesons, the mathematical expectations α_m, α, and α' may increase.

2. As mentioned at the beginning of the article, for small deflection angles of the excited centers from the direction of the primary nucleon, an azimuthal anisotropy of the secondary particle may arise, and the mesons will concentrate in the plane containing the momenta of the primary nucleon and of the isobar. Such an anisotropy will first lead to an increase of $M(\alpha_m)$ for $m > 2$, but will not greatly affect the mean values of α_2, α, and α' since the azimuthal angular distribution remains symmetric. It should be noted that, since the primary particles are described by a plane wave, there will be no azimuthal anisotropy of shower particles in the set of the single showers which is characterized only by the same momenta of primary nucleons but for which the concept of probability has no meaning (see reference 2). The tendency of mesons to be coplanar will make itself felt as a statistical dependence of the angles φ_i of secondary particles, which tends to increase $M(\alpha_m)$. We therefore believe that it is useful to differentiate sharply between the azimuthal anisotropy and the statistical dependence of the secondary-particle angles.

3. The presence of narrowly-correlated groups of particles in their angular distribution, originating from the decay of short-lived particles or isobars, tends to increase the mathematical expectations α_m, α, and α' [see Eqs. (1), (13), and (21)]. The quantities $M(\alpha_m)$, $M(\alpha)$, and $M(\alpha')$ would be much greater than one if the fast moving "fireballs" produced in nucleon-nucleon high-energy collisions were deflected from the direction of the primary particle, and if they had an isotropic distribution of the statistically independent angles φ. If, as a result of a collision in the c.m.s., two nucleons excited in the same way and moving at large angles to the direction of the primary particles are produced, then $M(\alpha_2)$ evidently will not increase, in contrast to the other quantities discussed above which characterize the azimuthal angular distribution of shower particles.

After these remarks, we shall use the proposed method for the analysis of experimental data on the angular distribution of shower particles obtained by J. Pernegr. The values of α, α', and α_m were calculated for 52 showers produced in emulsion by singly-charged cosmic-ray particles having ≤ 5 strongly ionizing particles. The number of secondary shower particles n_s varies from 6 to 42, and the value of the Lorentz factor of the center-of-

*We have in mind the equality (14) and inequality (15).

†This assumption does not correspond with experimental data.

Experimental data on the azimuthal angular distribution of shower particles in 52 jets*

Group characteristics	$n_s \leqslant 12,\ \overline{n}_s = 9.3,\ n = 36$				$n_s \geqslant 14,\ \overline{n}_s = 19.5,\ n = 16$			
Azimuthal group characteristics	$\overline{\alpha}\ (\overline{\alpha}', \overline{\alpha}_m)$	$\sigma\ (\overline{\alpha})$	$\frac{\lvert\overline{\alpha}-1\rvert}{\sigma\ (\overline{\alpha})}$	P, %	$\overline{\alpha}\ (\overline{\alpha}', \overline{\alpha}_m)$	$\sigma\ (\overline{\alpha})$	$\frac{\lvert\overline{\alpha}-1\rvert}{\sigma\ (\overline{\alpha})}$	P, %
	1	2	3	4	5	6	7	8
α	0,94	<0.17	—	—	1.07	<0.25	—	—
α'	0,87	<0,17	—	—	1.04	<0,25	—	—
α_2	0.78	0.22	1.0	—	0.63	0.34	1,1	—
α_3	0.91	0.16	—	—	1.80	0.24	3,3	<0.1 (⩽9)
α_4	0.80	0.13	1.6	~6	1,41	0.20	2,1	~2
α_5	0.98	0,11	—	—	1.35	0.17	2,0	~2
α_6	1,01	0,10	—	—	1,68	0.15	4.4	$<10^{-3}$ (⩽5)
α_7	0.99	0,09	—	—	1.23	0.14	1,7	~5
α_8	0.91	0.08	1.1	—	1.37	0,13	2,9	~0,2
α_9	1.05	0.08	—	—	1.32	0,12	2.7	~0.4
α_{10}	0,98	0,07	—	—	1,45	0,11	3,9	$<10^{-2}$ (⩽7)

*Only the values $|\overline{\alpha} - 1|/\sigma(\overline{\alpha}) \geqslant 1$ and of $P < 10\%$ are given.

mass system γ_c from 2.4 to 150. The values of $\overline{\alpha}$, $\overline{\alpha}'$, and $\overline{\alpha}_m$ calculated by us* for two groups of showers with small and large n_s respectively are presented in the table (columns 1 and 5). Columns 2 and 6 show the values of σ, the standard deviation of the quantities $\overline{\alpha}$, $\overline{\alpha}'$, and $\overline{\alpha}_m$ calculated assuming azimuthal isotropy of secondary particles and statistical independence of their angles φ_i. When these conditions are satisfied, the probabilities P that $\overline{\alpha}$ ($\overline{\alpha}'$, $\overline{\alpha}_m$) are not smaller or not greater than an observed value can also be calculated using the Lyapunov theorem. The corresponding data are shown in columns 4 and 8 of the table; in the parenthesis, the values of P obtained by means of the Chebyshev inequality are shown.

Experimental data for groups of showers with $n_s \leq 12$ agree, within the limits of error, with the assumptions on the azimuthal isotropy of shower particles and the statistical independence of their angles φ_i. However, for large n_s, the values of $\overline{\alpha}_m$ averaged over 16 showers are considerably greater than unity if $m \geq 3$. This increase in $\overline{\alpha}_m$ cannot be due to the influence of the energy-momentum conservation law, since no increase in the quantities $\overline{\alpha}$, $\overline{\alpha}'$, and $\overline{\alpha}_2$ is observed in this shower group and, in addition, such an influence should be even stronger for smaller n_s. The deviation of the excited centers from the direction of primary nucleons, if it exists at all, also does not exhibit any marked influence on the azimuthal angular distribution of secondary particles, since the values of $\overline{\alpha}$ and $\overline{\alpha}'$ are close to unity.

The experimental data of showers with $n_s \geq 14$ points towards a symmetrical but anisotropic azimuthal angular distribution of secondary particles in individual showers, and the symmetry is apparently conserved for any angle θ. The tendency of the emitted mesons to possess an azimuthal anisotropy might possibly also be observed for smaller n_s. In that case, however, the mathematical expectations α_m according to Eq. (5) cannot be great. In addition, for a small number of particles, the influence of the energy-momentum conservation law may be felt, causing a decrease of the mathematical expectations of the quantities under consideration.

The experimental data given above contradict the hydrodynamical theory of jet production in head-on collisions for a cylindrical symmetry of secondary particles, and indicate a considerable role played by peripheral collisions. The data of the table correspond to the emission of mesons mainly around the plane perpendicular to the direction of the intrinsic moment of momentum of the excited center. This fact was predicted theoretically as a consequence of the two-center model of multiple particle production in nucleon-nucleon high-energy collisions.[1,2] For its confirmation, additional statistical material is necessary.

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*Azimuthal angles are measured from the plane perpendicular to the emulsion plane.

[1] W. L. Kraushaar and L. J. Marks, Phys. Rev. **93**, 326 (1954).

[2] Z. Koba and S. Takagi, Nuovo cimento **10**, 755 (1958); Z. Koba and S. Takagi, Fortschr. Physik **7**, 1 (1959).

[3] H. W. Meier, Nuovo cimento **11**, 307 (1959).

[4] V. Petržilka, Nuovo cimento **13**, 1300 (1959).

[5] B. A. Nikol'skiĭ and A. P. Mishakova, JETP **38**, 1507 (1960), Soviet Phys. JETP 11, 1087 (1960).

[6] Gurevich, Mishakova, Nikol'skiĭ, and Surkova, JETP **34**, 265 (1958), Soviet Phys. JETP 7, 185 (1958).

[7] N. V. Smirnov and I. V. Dunin-Barkovskiĭ, Краткий курс математической статистики для технических приложений (Short Course on Mathematical Statistics for Technical Applications), Fizmatgiz, 1959.

[8] S. N. Bernshteĭn, Теория вероятностей Theory of Probability, ONTI, 1934.

[9] S. A. Azimov and V. M. Chudakov, Doklady Akad. Nauk Uzbek SSR (in print).

[10] C. F. Powell, Paper presented at the Conference on High-Energy Particle Physics, Kiev, 1959.

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