DISPERSION RELATIONS IN THE NONRELATIVISTIC SCATTERING THEORY WITH SPIN-ORBIT INTERACTION

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Dispersion relations for the scattering of spin-$\frac{1}{2}$ particles by a potential containing the spin-orbit interaction are derived in the nonrelativistic theory. The behavior of the Green's function of the total Hamiltonian at large complex energies is investigated.

DISPERSION relations (d.r.) for the nonrelativistic scattering amplitude have been considered in a number of papers (see references 1 to 3 and others). These authors assumed that the Hamiltonian is independent of the spin. The simplest method of derivation was based on a study of the behavior of the Green's function of the total Hamiltonian at complex energies E.[3]

In the present paper we consider the scattering of a spin-$\frac{1}{2}$ particle by a potential which includes the spin-orbit interaction. A proof of the d.r. for this case will be given. First we shall study some properties of the Green's function, leaving out a large part of the mathematical details. The latter will be published in a different place. Here we shall make only a few general remarks concerning these questions.

Two estimates are important for the proof of the d.r. with the help of the Green's function: one on the positive half of the real axis, which requires a detailed analysis of the integral equation for the Green's function at finite energies, and another at large energies in the complex plane, based on the Born series, which converges rapidly in this region.

For finite values of E we can investigate the Green's function for our problem with the help of the usual methods.[4,5] However, at large energies these methods must be modified somewhat, since the Born series converges poorly in the presence of the lS interaction. In this connection we propose a certain transformation of the Born series which improves its convergence properties (for large E). This transformation is made possible by the fact that we have been able to calculate explicitly the first term of the asymptotic Green's function for $E \to \infty$.

1. PROPERTIES OF THE GREEN'S FUNCTION

The energy operator of the system has the form

$$H = H_0 + V, \tag{1}$$

where

$$H_0 = -\nabla^2, \qquad V = -iV_1(r)[\mathbf{r}\times\nabla]\mathbf{S} + V_0(r), \tag{2}$$

and **S** is the vector for spin $\frac{1}{2}$.

The Green's function satisfies the integro-differential equation

$$R(E) = R_0(E) - R_0(E)VR(E), \tag{3}$$

where

$$R_0(E) = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\exp(i\sqrt{E}\,|\mathbf{r}-\mathbf{r}'|). \tag{4}$$

We assume that

$$|V_0(r)| \leqslant \frac{Ce^{-\varkappa r}}{(1+r)^{3+\varepsilon}}; \qquad |V_1(r)|,\ |V_1'(r)| \leqslant \frac{Ce^{-\varkappa r}}{(1+r)^{4+\varepsilon}} \qquad (\varepsilon > 0,\ \varkappa \geqslant 0). \tag{5}$$

Equation (3) can be written as an integral equation

$$R(E) = R_0(E) - R_0^V(E)R(E), \tag{6}$$

where

$$R_0^V(E) \equiv -iV_1(r')[\mathbf{r}'\times\nabla']R_0(\mathbf{r},\mathbf{r}';E)\mathbf{S} + V_0(r')R_0(\mathbf{r},\mathbf{r}';E). \tag{7}$$

Equation (6) can be investigated by the same methods as in the case where $V_1(r) = 0$. It can be established that, first, $R(\mathbf{r}, \mathbf{r}', E)$ is analytic in the complex E plane with a cut along the positive part of the real axis with the exception of a finite number of negative poles E_j at the positions of the eigenvalues; second,

$$\operatorname{Res} R(\mathbf{r}, \mathbf{r}'; E)|_{E=E_j} = \sum_k \psi_{jk}(\mathbf{r}) \overline{\psi_{jk}(\mathbf{r}')}, \tag{8}$$

where ψ_{jk} is an orthonormalized eigenfunction corresponding to the eigenvalue E_j; and third,

$$|R(\mathbf{r}, \mathbf{r}'; E)| \leqslant K(|\mathbf{r}-\mathbf{r}'|^{-1}+1) \tag{9}$$

on the cut.

The direct method of successive approximations for Eq. (6) (Born series) converges poorly even for large energies, since the kernel $R_0^V(E)$ contains $\sqrt{E}$ linearly. This implies that all terms of the series contain contributions which do not vanish as $E \to \infty$. However, by integration by parts, these terms can be separated out explicitly and summed up. As a result we obtain the following expression* for the nonvanishing term $P(E)$ of the asymptotic Green's function $R(E)$ as $E \to \infty$:

$$P(\mathbf{r}, \mathbf{r}'; E) = R_0(\mathbf{r}, \mathbf{r}'; E) \exp\left\{\tfrac{1}{8} i \mathbf{S}[\mathbf{r}\times\mathbf{r}'] \int_{-1}^{1} V_1\left(\tfrac{1}{2}(1-\eta)\mathbf{r} + \tfrac{1}{2}(1+\eta)\mathbf{r}'\right) d\eta\right\}. \tag{10}$$

Formula (10) can be proved rigorously by transforming Eq. (6) to the form

$$R(E) = P(E) + \Pi(E) - \Pi^V(E) R(E), \tag{11}$$

$$\Pi(E) = R_0(E) - P(E) - P(E) V R_0(E), \tag{12}$$

where $\Pi^V(E)$ is constructed from $\Pi(E)$ in the same way as $R_0^V(E)$ is from $R_0(E)$.

The method of successive approximations applied to Eq. (11) at high energies converges in the same way as the Born series in the absence of the spin-orbit interaction. We thus obtain the estimate (for large E)

$$|R(\mathbf{r}, \mathbf{r}', E) - P(\mathbf{r}', \mathbf{r}'; E)| \leqslant F(\mathbf{r}, \mathbf{r}'; E) \exp(-\operatorname{Im}\sqrt{E}|\mathbf{r}-\mathbf{r}'|), \tag{13}$$

where $F(\mathbf{r}, \mathbf{r}'; E) \to 0$ for $E \to \infty$ uniformly for any finite region of variation of $\mathbf{r}$ and $\mathbf{r}'$.

2. DERIVATION OF THE DISPERSION RELATIONS

As a complete set of asymptotic states for the scattering we choose the set of functions

$$\Phi_a(\mathbf{r}, \sigma) = \exp(i\sqrt{E}\boldsymbol{\alpha}\mathbf{r})\chi_m^\alpha(\sigma). \tag{14}$$

Here $a = (E, \boldsymbol{\alpha}, m)$, $\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = 1$, $m = \pm\frac{1}{2}$, and $\chi_m^\alpha(\sigma)$ is a spinor which has the components $\delta_{m\sigma}$ in a system oriented along the vector $\boldsymbol{\alpha}$.

*Similar formulas [see also Eqs. (21) and (22)] have been considered by many authors for the case $V_1(r) = 0$ in the determination of correction terms to the Born approximation (see, for example, reference 6).

For the scattering amplitude we use the following expression:

$$f_{ba} = -\frac{1}{4\pi}(\Phi_b; V\Psi_a), \tag{15}$$

where $b = (E, \boldsymbol{\beta}, n)$, $\boldsymbol{\beta}\cdot\boldsymbol{\beta} = 1$, $n = \pm\frac{1}{2}$, and $\Psi_a(\mathbf{r}, \sigma)$ is a solution of the scattering problem

$$\Psi_a = \Phi_a - R(E+i0) V\Phi_a. \tag{16}$$

We then find for f_{ba}

$$f_{ba} = -(1/4\pi)(\Phi_b, V\Phi_a) + (1/4\pi)(V\Phi_b, R(E+i0)V\Phi_a). \tag{17}$$

We introduce the usual notations

$$\boldsymbol{\pi} = |\boldsymbol{\pi}|\mathbf{n} = \tfrac{1}{2}\sqrt{E}(\boldsymbol{\beta}+\boldsymbol{\alpha}); \quad \boldsymbol{\Delta} = \sqrt{E}(\boldsymbol{\beta}-\boldsymbol{\alpha}), \quad \mathbf{n}\boldsymbol{\Delta} = 0;$$
$$\boldsymbol{\nu} = [\boldsymbol{\beta}\times\boldsymbol{\alpha}]/\sin\theta, \quad \cos\theta = \boldsymbol{\beta}\boldsymbol{\alpha}. \tag{18}$$

We shall regard f_{ba} as a function of E, with fixed $\boldsymbol{\Delta}$, $\mathbf{n}$ $(\Delta \leq \kappa/2)$, and spinor variables (f_{ba} depends practically only on E, Δ, and the spinor variables).

We expand the spinors χ_m^α and χ_m^β in terms of the constant spinors $\chi_m^{\mathbf{n}}$:

$$\chi_m^\alpha = \sum_k \tau_{mk}\left(\frac{\theta}{2}\right)\chi_k^{\mathbf{n}}, \quad \chi_m^\beta = \sum_k \tau_{mk}\left(\frac{\theta}{2}\right)(-1)^{m-k}\chi_k^{\mathbf{n}},$$
$$\tau_{mk}(\theta) = \delta_{mk}\cos\frac{\theta}{2} + i\delta_{m,-k}\sin\frac{\theta}{2}. \tag{19}$$

It is now easily seen that the first term on the right-hand side of formula (17) is analytic in E with a cut along the positive half of the real axis and with a singularity of the type $1/\sqrt{E}$ at $E = 0$.

The second term has the following form in the new variables:

$$\frac{1}{4\pi}(V\Phi_b, R(E)V\Phi_a) = \frac{1}{4\pi}\sum_{k,k'} \tau_{mk}\left(\frac{\theta}{2}\right)\tau_{nk'}\left(\frac{\theta}{2}\right) \times \int d\mathbf{r}\,d\mathbf{r}' \exp\left\{-\frac{i}{2}\boldsymbol{\Delta}(\mathbf{r}+\mathbf{r}')\right\}\exp\left\{-i\sqrt{E-\frac{\Delta^2}{4}}\,\mathbf{n}\times(\mathbf{r}-\mathbf{r}')\right\}\{\chi_{k'}^{\mathbf{n}}, (\sqrt{E}V_1(r)[\mathbf{r}\times\boldsymbol{\beta}]\mathbf{S} + V_0(r))R(\mathbf{r}, \mathbf{r}'; E) \times (\sqrt{E}V_1(r')[\mathbf{r}'\times\boldsymbol{\alpha}]\mathbf{S} + V_0(r'))\chi_k^{\mathbf{n}}\}, \quad \sin\frac{\theta}{2} = \frac{\Delta}{2\sqrt{E}}. \tag{20}$$

If we separate out the poles of $R(\mathbf{r}, \mathbf{r}'; E)$ and apply the integral analog of the Weierstrass theorem,[7] using (9) and (13), we can conclude that the analytic properties of the second term differ from those of the first term described above only by the presence of the additional poles at the points E_j.

It follows from the estimate (13) that the function f_{ba} does not grow faster than E as $E \to \infty$; the leading term of the amplitude f_{ba}^0 for $E \to \infty$ can be easily calculated:

$$f_{ba}^0 = -\frac{1}{4\pi}\sqrt{E}(\mathbf{S}[\mathbf{r}\times\boldsymbol{\beta}]V_1\Phi_b, \Psi_a^0), \tag{21}$$

where

$$\Psi_a^0 = e^{i\sqrt{E}\alpha r} \exp\left\{\frac{i}{4} S[\alpha \times r] \int_0^\infty d\eta V_1(r - \alpha\eta)\right\} \chi_m^\alpha \quad (22)$$

is the leading term of the solution Ψ_a.

We separate out the dependence of $f_{nm}(\Delta, n; E)$ on n, m by writing $f_{nm}(\Delta, n; E)$ in the matrix form

$$f(\Delta, n; E) = A(\Delta, E) + B(\Delta, E)\nu S, \quad (23)$$

where the functions $A(\Delta, E)$ and $B(\Delta, E)$ do not depend any more on m and n. These functions have the properties

$$A(\Delta, E - i0) = \overline{A(\Delta, E + i0)},$$
$$B(\Delta, E - i0) = \overline{B(\Delta, E + i0)}, \quad (24)$$

which probably are most easily obtained by expanding $f(\Delta, E)$ in terms of spherical functions. We conclude from (24) that the residues of $A(\Delta, E)$ and $B(\Delta, E)$ at the points E_j are real.

Thus we have obtained the following dispersion relation for the function $g(E) = A(\Delta, E)$, $B(\Delta, E)$:

$$\operatorname{Re} g(E) = P\frac{1}{\pi}\int_0^\infty \frac{1}{E' - E}\frac{E - E_0}{E' - E_0} \operatorname{Im} g(E')\, dE'$$
$$+ \operatorname{Re} g(E_0) + \sum_j \frac{1}{E - E_j}\frac{E - E_0}{E_j - E_0} d_j^g, \quad (25)$$

where

$$d_j^g = \operatorname{Res} g(E)\big|_{E=E_j}, \quad (26)$$

and E_0 is an arbitrary positive energy. The residues d_j^g are given in terms of the eigenfunctions of the discrete spectrum.

The dispersion relation for $B(\Delta, E)$ can, apparently, be given in a different form, since $\operatorname{Im} B(\Delta, E)$ increases more slowly than $\sqrt{E}$; however, this requires a more detailed investigation of the asymptotic form of $B(\Delta, E)$.

The values of $\operatorname{Im} g(E)$ in the unphysical region $E < \frac{1}{4}\Delta^2$ which enter in the d.r. can be obtained in the usual manner[2,8] by analytic continuation from the physical region, using, for example, the expansion in spherical functions.

In conclusion we note that the modified Born approximation for the scattering amplitude (21) has possible interest for quantitative calculations in cases where the lS interaction cannot be neglected.

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