SOVIET PHYSICS JETP VOLUME 12, NUMBER 5 MAY, 1961

THE INFLUENCE OF IMPURITIES ON THE DE HAAS-VAN ALPHEN EFFECT

Yu. A. BYCHKOV

Institute of Physics Problems, Academy of Sciences of the U.S.S.R.

Submitted to JETP editor June 22, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 1401-1410 (November, 1960)

The methods of quantum field theory are used to study the influence of the elastic scattering by impurities on the quantum oscillations of the magnetic susceptibility of the electron gas in a metal (the de Haas-van Alphen effect). The quasi-classical case is considered, for which the condition $\hbar\omega/\zeta \ll 1$ is satisfied, where ω is the frequency of revolution in the magnetic field and ζ is the chemical potential; a quadratic dispersion law is assumed. It is shown that if $1/\omega\tau_0 \gg \hbar\omega/\zeta$ the effect of the scattering by impurities is described by the Dingle factor[3] with $\tau_D = 4\tau_0/\pi$ (τ_0 is the mean free time for $H = 0$). If $1/(\omega\tau_0) \lesssim \hbar\omega/\zeta$, a more complicated situation exists. In particular, if $\zeta = \hbar\omega(M_0 + 1/2) + \Delta$ and $|\Delta| \leq \hbar\omega/2$, then for $\hbar^2\omega^2/\zeta \ll |\Delta| \lesssim \hbar\omega/2$ the effect of impurities reduces, as before, to the Dingle factor. In the region $|\Delta| \lesssim \hbar^2\omega^2/\zeta$ the effect of the impurities is extremely complicated.

It is well known that at sufficiently strong magnetic fields, for which the quantization of the energy of an electron becomes important, the electron gas in a metal has an appreciable magnetic susceptibility.[1-2] The susceptibility $\chi(H)$ is the sum of two terms: $\chi(H) = \chi_1 + \chi_2(H)$, where χ_1 does not depend on the magnitude of the magnetic field, while $\chi_2(H)$ is an oscillating function of H.

In calculating $\chi_2(H)$ the overwhelming majority of authors have concerned themselves only with the dependence of χ_2 on the magnetic field and the topology of the Fermi surface and have not studied the effect of the scattering of the electrons on $\chi_2(H)$. An exception is the work of Dingle,[3] who has made an attempt to take into account the effect on χ_2 of the elastic scattering of the electrons by impurities. For this purpose Dingle used the artificial method of introducing into all the formulas for χ_2 a certain averaging factor, which subsequently received the name of the Dingle factor.

In the present paper we use the methods of quantum field theory, as developed for application to scattering by impurities by Edwards[4] and by Abrikosov and Gor'kov,[5] to study the problem of the effect of the elastic scattering of the electrons by impurities on the oscillating part of the susceptibility.

To find the magnetic moment of a system of electrons it is necessary to calculate the free energy F. Since we are considering only elastic collisions with the impurities, the distribution function is not changed, so that we can write at once:

$$F = N\zeta - 2kT\int_0^\infty \frac{dZ}{d\varepsilon}\ln\left(1 + e^{\frac{\zeta-\varepsilon}{kT}}\right)d\varepsilon, \qquad (1)$$

where ζ is the chemical potential of the system, N is the number of electrons per unit volume, and $dZ/d\epsilon$ is the number density of electron states per unit energy range, in the presence of the magnetic field H and chaotically distributed impurities. The factor 2 in the term with the integral is due to the spin of the electron (we are not concerned with the spin part of the susceptibility).

The problem thus reduces to that of calculating the density of states $dZ/d\epsilon$. For simplicity we shall adopt the dispersion law $\epsilon = p^2/2m$. The results can be extended in a very simple way to an arbitrary quadratic dispersion law.

In the absence of impurities

$$\left(\frac{dZ}{d\varepsilon}\right)_0 = \frac{m^{3/2}\omega}{2^{3/2}\pi^2}\sum_M \frac{1}{\sqrt{\varepsilon - \omega(M + 1/2)}} \qquad (2)$$

(we have set $\hbar = c = 1$). Here the summation over M is taken for positive values of the radicand, and $\omega = eH/m$ is the frequency of revolution of an electron in the magnetic field.

It is very convenient to represent the energy in the form

$$\varepsilon = \omega(M_0 + 1/2) + \Delta, \qquad |\Delta| \leqslant \omega/2. \qquad (3)$$

It can be seen from Eq. (3) that for $\Delta = 0$ the density of levels $dZ/d\epsilon$ is a discontinuous function: it approaches a finite limit for $\Delta \to -0$, and goes to infinity like $\Delta^{-1/2}$ for $\Delta \to +0$. Inclusion of the scattering by the impurities must destroy this discontinuity and make $dZ/d\epsilon$ a smooth function.

1. CALCULATION OF $dZ/d\epsilon$

Let us consider the Green's function of an electron

$$G(\mathbf{r}, \mathbf{r}'; \varepsilon) = \sum_n \frac{\psi_n(\mathbf{r})\,\psi_n^*(\mathbf{r}')}{E_n - \varepsilon - i\delta}, \qquad \delta \to 0. \tag{4}$$

Here n is a set of quantum numbers that characterize the state of the electron in the presence of the magnetic field H and of the impurities; E_n and $\psi_n(\mathbf{r})$ are the energy and wave function of the electron in the corresponding stationary state. If we now use the orthonormality of $\psi_n(\mathbf{r})$ and the definition of $dZ/d\epsilon$, according to which

$$dZ/d\varepsilon = \sum_n \delta(E_n - \varepsilon)$$

(where $\delta(x)$ is the usual δ function), it is easy to establish a connection between the density of states $dZ/d\epsilon$ and the Green's function $G(\mathbf{r}, \mathbf{r}'; \epsilon)$:

$$\frac{dZ}{d\varepsilon} = \frac{1}{\pi}\,\mathrm{Im}\int G(\mathbf{r}, \mathbf{r}; \varepsilon)\,d\mathbf{r}. \tag{5}$$

The Green's function $G(\mathbf{r}, \mathbf{r}'; \epsilon)$ satisfies the equation:[5]

$$G(\mathbf{r}, \mathbf{r}'; \varepsilon) = G_0(\mathbf{r}, \mathbf{r}'; \varepsilon) - \int G_0(\mathbf{r}, \mathbf{r}''; \varepsilon)\,V(\mathbf{r}'')\,G(\mathbf{r}'', \mathbf{r}'; \varepsilon)\,d\mathbf{r}'', \tag{6}$$

where $G_0(\mathbf{r}, \mathbf{r}'; \epsilon)$ is the Green's function of the electron in the absence of impurities and

$$V(\mathbf{r}) = \sum_\alpha v(\mathbf{r} - \mathbf{r}_\alpha) \tag{7}$$

is the potential of the field of the chaotically distributed impurities.

Let $E_n^0 = \omega(M + 1/2) + p_z^2/2m$ and $\psi_n^0(\mathbf{r})$ be the energies and wave functions of the stationary states in the absence of impurities. The set of quantum numbers n consists of the magnetic quantum number M, the momentum p_z (H is directed along the z axis), and a wave number p_y, which is connected with the center of the orbit of the electron on the x axis by the relation $x_0 = p_y/eH$. Let us expand $G(\mathbf{r}, \mathbf{r}'; \epsilon)$ in terms of these functions. Obviously

$$G_0(\mathbf{r}, \mathbf{r}'; \varepsilon) = \sum_n \psi_n^0(\mathbf{r})\,g_n\,\psi_n^{0*}(\mathbf{r}'),$$

$$G(\mathbf{r}, \mathbf{r}'; \varepsilon) = \sum_{n,m} \psi_n^0(\mathbf{r})\,G_{nm}\,\psi_m^{0*}(\mathbf{r}'), \tag{8}$$

where the summations over n, m mean summation over the magnetic quantum number M and integration over dp_z and dp_y; $g_n = (E_n^0 - \epsilon - i\delta)^{-1}$.

By using the relations (6) and (8), we easily get the equation that is satisfied by G_{nm}:

$$G_{nm} = \delta_{nm} g_n - g_n \sum_k V_{nk}\,G_{km}, \tag{9}$$

$$V_{nk} = \int \psi_n^{0*}(\mathbf{r})\,V(\mathbf{r})\,\psi_k^0(\mathbf{r})\,d\mathbf{r}.$$

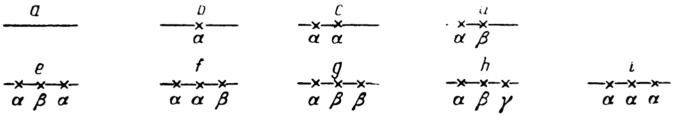

FIG. 1

Here V_{nk} is the matrix element of the operator for the interaction of the electron with the impurities. The function for the energy density of the states is simply related to G_{nn}:

$$\frac{dZ}{d\varepsilon} = \frac{1}{\pi}\,\mathrm{Im}\sum_n G_{nn}. \tag{10}$$

Let us now proceed to the solution of Eq. (9). For this purpose we use the results of the paper of Abrikosov and Gor'kov.[5] According to Eq. (7)

$$V_{nk} = \sum_\alpha v_{nk}^\alpha, \qquad v_{nk}^\alpha = \int \psi_n^{0*}(\mathbf{r})\,v(\mathbf{r} - \mathbf{r}_\alpha)\,\psi_k^0(\mathbf{r})\,d\mathbf{r}.$$

If in a diagram we use a cross with the index α to correspond to the expression v_{lk}^α, and a straight line to correspond to g_l, then G_{nm} can be represented up to terms of third order in V by the diagrams shown in Fig. 1. The transition from a diagram to the corresponding algebraic expression is made by means of a simple rule: one places in correspondence with each line a factor g_l and with each cross a factor v_{lk}^α; as we move along the diagram, the indices l and k change only when we pass through a cross; to the initial and final lines there correspond the indices n and m. After the expression is written out, it must be summed over all indices corresponding to intermediate lines and over the indices of the various impurities, which are denoted by Greek letters, and then multiplied by $(-1)^s$, where s is the number of crosses in the given diagram.

To illustrate this rule by examples, let us write out the expressions for diagrams c and e of Fig. 1:

$$g_n \sum_{k,\alpha} v_{nk}^\alpha g_k v_{km}^\alpha g_m, \tag{c}$$

$$-g_n \sum_{\substack{k,l \\ \alpha\neq\beta}} v_{nk}^\alpha g_k v_{kl}^\beta g_l\, v_{lm}^\alpha g_m. \tag{e}$$

From the diagrams drawn it is clear than we have first a perturbation-theory series with respect to the scattering by a given impurity (for example, b and c in Fig. 1), and second a perturbation-theory series corresponding to successive scatterings by different impurities (b, d, h in Fig. 1).

Obviously the summation of the diagrams of types b and c in Fig. 1 is simply the transition from v_{lk}^α to the complete amplitude for scattering of the electron by the given impurity, a_{lk}. We have

$$a^{\alpha}_{lk} = v^{\alpha}_{lk} - \sum_{s} v^{\alpha}_{ls} g_s v^{\alpha}_{sk} + \sum_{s,t} v^{\alpha}_{ls} g_s v^{\alpha}_{st} g_t v^{\alpha}_{tk} \dots \qquad (11)$$

We shall therefore not consider henceforth diagrams in which a given impurity is encountered two or more times in succession (types c, f, etc.), and shall put in correspondence with a cross the complete amplitude a_{lk} (there remain, however, diagrams like e in Fig. 1, in which, between successive scatterings by an impurity α, the electron is scattered by one or more impurities $\beta \neq \alpha$). If we make this stipulation, then up to the fourth order we need consider only the diagrams shown in Fig. 2, where crosses for the same impurity are connected by dashed lines.

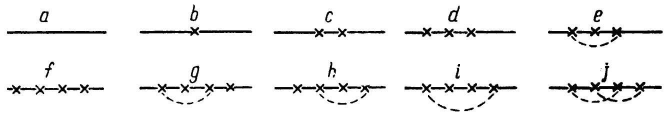

FIG. 2

Some remarks must be made regarding the total amplitude a_{lk} for scattering by a given impurity. For metals with a quadratic dispersion law, i.e., a small number of carriers, the wavelength of the electron is much larger than the range of the potential of an impurity. Therefore the potential of the impurity can be treated like a δ function. In this case the scattering amplitude has the form[6]

$$a^{\alpha}_{lk} = F(\varepsilon)\, \psi^{0*}_{l}(\mathbf{r}_{\alpha})\, \psi^{0}_{k}(\mathbf{r}_{\alpha}). \qquad (12)$$

The function $F(\epsilon)$ can be determined very simply. Obviously

$$a^{\alpha}_{lk} = \int \psi^{0*}_{l}(\mathbf{r})\, v(\mathbf{r} - \mathbf{r}_{\alpha})\, \psi_k(\mathbf{r})\, d\mathbf{r},$$

where $\psi_k(\mathbf{r})$ satisfies an integral equation that is equivalent to the Schrödinger equation:

$$\psi_k(\mathbf{r}) = \psi^{0}_{k}(\mathbf{r}) - \int G(\mathbf{r}, \mathbf{r}';\ \varepsilon)\, v(\mathbf{r} - \mathbf{r}_{\alpha})\, \psi_k(\mathbf{r})\, d\mathbf{r}.$$

It is clear that to find the amplitude a_{lk} it is necessary to know $\psi_k(\mathbf{r})$ in the region of action of the potential. But in this region, i.e., for small values of $\mathbf{r}$ and $\mathbf{r}'$, the Greens function satisfies

$$G(\mathbf{r}, \mathbf{r}';\ \varepsilon) \approx G_0(\mathbf{r}, \mathbf{r}') + g(\varepsilon),$$

where $G_0(\mathbf{r}, \mathbf{r}')$ is the Green's function of the electron for $\epsilon = 0$ in the absence of the magnetic field. If we take all these points into account, we get for $F(\epsilon)$:

$$F(\varepsilon) \approx f - f^2 g(\varepsilon), \qquad (13)$$

where f is the scattering amplitude of the electron for $\epsilon = 0$ and $H = 0$.

The expression (13) is nothing other than the expansion of $F(\epsilon)$ in powers of the small ratio of the scattering amplitude for $\epsilon = 0$ to the wavelength of the electron; it is incorrect only in the narrow range of values of the magnetic field in which $\Delta \ll \omega$,[6] since in this region $fg \gg 1$ and therefore it is necessary to take into account subsequent terms in an expansion of $F(\epsilon)$ in series of powers of fg. We shall assume, however, that when collisions with impurities are taken into account $fg \ll 1$. We shall discuss this condition later in more detail.

It is easy to establish the connection of $g(\epsilon)$ with the sum $\sum_n g_n$. It can be shown[6] that the separation out of $g(\epsilon)$ is equivalent to dropping from the sum $\sum_n g_n$ the divergent part that arises from summation over very large magnetic quantum numbers M.

In the specific calculation of G_{nm} we shall assume that the following conditions are satisfied: $\zeta \gg \omega$, i.e., we have the quasi-classical case, and $\omega\tau > 1$ (as we shall see later, for $\omega\tau \lesssim 1$ the quantum effects are exponentially small).

The simplest diagrams in Fig. 2 are b, c, d, and f. For these diagrams the averagings over the positions of the impurities for the various crosses are independent of each other. The averages can be taken over the whole volume of the crystal. In particular, for diagram c we have

$$g_n g_m \sum_{\substack{k \\ \alpha \neq \beta}} \langle a^{\alpha}_{nk} \rangle\, g_k\, \langle a^{\beta}_{km} \rangle = \delta_{nm} g^3_n c^2 F^2(\varepsilon), \qquad (14)$$

where c is the concentration of impurities. The relation (14) follows from the obvious formula

$$\sum_{\alpha} \langle a^{\alpha}_{nk} \rangle = F(\varepsilon)\, c \int \psi^{0*}_{n}(\mathbf{r}_{\alpha})\, \psi^{0}_{k}(\mathbf{r}_{\alpha})\, d\mathbf{r}_{\alpha} = F(\varepsilon)\, c\, \delta_{nk}.$$

Obviously to each impurity there must correspond a term cf^2g, since the term cf is simply the change of the reference level for the energy. If we take this into account, the expression (14) is equal to $\delta_{nm} g^3_n c^2 f^4 g^2$.

We can at once calculate the expression for an arbitrary diagram in which a given impurity is encountered only once (types b, c, and so on). We have

$$\delta_{nm} g^{s+1}_n c^s f^{2s} g^s (-1)^s. \qquad (15)$$

In order of magnitude such a diagram gives

$$\Delta^{-1} (cf^2 g/\Delta)^s, \qquad (16)$$

i.e., for $\Delta \sim cf^2g$ they give the same contribution.

Let us now study the simplest diagrams in which

FIG. 3

a given impurity is encountered twice (e in Fig. 2). It is easy to show that after the averaging the diagram e gives

$$-\delta_{nm}g_n^2c^2f^4g\frac{\partial g}{\partial \varepsilon}\sim\frac{1}{\Delta}\left(\frac{cf^2g}{\Delta}\right)^2.$$

For $\Delta \sim cf^2g$ this diagram is of the same order as diagram c. By similar arguments it is easy to show that for $\Delta \sim cf^2g$ we get the same contribution from all diagrams in which a given impurity is encountered not more than twice. These include, in particular, diagrams of the type a of Fig. 3.

An exception to this rule is formed by diagrams of the type j of Fig. 2. Simple estimates show that for arbitrary Δ the ratio of the expression corresponding to this diagram to that for diagram c of Fig. 2 is $1/M_0$, where $M_0 \sim \epsilon/\omega$. But since we are assuming that $M_0 \gg 1$, this means that the "overlapping" diagrams can be omitted. We note here that the dropping of diagrams $\sim 1/M_0$ is practically equivalent to the neglect of corrections $\sim 1/(\epsilon\tau)$ in comparison with $\sim 1/(\omega\tau_0)$. Since we are interested only in corrections to ω, we shall omit diagrams of the type j of Fig. 2.

Diagrams are possible in which a given impurity is encountered more than twice. An example is the diagram b of Fig. 3. A simple estimate shows that in order of magnitude this diagram is equal to $\Delta^{-1}(cf^2g/\Delta)^3fg$, i.e., as compared with the diagrams considered previously, it contains the extra factor $fg \ll 1$. This factor occurs in all diagrams in which a given impurity is encountered more than twice.

Our final conclusion is that in the expansion (10) we need include only terms in which each impurity is encountered not more than two times, and that we must omit overlapping diagrams of the type j of Fig. 2. In addition to this, after the averaging only the diagonal elements of the matrix G_{nm} will be different from zero:

$$\langle G_{nm}\rangle = \delta_{nm}G_n.$$

If we take all these facts into account, it is easy to get the equation that G_n satisfies. Obviously

$$G_n^{-1} = E_n^0 - \varepsilon + cF - cF^2(G-g), \qquad G = \sum_n G_n,$$

or, if we use the fact that $F = f - f^2g$,

$$G_n^{-1} = E_n^0 - \varepsilon + cf - cf^2G. \tag{17}$$

Finally we get for $G(\epsilon)$ the following equation (we have dropped the term cf, which means a change of the reference level for energies and is of no interest):

$$G = \sum_n (E_n^0 - \varepsilon - cf^2G)^{-1}.$$

If we perform the integration over p_z, we get

$$G = i\frac{m^{3/2}\omega}{2^{3/2}\pi}\left\{\sum_{M=0}^{\infty}[\varepsilon + cf^2G - \omega(M + 1/2)]^{-1/2} - A\right\}, \tag{18}$$

where the subtraction of A is equivalent to the subtraction of the divergent term caused by summation over very large values of M.

According to Eq. (9), the density of levels is connected with $G(\epsilon)$ by the relation

$$\frac{dZ}{d\varepsilon} = \frac{1}{\pi}\operatorname{Im} G(\varepsilon). \tag{10'}$$

Since the calculation is being made in the quasiclassical domain $\omega/\zeta \ll 1$, G is essentially equal to an integral with respect to M plus an added term G_1 which goes to zero for $H = 0$ and is in general small in comparison with the integral, except for a narrow range of values of the magnetic field, in which $|\Delta| \ll \omega$. We take for the integral the value $G_0 = im^{3/2}\epsilon^{1/2}(2\pi)^{-1/2}$, since we must neglect the corrections to ϵ caused by the scattering. Thus we have

$$G = G_0 + i\frac{m^{3/2}\omega}{2^{3/2}\pi}\left\{\sum_{M=0}^{\infty}\frac{1}{\sqrt{\varepsilon + cf^2G - \omega(M + 1/2)}} - \int_0^{\infty}\frac{dM}{\sqrt{\varepsilon + cf^2G - \omega M}}\right\}. \tag{19}$$

For not too small Δ the ratio G_1/G_0 is in order of magnitude $(\omega/\epsilon)^{1/2} \ll 1$. Thus in this range of values of H Eq. (19) can be solved by the method of successive approximations; that is, we simply substitute G_0 in the expression in curly brackets. Furthermore,

$$cf^2G_0 = i\pi/2\tau_0, \tag{20}$$

where τ_0 is the mean free time for $H = 0$.

The calculation of G_1 can be carried out by means of the Poisson summation formula:

$$\sum_{M=0}^{\infty} f(M + 1/2) = \sum_{-\infty}^{\infty}(-1)^l\int_0^{\infty} f(x)e^{2\pi ilx}dx,$$

from which we have

$$G(\varepsilon) = G_0(\varepsilon) + i\frac{m^{3/2}\omega}{2^{3/2}\pi}\sum_{-\infty}^{\infty}{}'(-1)^l\int_0^{\infty}\frac{e^{2\pi ilx}dx}{\sqrt{\varepsilon + i\pi/2\tau_0 - \omega x}}; \tag{21}$$

the prime on the summation sign means that the term with $l = 0$ is omitted.

For the oscillating part G_1 we have

$$i\,\frac{m^{3/2}\omega^{1/2}}{2^{3/2}\pi}\sum_1^{\infty}(-1)^l e^{2\pi il\varepsilon/\omega}\int_{-\infty}^{\infty}\frac{e^{-2\pi ilx}\,dx}{\sqrt{x + i\pi/2\omega\tau_0}}.$$

The calculation of this integral is not difficult, and therefore we simply state the result,

$$i\,\frac{m^{3/2}\omega^{1/2}}{2\pi}\sum_{1}^{\infty}\frac{(-1)^l}{\sqrt{l}}\exp\left(2\pi i l\,\frac{\varepsilon}{\omega}-i\,\frac{\pi}{4}-\frac{\pi^2 l}{\omega\tau_0}\right). \quad (22)$$

From this we get as the expression for $(dZ/d\epsilon)$:

$$\left(\frac{dZ}{d\varepsilon}\right)_{\mathrm{osc}}=\frac{m^{3/2}\omega^{1/2}}{2\pi^2}\sum_{1}^{\infty}\frac{(-1)^l}{\sqrt{l}}\exp\left(-\pi^2 l/\omega\tau_0\right)\cos\left(2\pi l\,\frac{\varepsilon}{\omega}-\frac{\pi}{4}\right). \quad (23)$$

The factor $\exp(-\pi^2 l/\omega\tau_0)$ is indeed the Dingle factor.[3]*

The maximum value of G_1 is reached near $\Delta = 0$; this maximum value is of the order of magnitude $m^{3/2}\omega\tau_0^{1/2}$, and therefore if $(\omega\tau_0)^{-1} \gg \omega/\epsilon$ Eq. (23) is valid for arbitrary values of Δ. A more complicated situation arises in the case $(\omega\tau_0)^{-1} \lesssim \omega/\epsilon$. Here too, however, Eq. (23) is valid as before for $|\Delta| \gg \omega^2/\epsilon$.

For small values of Δ ($|\Delta| \ll \omega$) Eq. (19) becomes much simpler, and takes the form

$$G = G_0 + \frac{i m^{3/2}\omega}{2^{3/2}\pi}\,\frac{1}{\sqrt{\Delta + cf^2 G}}. \quad (24)$$

Equation (24) is obviously an algebraic equation of the third degree in G. We shall only study some of its properties.

An elementary analysis shows that Im G_1 falls off rapidly with increase of $|\Delta|$, but remains positive. At $\Delta_0 = \pi(2\cdot 3^{1/2}\tau_0)^{-1} \ll \omega$ the function Im G_1 has a maximum, and

$$(\mathrm{Im}\,G_1)_{max} \approx \mathrm{Im}\,G_0 \qquad (1/\omega\tau_0 \sim \omega/\varepsilon). \quad (25)$$

In particular, for $(\omega\tau_0)^{-1} \ll \omega/\epsilon$

$$(\mathrm{Im}\,G)_{max} = \mathrm{Im}\,G_0\,\frac{\sqrt{3}}{\pi^{1/3}2^{4/3}}\left(\frac{\omega^2\tau_0}{\varepsilon}\right)^{1/3}. \quad (25')$$

Equation (24) is valid only in a narrow range of energies near $\Delta = 0$. In order to calculate the position of the minimum of Im G it is necessary to use Eq. (19). We shall not do this.

Let us now return to the condition (13). It is quite obvious that Eq. (13) will be satisfied if we have $f(\mathrm{Im}\,G)_{\max} \ll 1$. It is easy to show, when one uses Eq. (25), that this condition is equivalent to the following restriction on the concentration of impurities:

$$c \gg \sqrt{\sigma}\,(eH)^2, \quad (26)$$

where σ is the total cross section for elastic scattering of an electron by an impurity for $H = 0$, $\epsilon = 0$.

2. CALCULATION OF THE MAGNETIC SUSCEPTIBILITY AND THE MAGNETIC MOMENT

It now remains for us to calculate the formulas for the magnetic moment and the susceptibility. Since we are interested in only the oscillating parts of these quantities, we shall hereafter always give formulas for χ_2 and M_2. If we use the relations (1) and (10′), we get

$$M_2 = \frac{2kT}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{\partial G_1}{\partial H}\right)\ln\left(1+\exp(\zeta-\varepsilon)/kT\right)d\varepsilon. \quad (27)$$

It can be seen from Eq. (19) that G_1 is a function of the variables ω and $\Delta = \epsilon - \omega(M_0 + \frac{1}{2})$. It is obvious that in Eq. (27) we must differentiate G_1 only with respect to Δ, since then each differentiation with respect to H will involve multiplication by the quantity $M_0 \gg 1$, and therefore

$$\frac{\partial G_1}{\partial H} = -\left(\frac{e}{m}\right)\left(\frac{\varepsilon}{\omega}\right)\frac{\partial G_1}{\partial \varepsilon}. \quad (28)$$

This means that we must know the functional relationship $G_1(\Delta)$ over the whole periodicity interval $|\Delta| \leq \omega/2$. But for arbitrary Δ this can be obtained in closed form only in the case $(\omega\tau_0)^{-1} \gg \omega/\epsilon$ (cf. Eq. (23)):

$$M_2 = -\frac{m^{3/2}\omega^{1/2}}{\pi}\left(\frac{e}{m}\right)\left(\frac{\zeta}{\omega}\right)kT \times \sum_{1}^{\infty}\frac{(-1)^l\exp(-\pi^2 l/\omega\tau_0)}{\sqrt{l}\,\sinh(2\pi^2 l kT/\omega)}\sin\left(2\pi l\,\frac{\zeta}{\omega}-\frac{\pi}{4}\right). \quad (29)$$

Actually this formula is also valid for arbitrary $(\omega\tau_0)^{-1}$. In fact, it can be seen from Eq. (29) that $M_2 \sim (e/m)m^{3/2}\omega^{1/2}\zeta$. At the same time the contribution to M_2 given by the integration in Eq. (27) in the region $|\Delta| \sim \omega^2/\zeta$ is $(e/m)m^{3/2}\zeta\Delta^{1/2}$. But the ratio of this correction to M_2 is $\sim(\omega/\zeta)^{1/2} \ll 1$, so that the contribution from the region $\Delta \sim 0$ is relatively small, and thus M_2 depends only very weakly on the exact behavior of Im G_1 for $\Delta \sim 0$.

We have written out the expression for the magnetic moment in the case in which there is a single group of electrons with an isotropic dispersion law. This formula can be extended very simply to the case of several groups of electrons corresponding to different ellipsoids with arbitrary quadratic dispersion laws $\epsilon = \frac{1}{2}m_{ik}^{-1}p_i p_k$, where m_{ik} is the mass tensor. If we take the z axis in the direction of **H**, then the frequency of revolution of the electrons that belong to the ith ellipsoid is $\omega_i = eH(m_{zz}^i/|m^i|)^{1/2}$ ($|m^i|$ is the determinant of the mass tensor). With this notation we have instead of Eq. (29)

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*We note that the quantity τ_D, which in Dingle's paper[3] plays the role of a mean free time, is actually connected with the true mean free time τ_0 for $H = 0$ by the relation $\tau_D = 4\tau_0/\pi$.

$$M_2 = -e\sum_i \frac{|m^i|^{1/2}\omega_i^{1/2}}{\pi}\sqrt{\frac{m^i_{zz}}{|m^i|}}\left(\frac{\zeta}{\omega_i}\right)$$
$$\times kT\sum_1^\infty \frac{(-1)^l \exp(-\pi^2 l/\omega_i\tau_0)}{\sqrt{l}\sinh(2\pi^2 l\,kT/\omega_i)}\sin\left(2\pi l\frac{\zeta}{\omega_i}-\frac{\pi}{4}\right), \qquad (29')$$

where τ_0 is the mean free time in the absence of the magnetic field. Thus the effect of the scattering by impurities on the oscillating part M_2 of the magnetic moment has the character of a Dingle factor for an arbitrary value of the ratio $1/(\omega\tau_0)$.

Besides M_2, the susceptibility $\chi_2 = \partial M_2/\partial H$ is of interest. If we use the same arguments as in the derivation of Eq. (28), we easily find that

$$\chi_2 = -\frac{2}{\pi}\left(\frac{e}{m}\right)^2\left(\frac{\zeta}{\omega}\right)^2\int_0^\infty \operatorname{Im} G_1\frac{\partial f_0}{\partial\varepsilon}\,d\varepsilon$$
$$\left(f_0 = \left\{1+\exp\frac{\varepsilon-\zeta}{kT}\right\}^{-1}\right). \qquad (30)$$

In particular, for $T = 0$

$$\chi_2 = \frac{2}{\pi}\left(\frac{e\zeta}{m\omega}\right)^2 \operatorname{Im} G_1, \qquad (31)$$

i.e., χ_2 is directly connected with $\operatorname{Im} G_1$, whose properties we have already elucidated in the preceding section.

We can finally summarize as follows the results relating to the effect of the scattering by impurities on the magnetic susceptibility χ_2. If $(\omega\tau_0)^{-1} \gg \omega/\zeta$, the effect of the scattering is described by the Dingle factor for all values of the magnetic field **H**. For $(\omega\tau_0)^{-1} \lesssim \omega/\zeta$ and $|\Delta| \gg \omega^2/\zeta$ this is still true, but in the region $|\Delta| \lesssim \omega^2/\zeta$ the dependence on the impurities is more complicated.

We note here that for $T \gg \omega^2/\zeta$ and arbitrary $\omega\tau_0$ we have as a valid formula for χ_2:

$$\chi_2 = 2\frac{m^{3/2}}{\omega^{1/2}}\left(\frac{e}{m}\right)^2\left(\frac{\zeta}{\omega}\right)^2 kT\sum_1^\infty(-1)^l\sqrt{l}\,\frac{\exp(-\pi^2 l/\omega\tau_0)}{\sinh(2\pi^2 l kT/\omega)}$$
$$\times\cos\left(2\pi l\frac{\zeta}{\omega}-\frac{\pi}{4}\right). \qquad (32)$$

In conclusion I express my deep gratitude to Academician L. D. Landau, and also to A. A. Abrikosov, L. P. Gor'kov, and I. M. Khalatnikov for helpful advice and discussions.

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Translated by W. H. Furry
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