THE RADIATION FROM A CHARGE MOVING IN AN INHOMOGENEOUS MEDIUM

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The problem considered is that of the radiation from a uniformly moving charge in a medium with random inhomogeneities. Expressions are obtained for the intensity, the polarization, and the directivity of the radiation. It is pointed out that as the threshold for the Cerenkov effect is approached there is a logarithmic increase of the radiated intensity.

A charge moving uniformly in an inhomogeneous medium must give rise to radiation due to the polarization of the medium. If the inhomogeneities of the medium are randomly distributed, then we get incoherent radiation caused by the polarization of these inhomogeneities. This effect can be regarded as a special type of transition radiation at the inhomogeneities of the medium. It is obvious that these phenomena are analogous to the scattering of light and can be described in terms of the scattering of the waves that accompany the motion of the particle. Unlike the Cerenkov effect, the radiation at inhomogeneities of the medium will occur also at speeds smaller than the phase velocity of light in the medium.

For the calculation we represent the field caused by the motion of the charge as a sum of plane waves (cf. e. g., reference 1). When a charge moves with the velocity v the field produced is excited by the charge and current density distributions

$$\rho = e\delta(\mathbf{r} - \mathbf{v}t), \qquad j = e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t).$$

When the electromagnetic field from these sources is represented as Fourier integrals describing the variation with the coordinates, the field is described by the following Fourier components $\mathbf{A_k}$ of the vector potential and $\varphi_\mathbf{k}$ of the scalar potential

$$\mathrm{A_k} = \frac{e}{2\pi^2 c}\frac{\mathbf{v}}{k^2 - \omega^2\varepsilon/c^2} e^{-i\omega t}, \qquad \varphi_\mathbf{k} = \frac{e}{2\pi^2\varepsilon}\frac{1}{k^2 - \omega^2\varepsilon/c^2} e^{-i\omega t}.$$

From this we have for the spatial Fourier components of the electric field, which depend on the time,

$$\mathbf{E_k} = \frac{i\omega}{c}\mathbf{A_k} - i\mathbf{k}\varphi_\mathbf{k} = \frac{ie}{2\pi^2}\frac{\omega\mathbf{v}/c^2 - \mathbf{k}/\varepsilon}{k^2 - \omega^2\varepsilon/c^2} e^{-i\omega t}, \qquad (1)$$

where k is the wave vector; $\epsilon = \epsilon(\omega)$ is the dielectric constant at frequency ω; $\omega = \mathbf{k}\cdot\mathbf{v} = k_x v$; v is the speed of motion along the x axis.

Let us consider the radiation from the fields at the inhomogeneities of the medium, using the usual arguments of scattering theory. For scalar scattering on random inhomogeneities of the dielectric constant of the medium, the radiation is emitted from volumes that are small in comparison with the wavelength. This corresponds to Rayleigh scattering at small particles or at fluctuations of the density of the medium, on the assumption of dipole radiation from the scatterers.

The total intensity of the incoherent radiation in the frequency range $d\omega$ is obtained by summing the intensities radiated from the individual scatterers over the entire volume V:

$$dI_\omega = \frac{8\pi\omega^4\sqrt{\varepsilon}}{3c^3}\int_V N\,|\mathbf{P}_\omega|^2\,dV\,d\omega,$$

where N is the number of scattering centers in unit volume, and $\mathbf{P}_\omega$ is the Fourier component of the dipole moment of a scattering element with the polarizability α. Since $\mathbf{P}_\omega = \alpha\mathbf{E}_\omega$, the total intensity is given by

$$dI_\omega = \frac{8\pi\omega^4\sqrt{\varepsilon}N\alpha^2}{3c^3}\int |\mathbf{E}_\omega|^2\,dV\,d\omega.$$

We see that the scattering properties of the medium are entirely determined by the extinction coefficient

$$s = \frac{8\pi\omega^4}{3c^4}N\alpha^2 = \frac{\omega^4}{6\pi c^4}V\overline{\delta\varepsilon^2}. \qquad (2)$$

The extinction coefficient has the dimensions of an inverse length. The distance $L = 1/s = 1/N\sigma$ is essentially the range of radiation in the scattering medium; L is determined by the scattering either at fluctuations of ϵ or at scattering particles which are characterized by a cross section σ.

The integral of $|\mathbf{E}_\omega|^2$ over a volume that is unlimited in the y and z directions and is of length l in the x direction can be expressed in terms of the Fourier components E_k by the formula

$$\iiint |\mathbf{E}_\omega|^2 \, dx\, dy\, dz = \frac{4\pi^2}{v^2}\int_0^l dx \iint_{-\infty}^{+\infty} \mathbf{E}_k \mathbf{E}_k^* \, dk_y\, dk_z.$$

Thus we get for the radiation from unit path length in the frequency range $d\omega$ the expression

$$J_\omega = \frac{dI_\omega}{l\,d\omega} = \frac{4\pi^2 sc\sqrt{\varepsilon}}{v^2}\iint_{-\infty}^{+\infty} \mathbf{E}_k \mathbf{E}_k^* \, dk_y\, dk_z.$$

Using the relations (1) and (2), we break the integrand up into two parts,

$$J_\omega = \frac{2sc^2\sqrt{\varepsilon}}{c\beta^2\varepsilon^2}\iint\left\{\frac{(\omega/v)^2(\varepsilon\beta^2-1)^2}{[(\omega/v)^2(1-\varepsilon\beta^2)+k_y^2+k_z^2]^2}\right.$$
$$\left. + \frac{k_y^2+k_z^2}{[(\omega/v)^2(1-\varepsilon\beta^2)+k_y^2+k_z^2]^2}\right\} dk_y\, dk_z, \qquad (3)$$

where $\beta = v/c$. For the integration it is convenient to change to polar coordinates in the plane of k_y and k_z:

$$p^2 = k_y^2 + k_z^2.$$

In the first part of the integral we have separated out the terms that are due to the polarization of the volume by the field E_x directed along the motion of the charge. This integral converges at all values of p^2 and gives

$$J_{\omega x} = \frac{2\pi e^2 s}{c\beta^2\varepsilon^{3/2}}(1-\varepsilon\beta^2). \qquad (4)$$

The integration of the second part of Eq. (3) gives the radiation that is due to the polarization of the volume by cylindrical waves with the fields E_y, E_z,

$$J_{\omega yz} = \frac{2\pi e^2 s}{c\beta^2\varepsilon^{3/2}}\left(\ln\frac{p_\infty^2 v^2}{\omega^2(1-\varepsilon\beta^2)} - 1\right), \qquad (5)$$

where p_∞ denotes the upper limit of the integration with respect to p.

Let us examine the behavior of the integrals for $\epsilon\beta^2 < 1$, i.e., below the threshold of the Cerenkov effect. Unlike the integral (4), the integral (5) diverges logarithmically at the upper limit, which corresponds to small distances. At small distances, however, the microscopic treatment of the scattering is illegitimate, and therefore we can simply introduce a certain limiting frequency $\omega_\infty = p_\infty/v$, corresponding to this restriction. Thus the complete spectral density of the energy radiated by the charge per unit path length is given by

$$J = \frac{\pi e^2 s}{c\varepsilon^{3/2}\beta^2}\left(2\ln\frac{\omega_\infty}{\omega\sqrt{1-\varepsilon\beta^2}} - \varepsilon\beta^2\right). \qquad (6)$$

Let us find the distribution of the polarization and intensity of this radiation. In view of the dipole character of the radiation from the scatterers, we get for the angular distribution of the radiation

$$dJ = \frac{3}{8\pi}\left[J_{\omega x}\sin^2\vartheta + \frac{1}{2}J_{\omega yz}(1+\cos^2\vartheta)\right]d\sigma, \qquad (7)$$

where $d\sigma$ is an element of solid angle and ϑ is the angle between the direction of observation and the motion of the charge.

A quantity of practical importance is the ratio of the intensities emitted in directions along the path and transverse to the path, which is given by

$$\frac{J_\parallel}{J_\perp} = \frac{\ln[\omega_\infty/\omega\sqrt{1-\varepsilon\beta^2}] - 1/2}{\ln[\omega_\infty/\omega\sqrt{1-\varepsilon\beta^2}] - \varepsilon\beta^2 + 1/2}.$$

Since the radiation from J_x is directed only transverse to the path of the particle, the radiation directed along the path of the particle is not polarized, as is already clear from considerations of symmetry. The degree of polarization P of the radiation directed transverse to the path depends on the velocity:

$$\frac{P_\parallel}{P_\perp} = \frac{\ln[\omega_\infty/\omega\sqrt{1-\varepsilon\beta^2}] - 1/2}{1-\varepsilon\beta^2}. \qquad (8)$$

Let us express the formula (6) for the total intensity of the radiation in terms of the number dn of quanta of energy $\hbar\omega$ that are emitted in the frequency range $d\omega$ in unit path length:

$$dn = \frac{e^2 s\lambda}{\hbar c\pi\varepsilon\beta^2}\left[2\ln\frac{\omega_\infty}{\omega} - \ln(1-\varepsilon\beta^2) - \varepsilon\beta^2\right]\frac{d\omega}{c}, \qquad (9)$$

where $\lambda = 2\pi c/\omega\epsilon^{1/2}$ is the wavelength in the medium.

A basic feature of the radiation at random inhomogeneities of the medium is its logarithmic increase with approach to the condition $\epsilon\beta^2 \to 1$, since the velocity dependence of the other terms is small. The value of the limiting frequency ω_∞, or the corresponding value of the minimum impact parameter, determines the part of the radiation that has only a weak velocity dependence for large β.

The magnitude of the effect itself can be compared with the intensity of the Cerenkov radiation, which occurs in the medium for $\epsilon\beta^2 > 1$. In the notations we have been using the intensity of the Cerenkov radiation is

$$dn_C = \frac{e^2}{\hbar c\varepsilon\beta^2}(\varepsilon\beta^2 - 1)\frac{d\omega}{c}.$$

In the case of a gas the ratio of intensities n/n_C for $\beta \sim 1$ is given in order of magnitude by the ratio of the polarizability of the medium to its scattering properties,

$$n/n_C \approx \alpha/\lambda^3,$$

i.e., it is determined by the ratio of the polarizability χ of a molecule or particle to the cube of the wavelength, and does not depend on the number density.

The possibility of studying the effect depends on the size of the scattering in the medium. It is obvious that the size of the volume in which radiation from the charge occurs must be smaller than the range L of the radiation, in order for it to be permissible to neglect secondary scattering and depolarization of the radiation.

It would be interesting to observe the smearing of the Cerenkov radiation near the threshold owing to the logarithmic increase of the radiation from inhomogeneities of the medium.

If the medium contains atoms that give resonance scattering, they will also be excited, since the medium scatters strongly at these frequencies. This excitation of atoms will occur even if the density and effective cross section of these atoms does not lead to local reaching of the threshold for Cerenkov radiation at the resonance frequency. Thus there can be excitation of small impurities in a gas if the main substance does not have an anomalous polarizability at the frequency in question.

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[1] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media), Fizmatgiz, 1957.

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