QUANTUM OSCILLATIONS OF THERMODYNAMIC QUANTITIES FOR AN ARBITRARY FERMI SURFACE

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Thermodynamic quantities of an electron gas in a constant magnetic field **H** are calculated in the general case of a non-convex Fermi surface. It is shown that the presence of self-intersecting trajectories leads to quantum oscillations of these quantities as functions of H. It is pointed out that the experimentally observed oscillations corresponding to "anomalously weakly filled" bands may be due either to separated small surfaces, and are then described by the Lifshitz-Kosevich theory,[1] or to small bulges or depressions in the main large band, in which case they are described by the present theory.

1. INTRODUCTION

In a series of papers by I. Lifshitz and Kosevich (see, for example, reference 1), it has been shown that, owing to the strong Fermi degeneracy of the electron gas, quantization of the energy levels of an electron with an arbitrary dispersion law $\epsilon = \epsilon(\mathbf{p})$ (ϵ is the energy, $\mathbf{p}$ is the quasimomentum) in a constant magnetic field $\mathbf{H}(0, 0, H)$ leads to an increment, which is periodically dependent on 1/H to the thermodynamic quantities. The period in the reciprocal of the magnetic field $\Delta(1/H)$ is equal to

$$\Delta(1/H) = eh/cS_m(\zeta), \tag{1}$$

where $S_m(\zeta)$ is the extremal (with respect to p_z) area of the intersection of the boundary Fermi surface $\epsilon(p) = \zeta$ with the plane p_z = const. Only this cross sectional area enters, for the reason (as is easy to understand) that the fundamental contribution to the oscillating part of the statistical sum for the smooth function $S(\zeta, p_z)$ will be made by just those electrons of the narrow layer close to the extremal (but, naturally, not equal to zero) sections, where, in the classical case, the majority of the electrons move with a period close to the given period in the region

$$\Delta p_z \sim p_0(\mu H/\varepsilon_0)^{1/2}, \quad p_0(\mu H/\varepsilon_0) \ll \Delta p_z \ll p_0 \tag{2}$$

(p_0 is of the order of the limiting momentum in the direction p_z, ϵ_0 is the limiting energy, $\mu = e\hbar/m^*c$, and $m^* = (2\pi)^{-1}\partial S/\partial\epsilon$ is the effective mass of the electron).

It is clear from (2) that the relative contribution of the periodic part of the thermodynamical quantities (which is comparable to the part which is monotonically dependent on the magnetic field) is also of order $(\mu H/\epsilon_0)^{1/2}$. Inasmuch as the part Ω which depends monotonically on H is even and, consequently, is proportional to $(\mu H/\epsilon_0)^2$, the absolute value of the periodic increment to Ω is proportional to $(\mu H/\epsilon_0)^{5/2}$, and the increment to the magnetic moment, which is linear in H, is proportional to $(\mu H/\epsilon_0)^{3/2}$.

If the Fermi surface is convex, then the only non-monotonic part of the statistical sum is connected with the extremal cross sections. In the case of a non-convex surface, the area of the cross section is generally not a smooth function: at a certain point $p_z^0(\zeta)$, which corresponds to the self-intersecting classical orbit in a magnetic field, a transition takes place from one cross section to two separate cross sections (Fig. 1), while the derivatives $\partial S/\partial\zeta$, $\partial S/\partial p_z$ at this point go to infinity logarithmically for all three areas.

It is natural to expect that these "singular" cross sections also contribute an oscillating part to the statistical sum (see also reference 2). It is physically clear that inasmuch as the picture of levels takes on the ordinary "non-singular" form at distances as small as the order of the distance between levels, i.e., of the order of

$$\Delta p_z \sim p_0 \mu H/\varepsilon_0, \tag{3}$$

then the irregular part, which is connected with self-intersection, will be at least $(\epsilon_0/\mu H)^{1/2}$ times smaller than the periodic part due to extremal sections. The absolute value of the irregular increment to the linear momentum is consequently proportional at least to $(\mu H/\epsilon_0)^2$

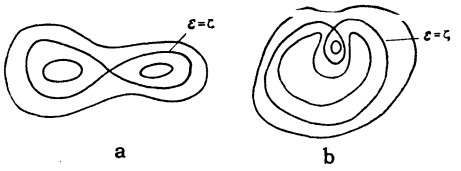

FIG. 1. a – case of "necking in," b – case of a "trough;" p_z = const.

[instead of $(\mu H/\epsilon_0)^{3/2}$ for the periodic part], while the contribution to Ω is proportional to $(\mu H/\epsilon_0)^3$ [instead of $(\mu H/\epsilon_0)^{5/2}$].

Thus the contribution to the oscillating terms from the cross sections with self-intersection is small in comparison with the known contribution of the extremal cross sections. However, the following circumstance must be kept in mind.

1. It is well known that the experimentally observed quantum oscillations (the De Haas-Van Alphen and the Shubnikov-De Haas effects) are brought about not by the main "large" bands, but by some anomalously weakly filled bands with density of the order of $10^{-2} - 10^{-6}$ electrons per atom. The genesis of these bands is completely unclear at the present time. It can be thought in each case that these bands are either separated small surfaces or small "bulges" or "depressions" in momentum space, which destroy the local bulge of the surface corresponding to a principal band.

Up to the present time only the first of these cases has been considered, although it does not follow at all that precisely this case takes place. Furthermore, it is difficult to ascertain experimentally which one of these actually does take place, since the oscillations have an entirely similar character in both cases, and the dependence of the amplitude of the oscillations on the magnetic field is difficult to determine unambiguously. The fact is that the amplitude of the oscillations is extremely sensitive to the mosaic structure, impurities, etc., and is least reliably established experimentally.

The interpretation of the observed effects can also be ambiguous. Thus, the directions in which one of the periods of the oscillations disappears can be interpreted either as the directions in which the cross sections with self-intersection disappear, or as the directions of open cross sections. Therefore, for explanation of this problem, it is necessary to draw upon resonance measurements in weak magnetic fields and on a study of quantum oscillations in high frequency and constant magnetic fields (see reference 3), in addition to a detailed investigation of the picture in strong magnetic fields. In such a case one makes use of the known noncentral character of cross sections with self-intersection and of the fact that the effective mass goes to infinity on these sections.

It is quite possible that both situations are realized in different metals.

2. Even if there is a case of a separated but non-convex surface, the "fine structure" due to self-intersection has quite an appreciable value, since for "small bands," $\epsilon_0/\mu H \sim 1 - 10^2$. Furthermore, the "fine structure" from similar bands can be larger (because of the smallness of the effective mass) than the principal structure from the "large" bands.

3. For a one-parameter family of directions of the constant magnetic field, there may in general be no cross section which is extremal in area, and the cross section with self-intersection may be the maximum in area. (Such a case exists, for example, for graphite.) In this case evidently only those oscillations remain which correspond to trajectories with self-intersection. In the investigation of the anisotropy of the effect, a sharp decrease in the amplitude of oscillations should be observed as one approaches a similar direction (this fall-off can also be ascribed to the approach to the open trajectories).

In the present paper, we calculate the thermodynamic potential Ω and the magnetic susceptibility in the general case of the presence of self-intersecting orbits.

2. GENERAL FORMULA FOR THE THERMODYNAMIC POTENTIAL Ω

1. It is well known that the thermodynamic quantities can be computed if the thermodynamic potential Ω, equal (per unit volume) to

$$\Omega = -\Theta \sum \ln\left[1 + \exp\left(\frac{\zeta - \varepsilon}{\Theta}\right)\right], \qquad \Theta = kT \tag{4}$$

is known. The summation is carried out over the quantum states, k is Boltzmann's constant, T is the temperature, ζ is the chemical potential.

Naturally, we will be interested only in the part of Ω connected with the electron gas, and therefore knowledge of the energy levels of the electrons in a constant magnetic field is sufficient in our case for the calculation of Ω. As was shown by the author,[2] the energy levels can be obtained from the two branches of the dispersion equation, which have the form

$$[2n_\pm(\varepsilon, p_z) + 1]\pi \equiv S_1 + S_2 + \varphi(k) \pm \cos^{-1}\{e^{-k\pi} \times (2\cos 2k\pi)^{-1/2}\cos(S_1 - S_2)\} = (2n+1)\pi, \tag{5}$$

in the case of the presence of "necking-in," and

in the case of the presence of a "trough"

$$[2n_\pm(\varepsilon, p_z)+1]\pi \equiv S_1 - S_2 + \varphi(k) \pm \cos^{-1}\{e^{k\pi}(2\cosh 2k\pi)^{-1/2} \times \cos(S_1+S_2)\} = (2n+1)\pi, \quad (6)$$

where n is an integer in each case,

$$\varphi(k) = 2\left\{k\ln\frac{|k|}{e} - \frac{1}{2i}\ln\frac{\Gamma(1/4+ik)}{\Gamma(1/4-ik)}\right\} - \tan^{-1}\tanh k\pi,$$

$$S(k) = \frac{c}{2e\hbar H}S(p_z), \quad k = \frac{\varepsilon-\varepsilon_0(p_z)}{2e\hbar H}c\sqrt{|m_x m_y|},$$

$$m_\alpha^{-1} = [\partial^2\varepsilon/\partial p_\alpha^2]_{p_x=p_y=0}, \quad (7)$$

$\epsilon_0(p_Z)$ is the energy for which self-intersection takes place for a given p_Z; the origin of the coordinates is located at the point of self-intersection for the given p_Z; the p_X and p_Y axes are directed along the bisectors of the angles formed by the trajectories at the point of intersection, so that $m_X > 0$, $m_Y < 0$. The quantity $S(p_Z)$ is the area in momentum space; in the case of "necking in," S_1 corresponds to the area bounded by the orbits to the left of the p_X axis and S_2 to the right of p_X; in the case of a "trough," S_1 corresponds to the total area bounded by the orbit, including the area up to the p_Y axis, while S_2 is the area of the "hole," including the area up to the p_Y axis (see Fig. 2). Equations (5), (6) are applicable in the quasi-classical case, that is, for $S_{1,2}(k) \gg 1$ and, consequently, $n \gg 1$.

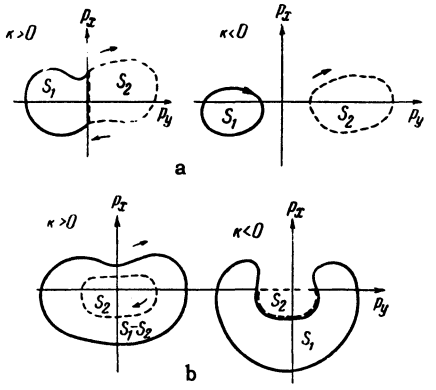

FIG. 2. a – case of necking in, b – case of a trough; the direction of the classical motion of the electron is shown by the arrow.

In accord with (5), (6), the energy depends on n and p_Z and, as a function of these quantities, divides into two branches:

$$\varepsilon = \varepsilon_\pm(n, p_z). \quad (8)$$

(For simplicity, we shall not write down the spin component $\pm e\hbar/2m_0c$, m_0 is the mass of the free electron, since it can be established that it has no effect on either the general course of the discussion nor on the final formula if we take into account the spin pair before the statistical sum.)

2. Corresponding to (8), we have

$$\Omega \sim -2\Theta\int_{-\infty}^{\infty}dp_z\sum_{\pm}\sum_{n_\pm^{\min}}^{\infty}\ln\left\{1+\exp\left(\frac{\zeta-\varepsilon_\pm(n,p_z)}{\Theta}\right)\right\}$$

$$(n_\pm^{\min} = \min_\varepsilon n_\pm(\varepsilon, p_z)). \quad (9)$$

The coefficient of proportionality is the same as in the case of free electrons considered by Landau,[4] and is equal to $eH/\hbar^2c$. This can be established, for example, by considering the electrons to be located in a box, where all the quantum numbers are discrete, and by calculating Ω as the dimensions of the box approach infinity. The thermodynamic potential is found either directly from Eq. (4) or by the equation

$$\Omega = -\int N(\zeta)\,d\zeta, \qquad N = \sum N_n|\psi_n|^2,$$

where $n \sim n_1, n_2, n_3$ are the quantum numbers, while ψ_n is the wave function (which is easily found in the quasi-classical region just as in reference 2). Detailed calculation shows that the Landau factor is not changed if there is degeneracy in the generalized momentum P_X, that is, if the levels are computed with accuracy up to $O(h^2)$, while the wave functions are computed with accuracy up to $O(h)$. Inasmuch as we are interested only in the oscillating part of $\Delta\Omega$ as a function of H^{-1}, and the lower levels ($n \sim 1$), as can be shown, give only the part of Ω proportional to H^2, summation in (9) with the previous accuracy can be carried out from n = 0. (Strictly speaking, this only makes it possible to calculate $\Delta\Omega$, since for $n \sim 1$ the fundamental equations (5), (6) are invalid.)

Thus Ω_1 (by Ω_1 we mean any function which has the same oscillating part as Ω) has the form

$$\Omega_1 = -\frac{2eH\Theta}{ch^2}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}dp_z\left[\ln\left(1+\exp\left\{\frac{\zeta-\varepsilon_+(n,p_z)}{\Theta}\right\}\right) + \ln\left(1+\exp\left\{\frac{\zeta-\varepsilon_-(n,p_z)}{\Theta}\right\}\right)\right]. \quad (10)$$

We now make use of Poisson's formula:

$$\sum_{n=0}^{\infty}\varphi(n) = \tfrac{1}{2}\varphi(0) + \int_0^\infty\varphi(n)\,dn + 2\,\mathrm{Re}\sum_{k=1}^{\infty}\int_0^\infty dn\,\varphi(n)\,e^{2\pi ikn}. \quad (11)$$

It can be shown that the first two components in (11) do not give terms which oscillate with the magnetic field and therefore can be omitted in Ω_1.

Integrating the remaining expression by parts, and carrying out the substitution of ϵ^+ for n in

the first integral, and ϵ^- in the second [such a substitution of variables is possible because of the monotonic character of $n_\pm(\epsilon)$, which is shown in reference 2, and which is determined by Eqs. (5) and (6)], and setting the lower limit of integration over ϵ_+ and ϵ_- at zero (since only $n \gg 1$ are important), we obtain

$$\Omega_1 = -\frac{2eH}{\pi ch^2}\,\mathrm{Im}\sum_{k=1}^{\infty}\frac{1}{k}\int_{-\infty}^{\infty}dp_z\int_0^{\infty}f_0\left(\frac{\varepsilon-\zeta}{\Theta}\right)\exp\{2\pi i k n_\pm(\varepsilon, p_z)\}\,d\varepsilon. \tag{12}$$

Making use of the formula

$$\sum_{k=1}^{\infty}\frac{\sin 2\pi kx}{\pi k} = \tfrac{1}{2} - x + [x] \equiv \psi(x) \tag{13}$$

(it is easy to establish the validity of this relation by expanding $\psi(x)$ in a Fourier series), where [x] is the largest integer contained in x, we can write

$$\Omega_1 = -\frac{2eH}{ch^2}\int_{-\infty}^{\infty}dp_z\int_0^{\infty}f_0\left(\frac{\varepsilon-\zeta}{\Theta}\right)d\varepsilon\,\{\psi(n_+(\varepsilon, p_z)) + \psi(n_-(\varepsilon, p_z))\}. \tag{14}$$

Knowing Ω_1 it is easy to obtain the oscillating part ΔN of the number of electrons N. For this purpose, it suffices to know the value of N_1 which has the same oscillating part as N:

$$N_1(\zeta, H, \Theta) = -\frac{\partial\Omega}{\partial\zeta} = -\frac{2eH}{ch^2}\int_{-\infty}^{\infty}dp_z\int_0^{\infty}f_0'(\varepsilon)\,d\varepsilon\,\{\psi(n_+) + \psi(n_-)\}. \tag{15}$$

The fact that we have obtained the oscillation of a number of electrons should not be remarkable, since all the quantities were determined in the convenient independent variables ζ and H. Actually, $\zeta = \zeta(H)$, where the form of the function must again be found from the requirement of the constancy of the number of particles; the oscillating part of $\zeta(H)$ is again determined by the absence of oscillations in the number of particles.

Equation (15) takes on an especially simple form at absolute zero temperature, when $f_0'(\epsilon) = -\delta(\epsilon-\zeta)$ (the right-hand side is the Dirac δ function), and

$$\Delta N(\zeta, H, 0) = \frac{2eH}{ch^2}\int_{p_z^{\min}}^{p_z^{\max}}dp_z\,\{\psi(n_+(\zeta, p_z)) + \psi(n_-(\zeta, p_z))\}$$

$$(p_z^{\min} = [\min_{n_\pm} p_z(\varepsilon, n_\pm)]_{\varepsilon=\zeta}). \tag{16}$$

Comparing (14) and (16), we find

$$\Delta\Omega(\zeta, H, \Theta) = \int_0^{\infty}f_0\left(\frac{\varepsilon-\zeta}{\Theta}\right)\Delta N(\varepsilon, H, 0)\,d\varepsilon. \tag{17}$$

Thus the determination of $\Delta\Omega$ reduces to finding the fluctuations of the number of particles at absolute zero as a function of the chemical potential and the magnetic field. It is easy to see that, with accuracy up to terms exponentially small in ζ/Θ,

$$\Delta\Omega(\zeta, H, \Theta) = \int_0^{\zeta}\Delta N(x)\,dx + \Theta\int_0^{\infty}f_0(x)\{\Delta N(\zeta+\Theta x) - \Delta N(\zeta-\Theta x)\}\,dx. \tag{18}$$

3. We now calculate how the limiting transition to the ordinary formula comes about at $\epsilon = \zeta$ for p_z far from the self-intersection. For such p_z, we have $|k| \gg 1$, and, throwing away only terms which are exponentially small in Eqs. (5) and (6), we obtain the following expression for the integrand in (16):

a) when $k > 0$ for "necking-in" or when $k < 0$ for a "trough":

$$\psi(n_+) + \psi(n_-) = \psi\left(\frac{S_1+S_2+\varphi}{\pi} - \frac{1}{2}\right), \tag{19}$$

since

$$\psi(x+1) = \psi(x); \quad \psi(x+\tfrac{1}{2}) + \psi(x) = \psi(2x); \tag{20}$$

b) when $k > 0$ for "troughs" or when $k < 0$ for "necking-in":

$$\psi(n_+) + \psi(n_-) = \psi\left(\frac{S_1+\varphi/2}{\pi} - \frac{1}{2}\right) + \psi\left(\frac{S_2+\varphi/2}{\pi} - \frac{1}{2}\right), \tag{21}$$

since

$$\cos^{-1}\{\cos(S_1-S_2)\} = \frac{\pi}{2} - (-1)^{[(S_1-S_2)/\pi]}\,\psi\left(\frac{S_1-S_2}{\pi}\right), \tag{22}$$

$$\psi\{\alpha - \tfrac{1}{2} - \tfrac{1}{2}(-1)^{[\beta]}\psi(\beta)\} + \psi\{\alpha + \tfrac{1}{4} + \tfrac{1}{2}(-1)^{[\beta]}\psi(\beta)\} = \psi\left(\alpha + \frac{\beta-1}{2}\right) + \psi\left(\alpha - \frac{\beta-1}{2}\right). \tag{23}$$

One can establish the validity of the functional relationship (20) by making use of the definition of the function ψ and considering separately the cases $x - [x] > \tfrac{1}{2}$ and $x - [x] < \tfrac{1}{2}$. The relation (23) is obtained if we consider the even and odd $[\beta]$ separately and make use of Eq. (20).

As has already been shown,[2] the function $\varphi(k)$, which approaches zero as $1/|k|$ when $|k| \to \infty$, is retained in (19) and (21) because it gives a small correction to the usual rule of quasi-classical quantization for $|k| \gg 1$. The fact that Eqs. (19) and (21) in the quasi-classical case, for $\varphi(k) = 0$ in the variable ζ, H, give a fluctuation in the number of levels (proportional to the number of electrons), is self-evident.

We note that for $|k| \gg 1$, Eqs. (19) and (21) coincide with (16), (5), and (6) within experimental accuracy.

4. We now transform (16) to a form which is convenient for the calculation of ΔN. For brevity, we shall demonstrate all the transformations by an example of the "necking-in." We make use of the equation

$$\psi(n_+) = \frac{1}{\pi} \operatorname{Im} \sum_{l=1}^{\infty} \frac{1}{l} \exp\{2\pi i l \ln_+(p_z)\}. \qquad (24)$$

Let the condition $n'_+(p_z^{(k)}) = 0$ be satisfied at the points $p_z^{(k)}$. We circle these points with cuts of length $\omega \to 0$, and outside of these cuts we displace the integration contours in (16) upwards, where the derivative $n'_+ > 0$, and downward where $n'_+ < 0$ (Fig. 3). On the displaced portions, the series (24) converges and can be summed:

$$\psi(n_+) = \frac{1}{\pi} \operatorname{Im} \ln(1 - e^{2\pi i n_+}). \qquad (25)$$

Inasmuch as the integral over cuts of length 2ω [both from (24) and (25)] tend to zero as $\omega \to 0$,

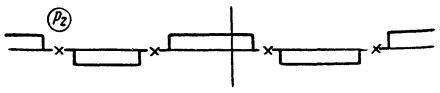

FIG. 3

on the contour the cuts can be moved on the real axis ($\omega = 0$), and we can write the expression (25) under the integral everywhere. We can now again restore the contour to the real axis everywhere except for points on the axis where n_+ is equal to an integer. These points must be passed around from above if $n'_+ > 0$ at them, and below if $n'_+ < 0$.

Similar considerations are carried out for $\psi(n_-)$.

Summing $\psi(n_+)$ and $\psi(n_-)$, we obtain

$$\Delta N = \frac{2eH}{ch^2} \operatorname{Im} \int dp_z \ln(1 + 2tx + t^2); \qquad (26)$$

$$t = e^{if} = \exp\{i[S_1 + S_2 + k \ln(k^2/e^2)]\} \frac{\Gamma(1/4 - ik)}{\Gamma(1/4 + ik)} \frac{\cosh k\pi - i \sinh k\pi}{\sqrt{\cosh 2k\pi}},$$

$$x = (2 \cosh 2k\pi)^{-1/2} e^{-k\pi} \cos(S_1 - S_2),$$

$$1 + 2tx + t^2 = (1 - e^{2\pi i n_+})(1 - e^{2\pi i n_-}),$$

$$n_\pm = \exp\{if \pm i \cos^{-1} x\}. \qquad (27)$$

The integration is carried along the real axis everywhere where the expression $1 + 2tx + t^2 \neq 0$; points where this expression vanishes are surrounded from above or below in correspondence with the rule given above.

Integrating (26) by parts and discarding the monotonic part outside the integral sign, we obtain

$$\Delta N = \frac{2eH}{h^2 c} \operatorname{Im} \int p_z dp_z (2tx + t^2)'_{p_z} / (1 + 2tx + t^2), \qquad (28)$$

where the rule of going around the poles of the denominator remains as before.

5. Thus, as is seen from the foregoing, the behavior of the functions $n'_\pm(p_z)$ plays a vital role. We therefore investigate this function. First, we note that since only the p_z close to $p_z^0(\epsilon)$ are important, i.e., close to those p_z for which, for a given ϵ, there is self-intersection (this statement is clear physically and will be rigorously demonstrated in what follows), while for $p_z = p_z^0(\epsilon)$, by definition, $k = 0$, $v_x = v_y = 0$ and $d\epsilon_0/dp_z = \partial\epsilon_0/\partial p_z = v_z(\epsilon)$, then we have

$$\frac{2e\hbar H}{c} k(p_z) = \frac{dk}{dp_z}(p_z - p_z^0(\varepsilon)) = -(p_z - p_z^0(\varepsilon)) \sqrt{|m_x m_y|}\, v_z(\varepsilon)$$

$$(\varepsilon \approx \zeta), \quad p_z = p_z^0(\zeta) - \frac{2\hbar\Omega(\zeta)}{v_z^0(\zeta)} k,$$

$$\Omega^{-1}(\zeta) = \frac{c}{eH} \sqrt{|m_x m_y|}\Big|_{\varepsilon = \zeta,\ p_z = p_z^0(\zeta)}. \qquad (29)$$

Here $v_z^0(\zeta) \neq 0$, since the four equalities $\epsilon = \zeta$, $v_x = v_y = v_z = 0$ are generally incompatible. It is clear that (29) is approximately true for any $|p_z - p_z^0(\epsilon)|$ which is small in comparison with the "width" in p_z of the surface $\epsilon(p) = \zeta$.

It is clear from (29) that $dn_\pm/dp_z \sim dn_\pm/dk$. From the definition of $n_\pm(k)$ according to (5), and from the form of S_1, S_2 for $k \ll k_0$, found in reference 2 [Eq. (1.10), where one must substitute the expression (29) in $S_{1,2}(\epsilon_0(p_z), p_z)$ and consider that $\epsilon_0(p_z) \approx \zeta$], it is easy to see that

$$n'_\pm(k) \sim [2 \ln k_0 + q(k)]/(2\pi), \quad q(k) \sim 1, \quad k_0 \sim \zeta/\hbar\Omega \gg 1.$$

For $k_0 \to \infty$, we have $n'_\pm(k) > 0$ for any k. Thus, in the case of interest to us, all the "dangerous" points on the p_z axis (for $k \ll k_0$) are bypassed from above.

3. CALCULATION OF THE FLUCTUATING PART OF Ω

1. We shall first compute the quantity ΔN. We divide the integration over p_z into three parts, corresponding to values of k in the interval $(-N, N)$ $(1 \ll N \ll k_0)$ and outside this interval.

For p_z corresponding to k outside the given interval, we have

$$1 + 2tx + t^2 = \begin{cases} 1 + e^{2i(S_1+S_2)}, & k \gg 1 \\ (1 + e^{2iS_1})(1 + e^{2iS_2}), & -k \gg 1 \end{cases}, \qquad (30)$$

and calculation can easily be carried out. In this region, naturally, terms appear which correspond

to $k = \pm N$ which are obviously contracted with similar terms obtained in integration in the interval corresponding to $-N \leq k \leq N$, and terms obtained by Lifshitz and Kosevich,[1] corresponding to the extremal values of the area at large distances from the cross section with self-intersection, where either $S_1'(p_z)$ or $S_2'(p_z)$ goes to zero for $k < 0$ or $(S_1 + S_2)'_{p_z}$ vanishes for $k > 0$.

One can show that, for example, for $v_z > 0$, the quantity $(S_1 + S_2)'_{p_z}$ vanishes in any case. In fact, for self-intersection, we have $p_z = p_z^0(\zeta)$ and $(S_1 + S_2)'_{p_z} = -\infty$. Consequently, with decrease of p_z, the positive quantity $S_1 + S_2$ close to this point increases and, since the quantity $S_1 + S_2$ must be equal to zero for $p_z = p_z^{min}$, then $(S_1 + S_2)'_{p_z}$ must be equal to zero for some p_z. This extremum of $S_1 + S_2$ must correspond to $k \sim 1$ only close to the chosen directions of the magnetic field. However, in this case also, the terms obtained by Lifshitz and Kosevich are absent only for the given surface, while for other surfaces (on which $\epsilon(\mathbf{p}) = \zeta$ is decomposed) they can be present.

We shall now make clear what yields the region corresponding to $-N \leq k \leq N$. In this region one can, by using Eqs. (26) and (29) and the fact that

$$S_{1,2} = S_{1,2}^0 + k \ln (k_0^{(1,2)} / e|k|), \qquad S_{1,2}^0 = S_{1,2}(\zeta, p_z^0(\zeta)),$$
$$k_0^{(1)} \sim k_0^{(2)} \gg 1, \qquad k_0^{(1,2)} \sim 1/H, \tag{31}$$

write ΔN in the form

$$\Delta N = \frac{2eH\Omega(\zeta)}{\pi h c v_z^0(\zeta)} \operatorname{Im} \int_{-N+i0}^{N+i0} \ln(1 + 2tx + t^2)\, dk;$$

$$t = \exp\{i(S_1^0 + S_2^0) + ik \ln k_0^{(1)} k_0^{(2)}\}$$
$$\times \frac{\Gamma(1/4 - ik)}{\Gamma(1/4 + ik)} \frac{\cosh k\pi - i \sinh k\pi}{\sqrt{\cosh 2k\pi}},$$

$$x = e^{-k\pi}(2 \cosh 2k\pi)^{-1/2} \cos(S_1^0 - S_2^0 + k \ln k_0^{(1)} / k_0^{(2)}). \tag{32}$$

In the calculation of the above integral we consider the contour shown in Fig. 4, where $1 \ll N' \ll k_0$. The integral over the upper horizontal is exponentially small (in N'), the integrals along the verticals, which are the continuation of the contour on the horizontal axis, are not of interest to us. Therefore, the integral from $-N$ to N is essentially equal to the integral along the contour

$(-N, N;\ N, N + iN';\ N + iN',\ -N + iN';$

$-N + iN',\ -N).$

Inside this contour the quantity $1 + 2tx + t^2$ has, for $k_0 \gg 1$, pairs of poles located close to one another at the points $(1 + n/2)\, i/4$ $(n = 0, 1, \ldots)$ and roots near these points. The integral along

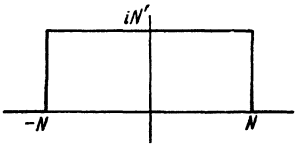

FIG. 4

the contour therefore reduces to the sum of integrals over contours surrounding each pair. Each of these integrals is equal to $2\pi i \Delta k$, where $\Delta k = k_p - k$ is the complex distance between the neighboring poles and the origin. Inasmuch as only the pair closest to the horizontal axis is important,

$$\Delta N = \frac{2\sqrt{\pi} eH\Omega(\zeta)}{hcv_z^0(\zeta)} (k_0^{(1)} k_0^{(2)})^{-1/4} \left\{ (k_0^{(2)} / k_0^{(1)})^{1/4} \sin\left(2S_1^0 - \frac{\pi}{4}\right) \right.$$
$$\left. + (k_0^{(1)} / k_0^{(2)})^{1/4} \sin\left(2S_2^0 - \frac{\pi}{4}\right) \right\} = \sum_{\alpha=1}^{2} A_\alpha \sin\left(2S_\alpha^0 - \frac{\pi}{4}\right); \tag{33}$$

it must be that

$$\Delta N \sim H / k_0^{1/2} \sim H^{3/2}.$$

Thus the amplitude of the fluctuations resulting from the self-intersecting cross sections is shown to be $(\epsilon_0/\mu H)$ times smaller than the amplitude of the oscillations from the extremal cross section; the oscillations have a simple periodic character with frequencies $2S_1^0$ and $2S_2^0$ [$S(k) \neq S(p_z)$!, see the definition (7) of the function $S(k)$].

In all the discussions given above it was not explicitly assumed that $\Omega \neq \infty$, that is, that neither m_x nor m_y is equal to zero. For the chosen directions of H (which form a single-parameter family) it is possible that $m_x = 0$ or that $m_y = 0$. Naturally this changes somewhat the structure of the levels and leads to an increase in the amplitude of oscillations [approximately by a factor of $(\epsilon_0/\mu H)^{1/4}$]. For isolated directions, where $m_x = m_y = 0$, the amplitude of the oscillations on cross sections with self-intersection can be shown to be of the same order as at the extremal.

Thus the picture of the quantum oscillations changes materially in the approach to certain chosen directions. We also note that in the case of a strong anisotropy, one of the values of $k_0^{(1,2)}$ can be shown to be of the order of unity. In this case, there arise complicated irregular oscillations.

2. Inasmuch as $S[\epsilon, p_z^0(\epsilon)]$ is a function of ϵ which does not have a logarithmic singularity, the calculation of $\Delta\Omega$ and its temperature dependence is carried out, starting from Eqs. (18), (33), with accuracy the same as given by Lifshitz and Kosevich,[1] and yields

$$\Delta\Omega(\zeta, H, \Theta) = \sum_{\alpha=1}^{2} \frac{2\pi e\hbar H}{\widetilde{m}_\alpha c} \frac{2\pi^2 c\Theta\widetilde{m}_\alpha / e\hbar H}{\sinh(2\pi^2 c\Theta\widetilde{m}_\alpha / e\hbar H)} A_\alpha \sin\left(2S_\alpha^0 - \frac{\pi}{4}\right), \tag{34}$$

$$\tilde{m}_\alpha = \frac{1}{2\pi}\frac{dS(\varepsilon, p_z^0(\varepsilon))}{d\varepsilon}. \quad (35)$$

From these formulas, it is easy to find all the thermodynamic quantities. Strictly speaking, one would have to compute the dependence of $\zeta(H)$, but it is easy to see that, as in reference 1, the consideration of this dependence does not change the form of the equations, in which one can write $\zeta(0)$ as before.

I am indebted to I. M. Lifshitz for valuable discussions.

[1] I. M. Lifshitz and A. M. Kosevich, JETP **29**, 730 (1959), Soviet Phys. JETP **2**, 636 (1956).

[2] M. Ya. Azbel', JETP (in press).

[3] M. Ya. Azbel', JETP **34**, 969, 1158 (1958), Soviet Phys. JETP **7**, 669, 801 (1958), Phys. Chem. Solids **7**, 105 (1958).

[4] L. D. Landau, Z. Physik **64**, 629 (1930).

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157