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ON THE SINGULARITIES OF COSMOLOGICAL SOLUTIONS OF THE GRAVITATIONAL EQUATIONS

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We obtain a broad class of cosmological solutions of the gravitational equations, which contains seven arbitrary physically different functions of the spatial coordinates. This number is only one less than the number of functions necessary for the description of an arbitrary initial distribution of the matter and gravitational field in the general case.

1. FORMULATION OF THE PROBLEM

THE particular classes of cosmological solutions of the equations of gravitation, obtained in the preceding communication[1] (referred to henceforth as I), show that the presence of singularities is, in any case, a rather broad property of such solutions. This is evidenced by various exact solutions (i.e., those valid over all of space at all instants of time), obtained by different authors under definite, sometimes special, assumptions concerning their form (see, for example, references 2 and 3).

However, all these solutions can by themselves not answer the main question of whether the presence of a singularity is a general property of the cosmological solutions, not connected with any specific assumption regarding the character of the distribution of matter and of the gravitational field. An affirmative answer to this question would mean that the equations of gravitation have a general solution with a singularity and with as many arbitrary functions of the coordinates as are necessary to specify the arbitrary initial conditions at a certain instant of time. To the contrary, the lack of a solution (with singularity) with this number of arbitrary functions would denote that the case of arbitrary distribution of matter and field does not, generally speaking, lead to the presence of a singularity.

We thus arrive at the following formulation of the problem: assuming the singularity to exist, it is required to find near the singularity the form of the broadest class of solutions of the equations of gravitation in such a way, that we can judge whether this solution is general from the number of the arbitrary functions of the coordinates contained in this solution.

Among the arbitrary functions contained in any given solution of the equations of gravitation there are, generally speaking, such whose arbitrariness is merely due to the arbitrary choice of reference frame. We obviously need be interested only in the "physically different" arbitrary functions, the number of which cannot be reduced by any choice of reference frame. From physical considerations it is readily seen that the number of such functions should in the general case be equal to eight: the arbitrary initial conditions should specify the initial spatial distribution of the density of matter, its three velocity components, and four additional quantities which determine the free gravitational field (i.e., the field not connected with the matter). We can arrive at this last number, for example, by considering weak gravitational waves: by virtue of their transverse nature, their field is determined by two quantities (the components g_{ik}) which satisfy a second-order equation (the wave equation), and therefore the initial conditions for them should be specified by four functions of the coordinates.

We shall use here, as in I, a reference frame subject to conditions I (1.3): $g_{0\alpha} = g_{00} = -1$. L. D. Landau has indicated long ago that in such a system one of the equations of gravitation [Eq. I (1.4)] makes it immediately possible to prove that the determinant g must vanish within a finite time (this was also noted by Komar[4]). This fact, however, does not in itself prove in any manner the necessity for the existence of a true physical singularity in the solutions, since the singularity (the vanishing of g) may prove to be fictitious, and may disappear on going to other reference frames. Furthermore, V. V. Sudakov has indicated that in the given case such a fictitious singularity should exist by virtue of the character of the chosen reference frame. It is easily seen that in this system the time lines (i.e., the lines $x^1, x^2, x^3 = \text{const}$) represent a family of geodesics. But the lines of such a family, on which no special parallelness conditions are imposed, will generally intersect each other on certain hypersurfaces — four-dimensional analogs of the caustic surfaces in geometrical optics. On the other hand, the intersection of the coordinate lines denotes the vanishing of the corresponding components of the metric tensor, and the determinant g also vanishes. The metric will therefore have a singularity on the indicated hypersurfaces, but not a physical one.*

In the present communication we give a very broad class of solutions of the equations of gravitation, obtained during the course of the indicated program. These solutions have physical singularities, but the class is still not general; it contains seven arbitrary functions, i.e., only one less than required in the general case.

2. CASE OF EMPTY SPACE

We begin the construction of this solution with the case of empty space.

In the absence of matter, the right halves of the general equations I (1.4) — (1.6) are replaced by zeroes; we rewrite these equations in the form

$$R_0^0 = \frac{1}{2}\frac{\partial^2}{\partial t^2}\ln(-g) + \frac{1}{4}\varkappa_\alpha^\beta \varkappa_\beta^\alpha = 0, \tag{2.1}$$

$$R_\alpha^0 = \frac{1}{2}\frac{\partial^2}{\partial x^\alpha \partial t}\ln(-g) - \frac{1}{2}\varkappa^\beta_{\alpha;\,\beta} = 0, \tag{2.2}$$

$$R_\alpha^\beta = P_\alpha^\beta + \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{-g}\varkappa_\alpha^\beta) = 0. \tag{2.3}$$

We seek a first-approximation solution of these equations near the singularity (principal terms of the expansion) in the form

$$g_{\alpha\beta} = t^{2p_1} l_\alpha l_\beta + t^{2p_2} m_\alpha m_\beta + t^{2p_3} n_\alpha n_\beta, \tag{2.4}$$

where **l**, **m** and **n** are three-dimensional vectors, which are functions of the coordinates. The exponents p_1, p_2, and p_3 may also be functions of the coordinates. The determinant of this tensor is

$$-g = (\mathbf{l}[\mathbf{m}\times\mathbf{n}])^2 t^{2(p_1+p_2+p_3)}. \tag{2.5}$$

The tensor $g^{\alpha\beta}$, which is the reciprocal of the tensor (2.4), can be written in the form†

$$g^{\alpha\beta} = \sum t^{-2p_1}\tilde{l}_\alpha \tilde{l}_\beta. \tag{2.6}$$

Here and below the summations are over the cyclic permutations of the vectors **l**, **m**, and **n** and of the numbers p_1, p_2, and p_3; we use the notation

$$\tilde{\mathbf{l}} = [\mathbf{m}\times\mathbf{n}]/(\mathbf{l}[\mathbf{m}\times\mathbf{n}]),\ \tilde{\mathbf{m}} = [\mathbf{n}\times\mathbf{l}]/(\mathbf{l}[\mathbf{m}\times\mathbf{n}]),\ \tilde{\mathbf{n}} = [\mathbf{l}\times\mathbf{m}]/(\mathbf{l}[\mathbf{m}\times\mathbf{n}]) \tag{2.7}$$

for the vectors which are "reciprocal" to the vectors **l**, **m**, and **n** (so that $\mathbf{l}\cdot\tilde{\mathbf{l}} = 1$, $\tilde{\mathbf{l}}\cdot\tilde{\mathbf{m}} = \tilde{\mathbf{l}}\cdot\tilde{\mathbf{n}} = 0, \ldots$). We have, furthermore,

$$\varkappa_{\alpha\beta} = \sum 2p_1 t^{2p_1-1} l_\alpha l_\beta, \qquad \varkappa_\alpha^\beta = t^{-1}\sum 2p_1 l_\alpha \tilde{l}_\beta,$$
$$\varkappa^{\alpha\beta} = \sum 2p_1 t^{-2p_1-1}\tilde{l}_\alpha \tilde{l}_\beta. \tag{2.8}$$

Substitution of (2.5) and (2.8) in (2.1) leads to

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2\,. \tag{2.9}$$

We now make the assumption that in Eq. (2.3) the three-dimensional curvature tensor P_α^β does not contribute to the principal terms of the equation.* Then the principal terms are of order t^{-2} and vanish if $p_1 + p_2 + p_3 = 1$. Together with relation (2.9) we obtain, therefore,

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^3 = 1. \tag{2.10}$$

These two equations relate the three functions p_1, p_2, and p_3 and consequently only one of these is independent.† With this, p_1, p_2, and p_3 never

*The analytic form of the metric near such a fictitious singularity will be indicated in another communication.

†The vector-operation symbols (vector products, the operations curl, grad, etc.) must be understood throughout in a purely formal manner, as operations on the components of the vectors **l**, **m**, and **n** as if the coordinates x^1, x^2, x^3 were Cartesian.

*There are apparently no broad classes of solutions which do not satisfy this condition. A relatively narrow class (which will be given later on) is obtained for constant values of p_1, p_2, and p_3 equal to s_1, s_2, and 1 respectively, where s_1 and s_2 are two numbers that satisfy the condition $s_1 + s_2 = s_1^2 + s_2^2$.

Relation (2.9) [i.e., Eq. (2.1)] is satisfied also when $p_1 = p_2 = p_3 = 1$, i.e., when $g_{\alpha\beta} = t^2 a_{\alpha\beta}$, where the $a_{\alpha\beta}$ are functions of the coordinates. Then Eq. (2.3) yields $P_{\alpha\beta} = -2a_{\alpha\beta}$; this means that the space has a constant negative curvature (independent of the coordinates of the point). The corresponding space-time metric can be written with the aid of "four-dimensional spherical coordinates" χ, θ, φ in the form

$$-ds^2 = -dt^2 + t^2[d\chi^2 + \mathrm{sh}^2\chi\,(d\theta^2 + \sin^2\theta\cdot d\varphi^2)]$$

(see, for example, reference 5, Sec. 104). But the transformation $r = t\sinh\chi$, $\tau = t\cosh\chi$ transforms such a metric simply to the Galilean metric

$$-ds^2 = -d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

†The constant numbers p_1, p_2, and p_3, which are related by Eqs. (2.9), were first used in the exact solution of Eqs. (2.1) – (2.3), indicated by Taub,[6] corresponding to a completely homogeneous (but not isotropic) empty space:

$$-ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2.$$

When $p_1 = p_2 = 0$ and $p_3 = 1$, the transformation $x_1 = x$, $x_2 = y$, $t\sinh x_3 = z$, $t\cosh x_3 = \tau$ makes this metric Galilean, i.e., the singularity is fictitious. At these values of p_1, p_2, and p_3 the singularity is also fictitious for the metric (2.4) (although the latter is, naturally, not Galilean). We exclude these values from further consideration.

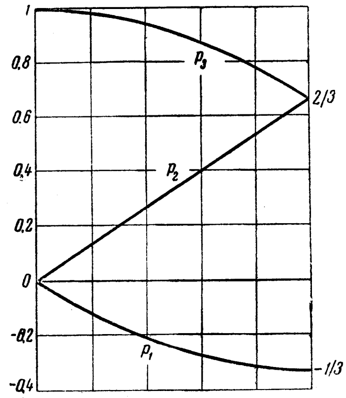
have the same value simultaneously, and two of them are equal only in the triplets 0, 0, 1 and $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$. In all other cases p_1, p_2, p_3 are all different, one being negative and the other two positive. We shall arrange these values in the order $p_1 < p_2 < p_3$. The quantities p_1, p_2, and p_3 run through their values in the following respective intervals

$$-{}^1/_3 \leqslant p_1 \leqslant 0, \qquad 0 \leqslant p_2 \leqslant {}^2/_3, \qquad {}^2/_3 \leqslant p_3 \leqslant 1.$$

They can be represented in parametric form as

$$p_1 = \frac{-s}{1+s+s^2}, \qquad p_2 = \frac{s(1+s}{1+s+s^2},$$
$$p_3 = \frac{1+s}{1+s+s^2}, \tag{2.11}$$

as the parameter s runs through values from 0 to 1. The figure shows the curves that determine any two of the values p_1, p_2, or p_3 once the third is specified (the three values lie on one vertical line).



The conditions (2.10) ensure the vanishing of the contribution $\sim t^{-2}$ from the second term in Eq. (2.3); in accordance with the assumption made, it is necessary also to ensure the absence of terms of the same order from the tensor P_α^β.

Inasmuch as the metric has an essentially different time dependence along the directions **l**, **m**, and **n**, it is convenient to "project" all the tensors on these directions. Denoting the corresponding projections by the subscripts l, m, and n we determine them in the following manner:

$$P_{ll} = P_{\alpha\beta}\tilde{l}_\alpha\tilde{l}_\beta, \qquad P_{lm} = P_{\alpha\beta}\tilde{l}_\alpha\tilde{m}_\beta, \ldots \tag{2.12}$$

In this notation we have, in particular,

$$g_{ll} = t^{2p_1}, \qquad g_{mm} = t^{2p_2}, \qquad g_{nn} = t^{2p_3}.$$

The "mixed" tensor components are defined accordingly as

$$P_l^l = P_{ll}/g_{ll} = t^{-2p_1}P_{ll}, \qquad P_l^m = P_{lm}/g_{mm} = t^{-2p_2}P_{lm}, \ldots \tag{2.13}$$

The calculation of the components of the tensor $P_{\alpha\beta}$ from the general formulas with the aid of the metric tensor (2.4) leads to the following expressions for the highest-order terms:

$$P_l^l = -P_m^m = -P_n^n = \frac{(\mathbf{l}\ \mathrm{curl}\ \mathbf{l})^2}{2(\mathbf{l}\,[\mathbf{m}\times\mathbf{n}])^2}\,t^{-2(1-2p_1)},$$
$$P_{lm} = 2\frac{(\mathbf{l}\ \mathrm{curl}\ \mathbf{l})\,p_{1,\,n}}{(\mathbf{l}\,[\mathbf{m}\times\mathbf{n}])}\ln t\cdot t^{2(p_1-p_3)},$$
$$P_{ln} = -2\frac{(\mathbf{l}\ \mathrm{curl}\ \mathbf{l})\,p_{1,\,m}}{(\mathbf{l}\,[\mathbf{m}\times\mathbf{n}])}\ln t\cdot t^{2(p_1-p_2)},$$
$$P_{mn} = 2\ln^2 t\cdot(p_{2,\,n}p_{1,m} + p_{3,m}p_{1,\,n} - p_{1,m}p_{1,\,n}). \tag{2.14}$$

The letters l, m, and n following the comma in the subscripts denote here differentiation in the corresponding direction, in accordance with the definition

$$f_{,l} = \tilde{l}_\alpha \partial f/\partial x^\alpha, \ldots$$

Since $p_1 < 0$, we see that the power of $1/t$ in the diagonal components $P_l^l, \ldots$, does not exceed 2. To satisfy Eqs. (2.3) it is therefore necessary in any case that these terms vanish, i.e., we must have

$$\mathbf{l}\ \mathrm{curl}\ \mathbf{l} = 0. \tag{2.15}$$

We note that this condition has a simple geometrical meaning. The vector satisfying this condition can be represented in the form $\mathbf{l} = \psi\ \mathrm{grad}\ \varphi$ (ψ and φ are two scalar functions), so that $l_\alpha l_\beta dx^\alpha dx^\beta = \psi^2 d\varphi^2$. This means that the direction of the vector **l** at each point of space can be chosen as the direction of the coordinate x^1 (so that the surfaces φ = const become the surfaces x^1 = const). It is well known that in the general case of an arbitrary three-dimensional vector field this is, generally speaking, impossible.

If condition (2.15) is satisfied, the principal terms in the components of the tensor $P_{\alpha\beta}$ are found to be of the following order of magnitude

$$P_l^l \sim P_m^m \sim P_n^n \sim \ln^2 t,$$
$$P_{lm} \sim t^{2(p_2-p_3)}\ln t, \qquad P_{ln} \sim P_{mn} \sim \ln^2 t, \tag{2.16}$$

and in the main do not influence the equations (2.3).

It remains for us to satisfy Eqs. (2.2). The largest terms in these equations could have an order $t^{-1}\ln t$: these terms appear when differentiating the exponents in the derivatives of $g_{\beta\gamma}$ with respect to the coordinates contained in the expression

$$\varkappa_{\alpha;\,\beta}^{\beta} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}(\sqrt{-g}\varkappa_\alpha^\beta) - \frac{1}{2}\varkappa^{\beta\gamma}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}.$$

However, by virtue of (2.10), these terms cancel out identically:

$$\varkappa^{\beta\gamma}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} = 4\sum p_l t^{-2p_l-1}\tilde{l}_\beta\tilde{l}_\gamma\cdot\sum t^{2p_l}\ln t\,\frac{\partial p_l}{\partial x^\alpha}l_\beta l_\gamma$$
$$= 4\frac{\ln t}{t}\sum p_1\frac{\partial p_1}{\partial x^\alpha} = 2\frac{\ln t}{t}\frac{\partial}{\partial x^\alpha}(p_1^2+p_2^2+p_3^2) = 0.$$

Therefore the principal terms are those of order $1/t$. The first term in (2.2) in this approximation vanishes, and the calculation of the derivative $\kappa^{\beta}_{\alpha;\beta}$ leads to

$$R^0_\alpha = -\frac{1}{t\,(\mathbf{l}\,[\mathbf{m}\times\mathbf{n}])}\sum l_\alpha\{[\mathbf{m}\times\mathbf{n}]\,\nabla p_1 + (p_3 - p_1)\,\mathbf{m}\ \mathrm{curl}\ \mathbf{n} + (p_1 - p_2)\,\mathbf{n}\,\mathrm{curl}\,\mathbf{m}\} = 0. \tag{2.17}$$

Projecting this equation on the directions **l**, **m**, and **n** we obtain the three relations

$$\begin{aligned}&[\mathbf{m}\times\mathbf{n}]\,\nabla p_1 + (p_3 - p_1)\,\mathbf{m}\,\mathrm{curl}\ \mathbf{n} + (p_1 - p_2)\,\mathbf{n}\,\mathrm{curl}\,\mathbf{m} = 0,\\ &[\mathbf{n}\times\mathbf{l}]\,\nabla p_2 + (p_1 - p_2)\,\mathbf{n}\,\mathrm{curl}\ \mathbf{l} + (p_2 - p_3)\,\mathbf{l}\,\mathrm{curl}\,\mathbf{n} = 0,\\ &[\mathbf{l}\times\mathbf{m}]\,\nabla p_3 + (p_2 - p_3)\,\mathbf{l}\ \mathrm{curl}\,\mathbf{m} + (p_3 - p_1)\,\mathbf{m}\,\mathrm{curl}\,\mathbf{l} = 0.\end{aligned} \tag{2.18}$$

The following terms of the expansion of the metric tensor

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta} \tag{2.19}$$

[where $g^{(0)}_{\alpha\beta}$ is given by (2.4)] are expressed in terms of the quantities contained in (2.4). We shall not repeat here the corresponding calculations, but will indicate only that the first correction terms have the following orders of magnitude:

$$h^l_l \sim h^m_m \sim h^n_n \sim h^n_l \sim h^n_m \sim t^{2(1-p_3)}\ln^2 t \tag{2.20}$$

(the component h_{lm} is found to be of relatively higher order of smallness and in this sense enters in the next approximation).

Expression (2.4) contains a total of ten different functions of the coordinates: three components for each of the three vectors **l**, **m**, and **n**, and one of the functions p_1, p_2, or p_3. These ten functions are connected by the four relations (2.15) and (2.18). In addition, the reference frame which we are using admits also of arbitrary transformation of the three spatial coordinates in terms of each other. Therefore the solution obtained contains merely $10 - 4 - 3 = 3$ physically different arbitrary functions of the coordinates. This is one less than required to specify the arbitrary initial conditions in vacuum.*

3. SOLUTION IN A SPACE FILLED WITH MATTER

Let us show now that the presence of matter does not change the solution obtained, and that the initial conditions for the distribution and motion of the matter can be specified in a fully arbitrary manner.

To gain an idea of the orders of magnitude of the energy density ϵ and of the components of the four-velocity of the matter u_i, it is convenient to use the hydrodynamic equations of motion of matter, which are contained, as is well known, in the equations of gravitation (the equations $T^k_{i,k} = 0$):

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,u^i\sigma\right) = 0, \tag{3.1}$$

$$(p+\varepsilon)\,u^k\left\{\frac{\partial u_i}{\partial x^k} - \frac{1}{2}u^l\frac{\partial g_{kl}}{\partial x^i}\right\} = -\frac{\partial p}{\partial x^i} - u_i u^k\frac{\partial p}{\partial x^k} \tag{3.2}$$

(see, for example, reference 7, Sec. 125). Here σ is the entropy density; for the ultrarelatistic equation of state $p = \epsilon/3$ the entropy is $\sigma \sim \epsilon^{3/4}$.

We make an assumption (which will be confirmed by the results obtained) that the principal terms in (3.1) and (3.2) are those which contain the derivatives with respect to time; then Eq. (3.1) and the spatial components of (3.2) (the temporal component yields nothing new) give:

$$\frac{\partial}{\partial t}\left(\sqrt{-g}\,u_0\varepsilon^{3/4}\right) = 0, \qquad 4\varepsilon\frac{\partial u_\alpha}{\partial t} + u_\alpha\frac{\partial\varepsilon}{\partial t} = 0,$$

hence

$$tu_0\varepsilon^{3/4} = \mathrm{const}, \qquad u_\alpha\varepsilon^{1/4} = \mathrm{const},$$

where the symbol "const" stands for quantities independent of the time. In addition, we have from the identity $u_i u^i = -1$, considering that all the covariant components u_α are of the same order

$$u_0^2 \approx u_n u^n = u_n^2 t^{-2p_3}$$

(we again use the projections on the directions **l**, **m**, and **n**, i.e., we represent the three-dimensional vector **u** in the form $\mathbf{u} = u_l\mathbf{l} + u_m\mathbf{m} + u_n\mathbf{n}$).

From these relations we obtain

$$\varepsilon \sim t^{-2(1-p_3)}, \qquad u_0^2 \sim t^{-(3p_3-1)}, \qquad u_\alpha \sim t^{(1-p_3)/2}, \tag{3.3}$$

after which we can readily verify that the terms discarded in (3.1) and (3.2) are actually small compared with those retained.

We now estimate the components of the energy-momentum tensor T^k_i, contained in the right halves of Eqs. I (1.4) — (1.6). In Eq. I (1.4) we have

$$T^0_0 \sim \varepsilon u_0^2 \sim t^{-(1+p_3)}.$$

Inasmuch as $p_3 < 1$, this quantity is of lower order in $1/t$ than the principal terms in the left side of the equation ($\sim t^{-2}$). The same holds for Eqs. I (1.6); the spatial components of the tensor T^k_i, "projected" on the directions **l**, **m**, and **n**, are of the order of magnitude

$$T^l_l \sim \varepsilon \sim t^{-2(1-p_3)}, \qquad T^m_m \sim \varepsilon u_m u^m \sim t^{-(1+2p_2-p_3)},$$

$$T^n_n \sim \varepsilon u_n u^n \sim t^{-(1+p_3)}, \tag{3.4}$$

*The solution with two arbitrary functions, obtained in Section 4 of I for the case of empty space, corresponds to the particular case of constant values p_1, p_2, and p_3, equal to $-1/3$, $2/3$, and $2/3$.

Recently Harrison[3] found a series of exact solutions of a special type. These solutions have singularities which can be reduced to the type (2.4) (with different constant values of p_1, p_2 and p_3) or to the type mentioned in footnote *, page 559. We are grateful to Harrison for a preprint of his paper.

which are all smaller than t^{-2}.

On the other hand, in Eq. I (1.5) we have

$$T^0_\alpha \sim \varepsilon u_0 u_\alpha \sim 1/t,$$

i.e., the same order of magnitude as in the left side of the equation. This circumstance, however, also leaves the character of the solution unchanged. Actually, in accordance with (3.3), we write

$$\varepsilon = \varepsilon^{(0)} t^{-2(1-p_3)}, \qquad \mathbf{u} = \mathbf{u}^{(0)} t^{(1-p_3)/2} \tag{3.5}$$

for the first terms of the expansion of these quantities; here

$$u_0^2 \approx u_n^{(0)2} t^{-(3p_3-1)}.$$

Equating expression (2.17) for R^0_α to the quantity $T^0_\alpha = 4\epsilon u_\alpha u^0/3$, we obtain in lieu of (2.18)

$$[\mathbf{m}\times\mathbf{n}]\,\nabla p_1 + (p_3 - p_1)\,\mathbf{m}\,\mathrm{curl}\,\mathbf{n} + (p_1 - p_2)\,\mathbf{n}\,\mathrm{curl}\,\mathbf{m}$$
$$= -\tfrac{4}{3}\varepsilon^{(0)} u_l^{(0)} u_n^{(0)}, \ldots \tag{3.6}$$

Thus only the connection between the functions contained in (2.4) changes, and this connection now includes the new functions $\epsilon^{(0)}$ and $\mathbf{u}^{(0)}$.

The form of the following terms of the expansion of the metric tensor also changes, and the first terms following (2.4) are precisely the terms connected with the presence of matter.

To calculate these terms, we write $g_{\alpha\beta}$ in the form (2.19). Here

$$g^{\alpha\beta} = g^{(0)\alpha\beta} - h^{\alpha\beta}, \qquad \varkappa_{\alpha\beta} = \varkappa^{(0)}_{\alpha\beta} + \dot h_{\alpha\beta},$$
$$\varkappa^\beta_\alpha = \varkappa^{(0)\beta}_\alpha + \dot h^\beta_\alpha - \varkappa^{(0)\gamma}_\alpha h^\beta_\gamma + \varkappa^{(0)\beta}_\gamma h^\gamma_\alpha \tag{3.7}$$

(the dot denotes differentiation with respect to t). From I (2.1) and I (2.3) we obtain the following equations for $h_{\alpha\beta}$:

$$R^0_0 = \tfrac{1}{2}(\ddot h + \varkappa^{(0)\beta}_\alpha \dot h^\alpha_\beta) = T^0_0, \tag{3.8}$$

$$R^\beta_\alpha = \frac{1}{2}\left(\ddot h^\beta_\alpha + \frac{1}{t}\dot h^\beta_\alpha + \frac{1}{t}\varkappa^{(0)\beta}_\alpha \dot h - \varkappa^{(0)\gamma}_\alpha \dot h^\beta_\gamma + \varkappa^{(0)\beta}_\gamma \dot h^\gamma_\alpha\right) = T^\beta_\alpha, \tag{3.9}$$

where $h = h^\alpha_\alpha$ [in the calculation of R^β_α it is necessary to take into account the fact that the $\kappa^{(0)\beta}_\alpha$ are proportional to 1/t, whereas $\kappa^{(0)\gamma}_\gamma = 2/t$; the contribution to R^β_α from the "perturbation" of the tensor P^β_α is of smaller order of magnitude than the terms in (3.9)]. Since these equations do not contain derivatives with respect to coordinates, we can directly change in these equations to projections on $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$; considering that only

$$\varkappa^{(0)l}_l = 2p_1/t, \qquad \varkappa^{(0)m}_m = 2p_2/t, \qquad \varkappa^{(0)n}_n = 2p_3/t,$$

differ from zero, we obtain from (3.9) equations of the form

$$\frac{1}{2}\left(\ddot h^l_l + \frac{1}{t}\dot h^l_l + \frac{p_1}{t}\dot h\right) = T^l_l, \ldots \tag{3.10}$$

$$\frac{1}{2}\left(\ddot h^m_l + \frac{1+2p_2-2p_1}{t}\dot h^m_l\right) = T^m_l, \ldots \tag{3.11}$$

Of the three "diagonal" components (3.4) of the energy-momentum tensor, the one containing the highest power of 1/t is T^n_n. Therefore in calculating h^l_l, h^m_m and h^n_n, we can omit T^l_l and T^m_m for the right halves of (3.10), and retain only $T^n_n = 4\epsilon u_n u^n/3$. As a result we obtain

$$h^l_l = -\frac{p_1}{1-p_3}h, \qquad h^m_m = -\frac{p_2}{1-p_3}h, \qquad h^n_n = 2h,$$
$$h = \frac{8\varepsilon^{(0)} u_n^{(0)2}}{3(1-p_3)(2-p_3)} t^{1-p_3}, \qquad h^n_l = \frac{8\varepsilon^{(0)} u_l^{(0)} u_n^{(0)}}{3(1-p_3)(1+p_3-2p_1)} t^{1-p_3},$$
$$h^n_m = \frac{8\varepsilon^{(0)} u_m^{(0)} u_n^{(0)}}{3(1-p_3)(1+p_3-2p_2)} t^{1-p_3}. \tag{3.12}$$

These corrections are of higher order of magnitude than the first correction terms in the absence of matter (on the other hand, the component h_{lm} is again found to be of relatively higher order of smallness).

Equation (3.8) is satisfied by the expressions (3.12) identically. On the other hand, the equation $R^0_\alpha = T^0_\alpha$, which we did not write out, would come in only in the determination of the following expansion terms of the energy and velocity.

Thus, the solution obtained for the gravitation equations represents a very broad class of solutions with singularities. It contains seven arbitrary functions of the coordinates: the three functions which enter in the absence of matter, the function ϵ^0 and the three functions $u^{(0)}_\alpha$.*

The character of variation of the metric $t \to 0$ in this solution is such that at each point of space the linear distances diminish along two directions (as t^{p_2} and t^{p_3}) and increase along the third (as $t^{-|p_1|}$); the volumes decrease here in proportion to t. The laws of these variations (i.e., the values of p_1, p_2, and p_3) vary in space and are determined by the initial conditions.

The density of matter becomes infinite at each point in space as $\epsilon \sim t^{-2(1-p_3)}$. This is clear evidence of the physical (not fictitious) nature of the singularity in the given solution (we note also that in the case of empty space the singularity is not fictitious because the scalars made up of the components of the curvature four-tensor R_{iklm}, for example, the scalar $R_{iklm}R^{iklm}$, do not become infinite). The velocity of motion of matter tends in this solution (in the reference frame considered here) to the velocity of light as $t \to 0$.† Actually,

*It can be shown that the higher terms of the expansion of the metric contain no other arbitrary function.

†In the particular case when p_1, p_2 and p_3 have the constant values $-1/3$, $2/3$, and $2/3$, the matter can be "written in" into the solution (20.4) in another, particular manner, by which its velocity tends to zero as $t \to 0$. This is the solution developed in I [formulas I(4.2) and (4.3)]; in this solution the matter brings only two, and not four arbitrary new functions, i.e., the initial conditions for this solution should have a certain particular character.

as $t \to 0$, the three-dimensional scalar $u_\alpha u^\alpha \approx u_n u^n$ goes to infinity as $t^{-3(p_3-1)}$. This means that the matter moves at any point essentially along the direction **n**, while the absolute magnitude of its ordinary three-dimensional velocity **v** tends to unity as

$$\sqrt{1-v^2} \sim t^{(3p_3-1)/2}.$$

The proper time τ of the moving matter is connected with the time t by means of $d\tau = dt\sqrt{1-v^2}$. Therefore

$$\tau \sim t^{(3p_3+1)/2}.$$

In the attached reference frame, the energy density goes to infinity, consequently, in accordance with the law

$$\varepsilon \sim \tau^{-4(1-p_3)/(3p_3+1)}.$$

The solution obtained, however, is still not general, for the general solution should contain eight arbitrary functions of the coordinates. The fact that this solution is incomplete manifests itself, in particular, in its stability properties. The general solution, by definition, is completely stable: no small perturbations can alter its character, since it admits of arbitrary initial conditions. The present solution, however, is unstable (if terms quadratic in the perturbations are taken into account) under perturbations of a definite type — perturbations connected with the appearance of the nonvanishing quantity **l** curl **l**. The question of the existence or absence of a general solution with singularity is closely related with the problem of the character of this instability, and calls for a separate investigation.

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[1] E. M. Lifshitz and I. M. Khalatnikov, JETP **39**, 149 (1960), Soviet Phys. JETP **12**, 108 (1961).

[2] E. Schucking and O. Herkmann, Onzieme conseil de physique Solvay. Bruxelles (1958).

[3] B. K. Harrison, Phys. Rev. **116**, 1285 (1959).

[4] A. Komar, Phys. Rev. **104**, 544 (1956).

[5] L. D. Landau and E. M. Lifshitz, Теория поля (Field Theory), 3d Ed. Fizmatgiz, 1960.

[6] A. H. Taub, Ann. of Math. **53**, 472 (1951).

[7] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media), Gostekhizdat, 1954.

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