THE DECAY OF ACOUSTIC EXCITATIONS IN CRYSTALS

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The properties of the acoustic excitation spectra in crystals near the decay threshold are considered. The longitudinal-phonon attenuation due to the decay of the longitudinal phonon into transverse phonons, is shown to be proportional to k^5. The effect of anisotropy on the phonon decay of transverse excitations is investigated. The weak coupling between acoustic vibrations causes a characteristic splitting of the spectrum near the decay threshold into two excitations with non-zero momenta. This can show up in neutron scattering experiments, where it causes the simultaneous existence of two peaks in the energy distribution of neutrons scattered at an angle close to critical.

THE singularities of the spectrum of elementary excitations near their decay threshold have been recently considered by Pitaevskiĭ[1]. He treated mainly the case of a Bose liquid. Although the qualitative picture also remains in force for acoustic excitations in crystals, there are in this case a number of particular circumstances, such as the existence of three branches of vibrations, anisotropy, and weak interaction (due to anharmonic effects) between the elementary excitations.

We write the phonon-interaction Hamiltonian in the form

$$H_{int} = \frac{\gamma}{\sqrt{V}} \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3} \sqrt{\omega_{p_1}\omega_{p_2}\omega_{p_3}}\, a^+_{p_1} a^+_{p_2} a_{p_3} + \text{Herm. adj.} \tag{1}$$

where the interaction constant $\gamma = \hbar^{3/2}(\rho c^2)^{-1/2}$, ρ is the density, c is the velocity of sound. The summation over the polarizations is omitted.

1. THE DECAY OF A LONGITUDINAL PHONON INTO TWO TRANSVERSE PHONONS

It is apparent that the decay of a fast phonon into phonons with smaller propagation velocities is possible kinematically. The polarization selection rules do not forbid such a decay, because along the crystallographic axes the velocity of longitudinal sound is greater than that of transverse sound. Such a decay causes the attenuation of longitudinal sound from the very beginning.

To calculate this attenuation or decay we will find the correction to the frequency $\omega_{\parallel}(q)$, which is described by the diagram in Fig. 1. We have

$$\Sigma(p) = G^{-1}(p) - G_0^{-1}(p)$$
$$= \frac{i\gamma^2\omega_{\parallel}^2(p)}{(2\pi)^4\hbar^2}\int \frac{\omega_{\perp}^2(q)\,\omega_{\perp}^2(\mathbf{p}-\mathbf{q})\,d^3q\,d\omega}{[\omega^2-\omega_{\perp}^2(q)-i\delta]\,[(\omega-\varepsilon)^2-\omega_{\perp}^2(\mathbf{p}-\mathbf{q})-i\delta]}, \tag{3}$$

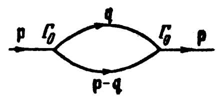

FIG. 1

where $\omega_{\parallel}(p)$ and $\omega_{\perp}(p)$ are respectively the frequencies of the longitudinal and transverse branches. After integrating with respect to ω and changing the variables, (3) can be brought to the form

$$\Sigma(p) = -\frac{\gamma^2\omega_{\parallel}^2(p)}{p\pi^2\hbar^2}\int_0^{q_m}\omega_{\perp}(q)\,q\,dq$$
$$\times \int_{|p-q|}^{p+q} \frac{\omega_{\perp}(u)\,(\omega_{\perp}(u)+\omega_{\perp}(q))\,u\,du}{(\omega_{\perp}(u)+\omega_{\perp}(q)+\varepsilon)\,(\omega_{\perp}(u)+\omega_{\perp}(q)-\varepsilon-i\delta)}. \tag{4}$$

To calculate the attenuation in the phonon part of the spectrum, we put

$$\omega_{\parallel}(p) = c_{\parallel}p, \qquad \omega_{\perp}(p) = c_{\perp}p. \tag{5}$$

The results thus obtained will also give the correct order of magnitude when dispersion is present. By substituting (5) in (4), and calculating the imaginary part as half the residue during the integration over u, we obtain

$$\operatorname{Im}\Sigma(p) = -2iAc_{\parallel}p^6,$$
$$A = \frac{\gamma^2 c_{\parallel} c_{\perp}}{4\hbar^2}\left(\frac{1}{80} - \frac{1}{24}\left(\frac{c_{\parallel}}{c_{\perp}}\right)^2 + \frac{1}{16}\left(\frac{c_{\parallel}}{c_{\perp}}\right)^4\right). \tag{6}$$

Hence, we have for the spectrum of elementary excitations

$$\omega_{\parallel}(p) = c_{\parallel}p - iAp^5. \tag{7}$$

The attenuation is everywhere small. For p ~ 1/a (a is the lattice constant) we have Im ω

$\times(p)/\omega(p) \sim \alpha = \hbar/\rho c a^4$. The parameter α has the value 10^{-1} to 10^{-2} for light elements and 10^{-3} to 10^{-4} for heavy elements. It is not difficult to see that anisotropy does not introduce important changes in the result obtained.

2. ALLOWANCE FOR ANISOTROPY IN THE PHONON DECAY OF EXCITATIONS

The transverse acoustic branches, which are stable at the start, can split up into two excitations, one of which is a phonon. We will find the kinematic conditions for such a decay in the anisotropic case. The laws of conservation of energy and momentum give

$$\Phi_{\mathbf{p}}(\mathbf{q}) = \varepsilon(\mathbf{q}) + \varepsilon(\mathbf{p}-\mathbf{q}) - \varepsilon(\mathbf{p}) = 0, \qquad \varepsilon(\mathbf{p}) = \hbar\omega(\mathbf{p}). \tag{8}$$

For small values of q we obtain

$$\Phi_{\mathbf{p}}(\mathbf{q}) = (c(\mathbf{n}) - \mathbf{v}_{\mathbf{p}}\mathbf{n})\,q,$$

where $c(\mathbf{n})$ is the velocity of sound along the direction $\mathbf{n}$; $\mathbf{q} = \mathbf{n}q$: $\mathbf{v}_{\mathbf{p}} = \partial\epsilon/\partial\mathbf{p}$. If $c(\mathbf{n}) > \mathbf{v}_{\mathbf{p}}\mathbf{n}$ for all $\mathbf{n}$, (8) has only the trivial solution $q = 0$. If p is increased along some direction, decay starts at the value of p for which the equation $c(\mathbf{n}) = \mathbf{v}_{\mathbf{p}} \cdot \mathbf{n}$ is first satisfied, at least for one direction of $\mathbf{n}$.

We introduce the function $\varphi(\mathbf{p})$ as follows:

$$\varphi(\mathbf{p}) = \min \frac{c(\mathbf{n})}{\mathbf{v}_{\mathbf{p}}\mathbf{n}},$$

i.e. $\varphi(\mathbf{p})$ is the minimum value of the ratio given, considered as a function of $\mathbf{n}$ for a given $\mathbf{p}$. The equation of the threshold surface then takes the form $\varphi(\mathbf{p}) = 1$.

We investigate the attenuation of excitations close to the threshold of phonon creation. Taking into account, as in the preceding case, only the contribution from the diagram in Fig. 1, we obtain after integration over ω:

$$\Sigma(p) \sim \gamma^2 \int \frac{q^3 dq\, d\cos\theta\, d\varphi}{x + (c(\mathbf{n}) - \mathbf{v}_c\mathbf{n})\,q - 2\beta_{ik}\Delta p_i q_k + \beta_{ik} q_i q_k - i\delta},$$

$$\beta_{ik} = \frac{1}{2}\frac{\partial^2\varepsilon}{\partial p_i \partial p_k}\bigg|_{\mathbf{p}=\mathbf{p}_c}, \qquad x = \mathbf{v}_c\Delta\mathbf{p} + \beta_{ik}\Delta p_i \Delta p_k - \Delta\varepsilon. \tag{9}$$

Attenuation occurs in cases where the denominator of the expression under the integral in (9) has zeros in the region of integration and is determined by half the residue at the corresponding pole. Close to a pole we always have $x \ll \Delta p$. This means that the correction to $\Delta\epsilon$ due to the decay is smaller than $(\Delta p)^2$, which will be confirmed by the result.

For the denominator in (9) to tend to zero requires apparently (for $\beta_i = \beta_{ik} n_k$):

$$c(\mathbf{n}) - \mathbf{v}_c\mathbf{n} \sim \beta_i \Delta p_i$$

(here the sign $\sim$ means agreement in order of magnitude). But the function $c(\mathbf{n}) - \mathbf{v}_c\mathbf{n}$ by hypothesis has a minimum at $\mathbf{n} = \mathbf{n}_c$ and close to the latter $c(\mathbf{n}) - \mathbf{v}_c\mathbf{n}$ takes the form

$$c(\mathbf{n}) - \mathbf{v}_c\mathbf{n} = Q(\Delta\theta, \Delta\varphi),$$

where $Q(x, y)$ is some positive determinate quadratic form. Hence, the important range of integration over θ and φ is found to be $\Delta\theta, \Delta\varphi \sim \sqrt{\beta_i \Delta p_i}$.

Having calculated the residue in (9), we obtain near the threshold

$$\varepsilon(p) = \varepsilon(p_c) + \mathbf{v}_c\Delta\mathbf{p} + \beta_{ik}\Delta p_i \Delta p_k - iB(\beta_i \Delta p_i)^3. \tag{10}$$

We note that in the isotropic case for a threshold for decay into phonons to appear on the $\epsilon(\mathbf{p})$ curve, at least a point of inflection must exist. In the anisotropic case it is much easier to satisfy the conditions of decay. In particular, the presence of a point of inflection on the $\epsilon(\mathbf{p})$ curves along a given direction is not required.

3. THE SPECTRUM CLOSE TO THE THRESHOLD OF DECAY INTO EXCITATION WITH NON-ZERO MOMENTA

Pitaevskiĭ has shown[1] that the decay of elementary excitations into two excitations with momenta not equal to zero is also possible. In this case the spectrum of elementary excitations breaks up at the decay point (p_c, ϵ_c). Its behavior beyond the decay point cannot be clarified without resorting to perturbation theory.

In the case we are considering the treatment can be complete because the interaction of the elementary excitations is weak, which allows perturbation theory to be used, We will limit ourselves to the contribution to Σ of the lowest order in the coupling constant given by the same diagram (see Fig. 1). The Green's function in the approximation taken agrees with that found by Pitaevskiĭ[1] and has the form

$$G^{-1}(p) \approx v_0\Delta p - \Delta\varepsilon + 2\alpha\sqrt{\varepsilon_c(v_c\Delta p - \Delta\varepsilon)}, \tag{11}$$

where v_0 is the velocity of the elementary excitation at the threshold point without taking interaction into account; v_c is the velocity of the excitations at which decay occurs; $\Delta\mathbf{p}$ and $\Delta\epsilon$ are the momenta and energy referred to the threshold point;* $\alpha \sim \hbar/\rho c a^4 > 0$. We choose in (11) that the branch of the root which is positive on the positive axis and which has a cut along the negative axis.

*In (11) only the singular part of $\Sigma(p)$ is taken into account. The inclusion of the regular part causes an unimportant displacement of the threshold point.

We shall show that $v_0 > v_c$. The function $F(p)$ is introduced as follows:

$$F(p) = \varepsilon(q_c) + \varepsilon(p - q_c) - \varepsilon(p). \tag{12}$$

Of course, $F(q_c) = 0$. The decay threshold p_c is that zero of the functions $F(p)$, which is closest to q_c (for $p_c > q_c$), where the condition below must be satisfied;

$$\varepsilon'(q_c) = \varepsilon'(p_c - q_c) \equiv v_c < c, \tag{13}$$

where c is the velocity of sound. The latter inequality is necessary to make decay with phonon creation impossible. It is seen that, by virtue of (13), $F(p) > 0$ in the range $q_c < p < p_c$. Since $F(p_c) = 0$, we have $F'(p_c) < 0$ and, consequently, $v_0 > v_c$.

The energy of the elementary excitations is determined by the poles of the Green's function, i.e., by the zeros of the double-valued function (11), which lie either on the real ϵ axis on the first sheet of the Riemann surface to the left of the cut, or in the lower half-plane of the second sheet close to the cut.

It is convenient to introduce new variables

$$x = (v_0 - v_c)\Delta p / \varepsilon_c, \qquad y = (\Delta\varepsilon - v_c\Delta p)/\varepsilon_c, \tag{14}$$

in which the equation $G^{-1}(p, \epsilon) = 0$ has the form

$$x - y - 2i\alpha\sqrt{y} = 0. \tag{15}$$

Here $\arg\sqrt{y} = \frac{1}{2}\arg y$, where in the first sheet we take $0 < \arg y < 2\pi$. Putting $y = re^{i\varphi}$, we obtain from (15)

$$x - r\cos\varphi + 2\alpha\sqrt{r}\sin(\varphi/2) = 0, \tag{16}$$

$$r\sin\varphi + 2\alpha\sqrt{r}\cos(\varphi/2) = 0. \tag{17}$$

Equation (17) has solutions: $\varphi = \pm\pi$ and $\sin(\varphi/2) = -\alpha/\sqrt{r}$. Substituting $\varphi = +\pi$ in (16), we obtain

$$x + r + 2\alpha\sqrt{r} = 0, \tag{18}$$

which is only possible for negative values of x. It is not difficult to verify that this is the only solution on the first sheet and that for $x > 0$ there is no solution of (15) on the first sheet. We note that $r \to 0$ as $x \to 0$, so that near p_c the dispersion law has the form $\Delta\epsilon = v_c\Delta p + 0(\Delta p^2)$, the same as that of Pitaevskiĭ.[1]

Further, putting $\varphi = -\pi$, we obtain real solutions on the second sheet:

$$\sqrt{r} = \alpha \pm \sqrt{\alpha^2 - x}. \tag{19}$$

The solution with the positive sign has a meaning when $-\infty \leq x \leq \alpha/\sqrt{r}$; the solution with the minus sign has a meaning only when $0 \leq x \leq \alpha^2$.

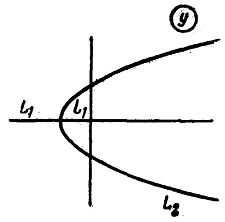

FIG. 2

We consider finally the case when $\sin(\varphi/2) = -\alpha/\sqrt{r}$. In this case all the roots of (15) lie on the second sheet and satisfy the equality $r = x$. It is necessary, apparently, to satisfy the condition $x \geq \alpha^2$, for which real solutions of (15) do not exist. The lines on which the poles of the Green's function are situated are shown in Fig. 2. The energies of the elementary excitations coincide with the poles situated on the line L_1 and the branch L_2 in the lower half-plane.

Transforming again to Δp, $\Delta\epsilon$, we obtain for $\Delta p < 0$

$$\Delta\varepsilon = v_0\Delta p + 2\alpha\varepsilon_c\left(\sqrt{\alpha^2 - (v_0 - v_c)\Delta p/\varepsilon_c} - \alpha\right). \tag{20}$$

Equation (20) shows that $\Delta\epsilon = v_0\Delta p$ for $|\Delta p| \gg \alpha^2\epsilon_c/(v_0 - v_c)$ and $\Delta\epsilon = v_c\Delta p$ for $|\Delta p| \ll \alpha^2\epsilon_c/(v_0 - v_c)$. We note, however, that as $\Delta p \to 0$ the residue of the Green's function diminishes as Δp, which causes excitations with dispersions $\Delta\epsilon = v_c\Delta p$ to have diminishing weight.

For $\Delta p \gtrsim \alpha^2\epsilon_c/(v_0 - v_c)$ the energy of the elementary excitations is given by the equation

$$\Delta\varepsilon = v_0\Delta p - 2\alpha^2\varepsilon_c - 2i\alpha\varepsilon_c\sqrt{[(v_0 - v_c)\Delta p/\varepsilon_c] - \alpha^2}. \tag{21}$$

For $\Delta p \gg \alpha^2\epsilon_c/(v_0 - v_c)$ formula (21) determines the energy and attenuation of the elementary excitations. The latter is proportional to $\sqrt{\Delta p}$.

In the region $(v_0 - v_c)\Delta p \sim \alpha^2\epsilon_c$ the poles of the Green's function are close to the branch point and both formulae (20) and (21) lose their meaning. To clarify the situation in this region we use a representation of the wave function of the excited system with the aid of the Green's function, which has been given by Galitskiĭ and Migdal:[2]

$$\langle\psi_p(t)\psi_p(0)\rangle = -iG(p, t) = i\int_0^\infty \operatorname{Im} G(p, \varepsilon)e^{-i\varepsilon t}d\varepsilon. \tag{22}$$

The integral (22) reduces to half the residue of $G(p, \epsilon)$ relative to the pole situated on the real axis [this pole is determined by Eq. (20)] and to the integral along the upper side of the cut.

In the variables x, y we have (for $\Delta p < 0$):

$$\langle\psi_p(t)\psi_p(0)\rangle = -i\pi\frac{xe^{-i\varepsilon(p)t}}{\sqrt{\alpha^2 - x}\,(\alpha + \sqrt{\alpha^2 - x})} + 2i\alpha\exp\{-i(\varepsilon_c + v_c\Delta p)t\}\int_0^\infty \frac{\sqrt{y}\,e^{-iy\varepsilon_c t}\,dy}{(x - y)^2 + 4\alpha^2 y}. \tag{23}$$

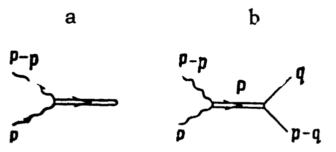

FIG. 3

The integral in (23) essentially diminishes in the time $\Delta t = 1/\alpha^2\epsilon_c$. In the course of this interval both terms in (23) play an equal role for $x \sim \alpha^2$. For $x \ll \alpha^2$ the second term is the most important and describes an excitation with energy ϵ_c and decay time $\sim 1/\alpha^2\epsilon_c$.

When $\Delta p > 0$, the first term in (23) disappears. The second term, for $\Delta p \gg \alpha^2\epsilon_c/(v_0 - v_c)$, as can be seen without difficulty, reduces to the residue with respect to the pole of the Green's function lying in the second sheet and determined by (21). When the coupling constant α becomes of the order of unity, the picture given above reduces to the case considered by Pitaevskiĭ.

In the case of weak coupling, the threshold effect gives a peculiar result in the neutron scattering spectrum. We will consider the scattering of neutrons by phonons in crystals of light elements (where anharmonic effects are comparatively large) at a temperature $T < \alpha^2 T_D$ (T_D is the Debye temperature).

Close to the threshold, the principal contribution to the scattering cross section is given by the diagrams in Fig. 3 (a, b). The diagram of Fig. 3,a contributes a sharp line, the intensity of which is given by the formulae

$$I \sim \begin{cases} 1 - (1 - (v_0 - v_c)\,\Delta p / \alpha^2)^{-1/2}, & \Delta p = p - p_c < 0, \\ 0 & \Delta p > 0, \end{cases} \tag{24}$$

and the position of ϵ is given by formula (20).

The diagram of Fig. 3,b makes a contribution to the neutron scattering cross section of the form

$$|G(p)|^2\,\delta(\varepsilon_P - \varepsilon_{\mathbf{P}-\mathbf{p}} - \varepsilon_q - \varepsilon_{\mathbf{p}-\mathbf{q}})\,d^3p\,d^3q, \tag{25}$$

Here $\mathbf{P}$ is the momentum of the neutron before scattering. After integrating the cross section (25) with respect to q, we obtain a formula for the distribution of scattered neutrons in energy for a given loss of momentum p:

$$dw = \frac{BF(p)\,d\varepsilon}{(v_0\Delta p - \Delta\varepsilon)^2 + 4\alpha^2(\Delta\varepsilon - v_c\Delta p)}, \tag{26}$$

$$F(p) = \begin{cases} \left(\dfrac{P^2}{2M} - \dfrac{(\mathbf{P}-\mathbf{p})^2}{2M} - \varepsilon_c - v_c\Delta p\right)\dfrac{1}{(ca)^2} & \text{for } \dfrac{P^2}{2M} - \dfrac{(\mathbf{P}-\mathbf{p})^2}{2M} > \varepsilon_c + v_c\Delta p \\ 0 & \text{for } \dfrac{P^2}{2M} - \dfrac{(\mathbf{P}-\mathbf{p})^2}{2M} < \varepsilon_c + v_c\Delta p. \end{cases} \tag{27}$$

Formula (26) corresponds to a line of width $\sim \alpha^2\epsilon_c$, which appears for $|v_c\Delta p| \sim \alpha^2\epsilon_c$ ($\Delta p < 0$).

Formulae (24) to (27) can be used to obtain the angular distribution and the distribution over energy of the scattered neutrons. Without giving the results, we will describe qualitatively the picture thus obtained.

For angles of scattering smaller than some φ_c, there is a sharp line in the energy distribution of neutrons at the energy $\epsilon_0(\varphi)$, the width of which depends on temperature and not on the angle of scattering. When the angle of scattering tends to φ_c, the intensity of this line diminishes as $\varphi_c - \varphi$. Apart from this line there is in the neutron energy distribution a background at energies greater than $\epsilon_0(\varphi)$. For $\varphi_c - \varphi \sim \alpha^2$ this background gradually gathers up into a line of width $\sim \alpha^2\epsilon_c$, while the intensity of the line increases as $\varphi - \varphi_c \to 0$. The center of this line lies at the energy $\epsilon_1(\varphi) > \epsilon_0(\varphi)$, where $\epsilon_1(\varphi) - \epsilon_0(\varphi) \sim \varphi_c - \varphi$. For $\varphi > \varphi_c$ there is only a smeared-out line.

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[1] L. P. Pitaevskiĭ, JETP **36**, 1168 (1959), Soviet Phys. JETP **9**, 830 (1959).

[2] V. M. Galitskiĭ and A. B. Migdal, JETP **34**, 139 (1958), Soviet Phys. JETP **7**, 96 (1958).

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