A RELATIVISTIC GENERAL THEORY OF REACTIONS

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The formulas of a relativistic general theory of reactions for the cross section and the polarization in terms of phase shifts may assume a different form for different definitions of the relativistic spin operator. However, in the rest system of the particle the spin operators coincide. This allows one to express the general theory in a form which is the same for all equivalent definitions of the spin.

INTRODUCTION

FOR the relativistic generalization of the formulas expressing the differential cross section and the polarization in terms of phases it is necessary to define the relativistic spin operator for the particle and to find the transformation functions for the transformation from a representation in terms of the momenta and the projections of the spins to a representation which is diagonal in the conserved total angular momentum of the system of interacting particles. In the papers of Chou Kuang-Chao and the author[1] and Yu. Shirokov[2] the Foldy—Yu. Shirokov representation[3] was used for the description of particles with spin. Chou Kuang-Chao and the author[1] arrived at this representation starting from a definition of the spin as the internal angular momentum of the particle with respect to its center of inertia.

However, other relativistic definitions of the coordinates of the center of inertia (see, for example, the papers of Pryce[4] and Yu. Shirokov[5]) are possible, for which the operator of the internal angular momentum is different from the spin of Pryce-Foldy-Shirokov (there is, for example, the possibility of a spin operator whose components do not commute, as the Pauli matrices; see references 4). Moreover, the Dirac spinors transform in a different way in going from one Lorentz frame to another than the spinor functions in the Foldy-Shirokov representation.* There exist still other possibilities of defining the relativistic spin operator (see, for example, references 5 and 6).

*It can be shown that these possibilities of a relativistic description of particles with spin correspond, mathematically, to non-unitary representations of the inhomogeneous Lorentz group (whereas the Foldy-Shirokov representation, which is particularly convenient for the general theory of reactions, is a unitary representation of this group).

Different definitions of the spin may give rise to different transformation functions and, correspondingly, to different expressions for the cross section and the polarization.* In the present paper we show that there exists a form of the general theory in which the transformation functions and formulas for the cross section and polarization are the same for all equivalent representations of the inhomogeneous Lorentz group (abbreviated ILG). We assume that the rest masses of all particles are different from zero and the spins are arbitrary.

We note that the wave functions describing the states of the interacting particles must transform according to representations of the ILG. The theory of representations of the ILG gives a description of systems all the states of which can be obtained from any arbitrary given state by translations, rotations, and Lorentz transformations (and superpositions of such states). There is thus no relativistically invariant difference between different states of the system.[7] The homogeneous Lorentz group does not contain translations and, therefore, cannot describe all states of the free particles.

THE SPIN IN THE REST SYSTEM OF THE PARTICLE AND THE GENERAL THEORY OF REACTIONS

1. The state of a free particle with spin is defined in the following way: 1) we give the momentum of the particle, **p**, (for example, in the c.m. system of the reaction), and 2) we give the spin

*It can, nevertheless, be shown that the angular correlations (for example, the azimuthal asymmetries in experiments on double and triple scattering) are the same, although they are expanded in terms of different complete systems of angular functions.

state of the particle in the Lorentz system in which it is at rest.

The spin operator of the particle in the rest system, $\mathbf{s}^0$, is equal to its total angular momentum M (since the orbital angular momentum in this system vanishes); hence all spin operators coincide in the rest system of the particle. The projection of the spin is defined as the eigenvalue of the operator $\Sigma = \mathbf{s}^0\mathbf{n} = \mathbf{Mn}$, where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{p}$ (with respect to the rest system).

2. The general principles of the formal theory of reactions have been given in a paper by the author[8] (Sec. 1) and in the paper of Jacob and Wick[9] (Introduction). The elements of the S matrix for a reaction of the type $a + b \rightarrow c + d$,

$$(m_c m_d \mathbf{p}' \,|\, S \,|\, m_a m_b \mathbf{p}) \tag{1}$$

must be expressed in terms of the phases* (more precisely, in terms of the elements of this matrix in the representation which is diagonal in the total angular momentum). The conservation of the total momentum is assumed to be already accounted for; $\mathbf{p}'$ and $\mathbf{p}$ are the relative momenta of the particles; the m's are the eigenvalues of the operators Σ of the separate particles. In contrast to reference 9, the spin functions which enter in the element (1) of the S matrix are referred to the c.m.s. of the particles (besides this, m_d in reference 9 is the eigenvalue of $\mathbf{M}_d(-\mathbf{n}')$, whereas here m_d denotes the eigenvalue of the operator $\mathbf{M}_d\mathbf{n}'$). However, the total angular momentum (being a quantity which is common to the initial and final states) must be referred to the same Lorentz frame (conveniently, the c.m.s. of the reaction) and, of course, to the same axis of quantization z.

The expression for the matrix element (1) in terms of the elements $(\lambda_c\lambda_d\mathbf{p}' \,|\, S_0 \,|\, \lambda_a\lambda_b\mathbf{p})$, which are referred to the c.m.s. of the reaction, must have the form

$$(m_c m_d \mathbf{p}' \,|\, S \,|\, m_a m_b \mathbf{p}) = q^*(m_c, m_d, p') \times (m_c m_d \mathbf{p}' \,|\, S_0 \,|\, m_a\, m_b\, \mathbf{p})\, q\,(m_a, m_b, p), \tag{2}$$

because we are dealing with Lorentz transformations with velocities parallel to the momenta of the particles (see the Appendix).

The elements of the matrix S_0 appearing in (2) can now be expressed in terms of the elements of S_0 in the representation determined by the squares and the z projections of the operator $\mathbf{J} = \mathbf{M}_1 + \mathbf{M}_2$, the total angular momentum (which is the spin of the system of interacting particles in the c.m.s. of the reaction), the operators Σ_1 and Σ_2, and the total energy E. The corresponding transformation function was obtained in the papers of Chou Kuang-Chao[10] and Jacob and Wick.[9] Its derivation (see reference 10, Sec. 2) does not rest on any assumptions about the particular representation in which the particles are described (in particular, even representations with vanishing rest mass are allowed). Using the fact that S_0 is diagonal with respect to the square, $J(J+1)$, and the projection, M, of the total angular momentum, we have (with the normalization of Jacob and Wick[9])

$$\begin{aligned}(m_c m_d \mathbf{p}' \,|\, S \,|\, m_a m_b \mathbf{p}) &= \frac{2J+1}{4\pi}\sum_{JM} D^J_{m_c+m_d,M}(-\pi, \vartheta', \pi-\varphi')\, q^*(m_c, m_d, p') \\ &\times (m_c m_d p' JM \,|\, S_0 \,|\, m_a m_b p JM)\, q\,(m_a, m_b, p) \\ &\times D^J_{M, m_a+m_b}(\varphi, \vartheta, 0).\end{aligned} \tag{3}$$

If we now introduce the matrix $\widetilde{S}$,

$$(m_c \,|\, \widetilde{S}^{JE} \,|\, m_a) = q^*(m_c, p')\,(m_c \,|\, S_0{}^{JE} \,|\, m_a)\, q\,(m_a, p), \tag{4}$$

we have completed the inverse transformation of (2), i.e., $\widetilde{S} = S$. We note that in the phase analysis we can only determine the product of all the factors on the right-hand side of (4), but not each factor separately.

The expression for the polarization tensors of the reaction products in terms of the elements (1) and the tensors of the beam and the target are determined by the nonrelativistic formulas (see, for example, reference 11). The elements $(m_c m_d \,|\, S^J \,|\, m_a m_b)$ can be introduced in these formulas. In particular, they appear in the expressions for the angular distribution and the polarization vector of reference 9. Of course, the polarization vector, for example, must be defined as the average value of the spin vector of the particle in its rest system. If it is initially known in some other Lorentz system, for example, in the laboratory system (polarized beam), one must find the corresponding expression in the rest system of the particle. To do this one may need a specific representation. In the remaining part of the paper we present a form of the general theory which is not significantly different from that discussed in detail by Jacob and Wick[9] and which is the same for different representations of particles with spin.

3. In this general theory of reactions still another problem comes up, the formulation and

*The invariance under four-dimensional rotations can also be expressed in the manner adopted by Stapp,[6] but only in terms of the phases can the unitarity of the S matrix be simply accounted for.

solution of which we shall demonstrate on the example of the double scattering of the proton.

Let us assume that we have found the polarization of the proton from the azimuthal asymmetry of the angular distribution of scattering II and that we want to use it for the phase analysis of scattering I. From the asymmetry of II one can find the components P_{z_2}, P_{y_2}, and P_{x_2} of the polarization vector in the direction of the proton momentum $\mathbf{p}_2$ in the c.m.s. of II (or in the laboratory system, since the target II is at rest), of the normal y_2 to the plane of scattering I, etc, as referred to the rest system of the proton K_2. For the phase analysis of scattering I we need the components P_{z_1}, P_{y_1}, and P_{x_1} (z_1 is parallel to the proton momentum $\mathbf{p}_1$ in the c.m.s. of I, $y_1 \parallel y_2$, etc.), referred to the rest system of the proton K_1, which differs from the rest system of the proton K_2 in the way explained below.

By rotating the components P_{z_2}, P_{y_2}, P_{x_2} about the angle between $\mathbf{p}_2$ and $\mathbf{p}_1$ we obtain the components P'_{z_1}, P'_{y_1}, P'_{x_1} referring to z_1, y_1, x_1 (for details see reference 11, Sec. 3), but expressed in the system K_2. In order to go from K_2 to K_1, we must carry out the following Lorentz transformations: 1) from K_2 to the laboratory system K_l by the velocity $\boldsymbol{\beta}_2 \parallel \mathbf{p}_2$, 2) from K_l to the c.m.s. of I by the velocity $\boldsymbol{\beta}$ parallel to the scattered beam I, and 3) from the c.m.s. of I to K_1. The corresponding velocity $\boldsymbol{\beta}_1$ is computed as the relativistic sum of the velocities $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}$. The product of these three transformations is a three-dimensional rotation (see reference 12, Sec. 22; reference 6; and also reference 1, footnote 5).

Thus the system of axes $z_1y_1x_1$ is oriented differently with respect to the spatial axes of K_2 than with respect to the axes of K_1. In other words, the vector $\mathbf{p}_1$ has different spherical angles with respect to K_2 than with respect to K_1.

The determination of the axis and the angle Ω of the above-mentioned rotation is a purely kinematical problem. In particular, Ω is the angle between the velocities ω and ω'' of Møller (reference 12, Sec. 22, formulas (59) and (59′), and to find $\sin\Omega$ it suffices to take the vector product of the expressions for ω and ω''.

The results are given in reference 1 in terms of the rotation of the spin vector with respect to the fixed spatial axes, which is equivalent to the above-mentioned rotation of the axes of K_1 with respect to the axes of K_2.

4. The general theory of reactions can thus be expressed in a form whose basic formula is (3), the same for different but equivalent representations of the inhomogeneous Lorentz group describing a particle with spin. Formula (3) agrees in form with the corresponding non-relativistic formula. In contrast to the non-relativistic case, the polarization tensors must be subjected to a certain rotation of relativistic origin in the phase analysis or in the determination of the angular correlations. The axis and the angle of this rotation are the same for different representations of particles with spin.

In conclusion I express my gratitude to Yu. M. Shirokov and I. V. Polubarinov for valuable comments.

APPENDIX

We presuppose the knowledge of the fundamentals of the theory of representations of the inhomogeneous Lorentz group (see, for example, references 13 and 14). The following considerations are valid for arbitrary representations of this group (not only for the unitary representations).

1. Since $\mathbf{s}^0$ is equal to M in the rest system, $[s_i^0, s_j^0] = i\epsilon_{ijk}s_k^0$. These commutation rules determine a representation of the three-dimensional rotation group, which can be assumed to be unitary.[15] Hence, s_k^0 can be assumed to be a Hermitian matrix.

2. We introduce the four-dimensional vector

$$\Gamma_\mu = \frac{1}{2i} \sum_{\nu,\sigma,\lambda} \varepsilon_{\mu\nu\sigma\lambda} M_{\nu\sigma} p_\lambda$$

($\epsilon_{\mu\nu\sigma\lambda}$ is the completely antisymmetric unit tensor of fourth rank), whose length Γ^2 is an invariant of the inhomogeneous Lorentz group.[13] In the rest system $\boldsymbol{\Gamma} = \kappa \mathbf{M}$, $\Gamma_4 = 0$ (κ is the rest mass of the particle), i.e., $\mathbf{s}^0 = \boldsymbol{\Gamma}/\kappa$. It follows from this that the square of the spin in the rest system, $(\mathbf{s}^0)^2$, is equal to the Lorentz invariant Γ^2/κ^2, which characterizes (together with the mass κ) a definite irreducible representation of the ILG ($\boldsymbol{\Gamma}^2$ determines, in particular, the number of components of the wave function of the particle).

3. Without using a definite representation we cannot determine the transformation properties of the spinor functions under changes of the Lorentz frame. But in our form of the general theory we need only know how the spinor functions transform under Lorentz transformations Λ with a velocity $\boldsymbol{\beta}$ parallel to the momentum $\mathbf{p}$ of the particle. The operator $\Sigma = \mathbf{s}^0\mathbf{n} = \Gamma_4/i\kappa\,|\mathbf{p}|$ is invariant under such transformations. Indeed, if

$\beta = \alpha \mathbf{p}/p_0$ ($0 < \alpha < 1$), then

$$\frac{\Gamma_4'}{|\mathbf{p}'|} = \frac{\gamma\{\Gamma_4 - i\beta\Gamma\}}{|\mathbf{p} + \beta[\beta\mathbf{p}(\gamma-1)/\beta^2 - p_0\gamma]|} = \frac{\Gamma_4}{|\mathbf{p}|}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(we made use of the equation $\sum_\mu \Gamma_\mu p_\mu = 0$).

Let us denote the representation of the group of transformations Λ by U_Λ, so that $\psi = U_\Lambda\psi'$. We have shown that $\Sigma' = U_\Lambda^{-1}\Sigma U_\Lambda = \Sigma$, i.e., $[\Sigma, U_\Lambda] = 0$. This implies that the matrix U_Λ is diagonal with respect to the eigenvalues of Σ, and the state $|\mathbf{p}, m\rangle$ goes over into

$$U_\Lambda|\mathbf{p}m\rangle = \sum_{m'}|\Lambda\mathbf{p}, m'\rangle Q_{m',m}(\mathbf{p}, \Lambda) = |\Lambda\mathbf{p}, m\rangle q(m, \mathbf{p}) \tag{A.1}$$

for transformations Λ.

We now show that the diagonal elements $q(m, \mathbf{p})$ of the spinor part of the transformation U_Λ depend only on $|\mathbf{p}|$.

The generator of U_Λ is an operator proportional to $\mathbf{pN}$, so that

$$U_\Lambda = \exp i\{-\tanh^{-1}\beta(\mathbf{pN})/|\mathbf{p}|\}.$$

The operator $\mathbf{p}\cdot\mathbf{N}$ is a three-dimensional scalar and, therefore, commutes with the three-dimensional rotation operators U_R. Hence

$$U_R U_\Lambda|\mathbf{p}m\rangle = U_\Lambda U_R|\mathbf{p}m\rangle$$

or*

$$R\Lambda\mathbf{p}, m\rangle q(m, \mathbf{p}) = |\Lambda R\mathbf{p}, m\rangle q(m, R\mathbf{p}), \tag{A.2}$$

from where we conclude

$$q(m, \mathbf{p}) = q(m, R\mathbf{p}) \equiv q(m, |\mathbf{p}|).$$

4. In conclusion we show that the proof of the equivalence of the irreducible representations of the ILG with the same values of κ^2 and Γ^2, given by Wigner,[14] can apparently be assumed to be valid for non-unitary representations as well. Wigner showed that an arbitrary representation is equivalent to the representation U_0, which is the product of the representation of some rotation within the "little group" (for particles with finite mass this group consists of the three-dimensional rotations in the space of wave functions with $p_0 = 0$) and a representation of the Lorentz transformation of the class Λ which acts only on the momentum variables [see formulas (67) and (67a) in reference 14]. More precisely, we have $Q^0_{m,m'} \times (\mathbf{p}, \Lambda(\mathbf{p})) = \delta_{m,m'}$ for transformations $\Lambda(\mathbf{p})$ from the rest system to a system where the momentum of the particle is equal to $\mathbf{p}$. If a given representation does not satisfy this requirement, then an equivalent representation which is obtained by a transformation which takes the function $\varphi(\mathbf{p}, m)$ into

$$\sum_{m'} Q_{m,m'}(\mathbf{p}_0, \Lambda^{-1}(\mathbf{p}))\varphi(\mathbf{p}, m')$$

does.

We emphasize that this transformation is among those generated by the operators (representing the transformations Λ) of the given representation.

For an arbitrary representation of the ILG one can thus find a transformation (not necessarily unitary) by which this representation is brought into the form (67a) of reference 14, which is identical for all representations.

*We fix the phases of states $|\mathbf{p}m\rangle$ with different m by defining $|\mathbf{p}m\rangle$ as

$$|\mathbf{p}m\rangle = \sum_n |\mathbf{p}n\rangle D_{n,m}(\mathbf{p}),$$

with $D(\mathbf{p})$ being the spinor part of U_R which depends on the Eulerian angles of rotation $\{-\pi, \vartheta, \pi - \varphi\}$, where ϑ and φ are the spherical angles of the momentum $\mathbf{p}$ in some fixed reference system, to which the projections n are also referred.[8] Then

$$U_R|\mathbf{p}m\rangle = \sum_{n',n}|R\mathbf{p}, n'\rangle D_{n',n}(R)D_{n,m}(\mathbf{p}) = \sum_{n'}|R\mathbf{p}, n'\rangle D_{n',m}(R\mathbf{p}) = |R\mathbf{p}, m\rangle.$$

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