CONTRIBUTION TO THE THEORY OF SIMPLE MAGNETOHYDRODYNAMIC WAVES

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The Riemann invariants are computed for simple magnetohydrodynamic waves. The change of velocity in fast and slow magnetoacoustic waves is determined for the case when the magnetic pressure in front of the wave is much smaller than the hydrostatic pressure.

1. BASIC EQUATIONS

We use the name "simple waves" for those solutions of the equations of magnetohydrodynamics, which have the form of a traveling wave[1]

$$u_i = f_i(x \pm Vt) \qquad (i = 1, 2, \ldots, n), \qquad (1)$$

where $u_1, u_2 \ldots, u_n$ is the totality of the magnetohydrodynamic quantities such as the pressure p, the density ρ, the three velocity components v_x, v_y, and v_z, and the two transverse components of the magnetic field H_y and H_z (we confine ourselves to plane one-dimensional waves, propagating in the x direction, so that H_x is constant). In the magnetohydrodynamics of an ideally conducting non-viscous medium, there are seven magnetohydrodynamic quantities n. It follows from (1.1) that in a simple wave all the magnetohydrodynamic quantities can be expressed in terms of one among them, for example ρ, which in turn depends on the coordinates and on the time.*

The quantity V, which enters in (1.1), is a function of ρ. This means that, in a simple wave, points which have different densities are displaced with different velocities. Thus, the profile of the simple wave is distorted as the wave propagates. It can be shown that the steepness of the wave decreases in the rarefaction regions ($d\rho/dt < 0$) and increases in the compression regions ($d\rho/dt > 0$).[4,6,7] This leads, in final analysis, to the formation of discontinuities (shock waves) in the compression regions.

The simple wave plays a special role in magnetohydrodynamics because in the absence of discontinuities, only a simple wave can have a boundary with a region of a steady flow.[3,5] It must be noted, however, that this theorem is somewhat of lesser importance in magnetohydrodynamics than in ordinary hydrodynamics. Thus, when a piston moves out in a non-conducting medium that obeys the equations of ordinary hydrodynamics, the region of steady flow along the characteristic $dx/dt = c$ (c is a velocity of sound) bounds on a simple wave that reaches the piston (see Fig. 1A). On the other hand, if a piston moves in a magnetohydrodynamic medium, two characteristics emerge from the point x = 0, t = 0: a 'fast' characteristic, $dx/dt = U_+$, and a 'slow' characteristic $dx/dt = U_-$ (U_+ and U_- are the propagation speeds of the fast and slow magnetoacoustic wave, see Fig. 1b). In the region enclosed between the two characteristics, a simple wave is produced, whereas in the region enclosed between the slow characteristic and the piston, the motion of the medium is in general not described by a simple wave.*

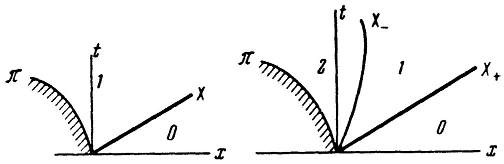

FIG. 1. Waves produced in the motion of the piston: a – in ordinary hydrodynamics, b – in magnetohydrodynamics. Π – line of motion of piston; X – characteristic in ordinary hydrodynamics, $dx/dt = v_x + c$; X_+ – 'fast' characteristic in magnetohydrodynamics, $dx/dt = v_x + U_+$; X_- – 'slow' characteristic in magnetohydrodynamics, $dx/dt = v_x + U_-$; O – standstill region; 1 – simple wave; 2 – non-simple wave.

However, if the piston moves with constant velocity, then the problem does not contain a parameter with the dimension of length. There-

*The first published paper on simple waves in magnetohydrodynamics is that of Owens.[2] The investigation of simple waves for $H_x \neq 0$ is the subject of papers by Friedrichs and Lax,[3] Kulikovskiĭ,[4] Akhiezer, Lyubarskiĭ, and Polovin.[5-7]

*This can be seen even from the fact that a simple wave is characterized by one arbitrary function,[5] whereas on the surface of the piston there should be satisfied boundary conditions[8] characterized by two arbitrary functions, $v_x(t)$ and $v_y(t)$.

fore all the magnetohydrodynamic quantities depend on the coordinates and the time only through the combination x/t. Such waves, called self-similar, are a particular case of simple waves. Self-similar waves are always rarefaction waves.[6,7]

In magnetohydrodynamics there are three types of simple waves:[5] fast and slow magnetoacoustic waves, Alfven waves, and entropy waves. The greatest interest attaches to magnetoacoustic waves. The latter are always plane: if the velocity and magnetic field vectors in the xy plane ($v_z = 0$, $H_z = 0$) at the initial instant then these relations will continue to be satisfied.

If the medium is described by the equation of state of an ideal gas, $p\rho^{-\gamma}$ = const, then, as shown by Friedrichs (see reference 9), the integration of the equation of simple waves reduces to the integration of a linear differential equation

$$dr/dq_\pm = \theta(rq_\pm^2 - 1)/q_\pm^2(q_\pm - 1),$$

$$r = c^2/U_x^2 \equiv 4\pi\gamma p/H_x^2, \qquad \theta = \gamma/(2-\gamma),$$

$$q_\pm = U_\pm^2/c^2,\ U_\pm = \{[U^2 + c^2 \pm \sqrt{(U^2+c^2)^2 - 4c^2U_x^2}\,]/2\}^{1/2}, \tag{1.2}$$

where c is the velocity of sound, equal to $\gamma p/\rho$; U is the Alfven velocity, equal to $\mathbf{H}/\sqrt{4\pi\rho}$. As can be seen from the definition, r is a dimensionless pressure.

Equation (1.1) makes it possible to determine the quantity $q_\pm$ as a function of r, after which the velocity components are found from the equations

$$dv_x/dr = \varepsilon c\sqrt{q_\pm}/\gamma r, \tag{1.3}$$

$$dv_y/dr = \mp \varepsilon c \gamma^{-1} r^{-1}(q_\pm - 1)^{1/2}(rq_\pm - 1)^{1/2}\,\mathrm{sign}(H_x H_{1y}). \tag{1.4}$$

The upper and lower signs in (1.4) correspond to the fast and slow waves, respectively; $\epsilon = +1$ for waves propagating in the direction of positive x and $\epsilon = -1$ corresponds to the opposite direction. The subscript 1 (designating H_y) pertains to the region in front of the wave.

The transverse magnetic field H_y is determined from

$$H_y = H_x\sqrt{(q_\pm - 1)(rq_\pm - 1)/q_\pm}\,\mathrm{sign}\,H_{1y}. \tag{1.5}$$

2. QUALITATIVE INVESTIGATION

From the equations of simple waves it follows that the transverse magnetic field Hy decreases in a fast self-similar wave and increases in a slow wave.[6,7] If the wave propagates in the direction of positive x, then the longitudinal velocity component

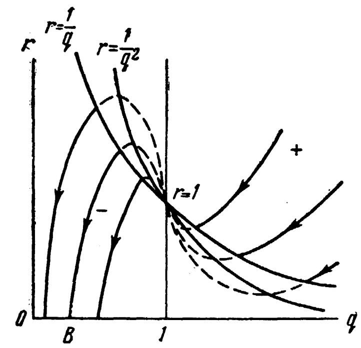

FIG. 2. Integral curves of Eq. (1.1). The + sign designates the fast waves and the − sign the slow waves. The non-existing portions of the integral curves (on which the transverse magnetic field H_y becomes imaginary) are shown dashed. The arrows indicate the direction of displacement in the plane (q, r) in the self-similar wave. The region of the abscissa axis on which a vacuum is produced behind the slow wave is designated by the letter B.

v_x decreases in both slow and fast magnetoacoustic waves. If, in addition, the quantity H_xH_y is positive, then the transverse velocity component v_y increases in the fast magnetoacoustic wave and decreases in a slow one.

From the definition of the quantities r and $q_\pm$ it follows that the inequalities

$$rq_+ \geqslant 1, \quad q_+ > 1, \qquad rq_+^2 > 1, \tag{2.1}$$

are satisfied in a fast magnetoacoustic wave, while in a slow wave we have

$$rq_- \leqslant 1, \quad q_- < 1, \qquad rq_-^2 < 1. \tag{2.2}$$

From inequalities (2.1) and (2.2) and from Eq. (1.2) it follows that $dr/dq_\pm > 0$. Since the pressure r decreases in self-similar waves, the quantities $q_\pm$ also decrease (in both slow and fast magnetoacoustic waves). The first inequality in (2.1) or (2.2), namely $rq_\pm \geq 1$, determines the simple wave with maximum amplitude. Magnetoacoustic waves in which $rq_\pm = 1$ will be called complete. It is seen from relations (2.1) and (2.2) that the equality $rq_\pm = 1$ can be reached behind a fast wave or in front of a slow wave.

When $rq_\pm = 1$ the transverse magnetic field vanishes. The point $r = 1$; $q_\pm = 1$ is a singular point (node) of (1.2). In the vicinity of the singular point, the equation of the integral curves has the form

$$r = 1 - \gamma(q-1)/(\gamma-1) + C(q-1)^\theta,$$

where C is a constant of integration. At the singular point all the integral curves have a common tangent, with a slope

$$dr/dq = -\gamma/(\gamma - 1).$$

At the point of intersection of the integral curves with the line $rq^2 = 1$, the derivative dr/dq vanishes and the integral curves have horizontal tangents.

The overall shape of the integral curves of (1.2) is shown in Fig. 2. The non-existent portions of the integral curves (corresponding to imaginary values of H_y) are indicated by dashed lines. The arrows indicate the direction of displacement in the plane (r, q) upon propagation of a self-similar wave. The fast and slow magnetoacoustic waves are marked "+" and "−" respectively. On the line $r = 0$ the pressure vanishes — cavitation sets in. This line is indicated in Fig. 2 by the letter B. As can be seen from Fig. 2, cavitation can take place only in a slow magnetoacoustic wave. It is also seen from Fig. 2 that the inequality $r < 1$ is satisfied behind a complete fast wave, and the inequality $r > 1$ is satisfied ahead of a complete slow wave.*

3. THE RIEMANN INVARIANTS

In ordinary hydrodynamics, the Riemann invariant is defined as a function J of the hydrodynamics quantities $u_1, u_2, \ldots, u_n$, which remains constant along a certain characteristic. According to this definition, there are no Riemann invariants in magnetohydrodynamics. In fact, the equations of magnetohydrodynamics can be written schematically in the form

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} X_{ik}(u_1, \ldots, u_n)\frac{\partial u_k}{\partial x} = 0 \quad (i = 1, \ldots, n). \tag{3.1}$$

Since the Riemann invariant $J(u_1, \ldots, u_n)$ is constant along the characteristic, we have

$$\frac{\partial J}{\partial t} + V(u_1, \ldots, u_n)\frac{\partial J}{\partial x} = 0, \tag{3.2}$$

where V is the slope of the characteristic.

Equation (3.2) should be the consequence of (3.1). There should therefore exist functions μ_i $(u_1, \ldots, u_n)$ such that the equation

$$\sum_i \mu_i \frac{\partial u_i}{\partial t} + \sum_{i,k} \mu_i X_{ik}\frac{\partial u_k}{\partial x} = 0. \tag{3.3}$$

becomes identical with Eq. (3.2), which we write in the form

$$\sum_i \frac{\partial J}{\partial u_i}\frac{\partial u_i}{\partial t} + \sum_k V\frac{\partial J}{\partial u_k}\frac{\partial u_k}{\partial x} = 0. \tag{3.4}$$

Comparing (3.3) and (3.4), we find

$$\mu_i = \partial J / \partial u_i, \tag{3.5}$$

$$\sum \mu_i X_{ik} = V \partial J / \partial u_k = V\mu_k. \tag{3.6}$$

From relations (3.5) we obtain $n(n-1)/2$ equations

$$\partial\mu_i/\partial x_k = \partial\mu_k/\partial x_i,$$

which must be satisfied by the functions μ. The system (3.6) contains an additional $(n-1)$ equation. Thus, the n functions $\mu_1, \ldots, \mu_n$ must satisfy $(n-1) + n(n-1)2 = (n-1)(n+2)/2$ equations, which is generally impossible if $n > 2$.* Consequently, the definition given above for the Riemann invariants cannot be extended to include magnetohydrodynamics.

However, it is possible to modify the definition of the Riemann invariant so as to make it meaningful in magnetohydrodynamics.[3] We shall define as a Riemann invariant a function of the magnetohydrodynamic quantities $J(u_1, \ldots, u_n)$, which remains constant in a simple wave. Inasmuch as simple waves are described by a system of ordinary differential equations, there exist $(n-1)$ integrals of the system of equations, and these are the Riemann invariants. To each simple wave there correspond thus $(n-1)$ Riemann invariants.

A simple magnetoacoustic wave, according to (1.2) — (1.4), has four Riemann invariants:

$$J_1 = r(q-1)^{-\theta} + \theta\int q^{-2}(q-1)^{-\theta-1}dq, \tag{3.7}$$

$$J_2 = v_x + \varepsilon\gamma^{-1}\int c\sqrt{q}\,r^{-1}\,dr, \tag{3.8}$$

$$J_3 = v_y \pm \varepsilon\gamma^{-1}\,\mathrm{sign}(H_x H_{1y}) \cdot \int (q-1)^{1/2}(rq-1)^{-1/2} c r^{-1}\,dr, \tag{3.9}$$

$$J_4 = p\rho^{-\gamma}. \tag{3.10}$$

When $\gamma = 5/3$, the invariant J_1 is expressed in terms of elementary functions:

$$J_1 = (r-1)(q-1)^{-5} + \frac{5}{2}(q-1)^{-4} - 5(q-1)^{-3} + 10(q-1)^{-2} - 25(q-1)^{-1} - 5q^{-1} + 30\ln(q/|q-1|). \tag{3.11}$$

The invariants J_2 and J_3 make it possible to determine the discontinuities in the quantities v_x and v_y:

$$\Delta_\pm v_x = -\varepsilon\gamma^{-1}\int_{r_2}^{r_1} c\sqrt{q}\,r^{-1}dr, \tag{3.12}$$

*It follows therefore that the fast and slow waves cannot be simultaneously complete. A complete slow wave can exist only when there is no fast wave and when the transverse magnetic field H_y in front of the wave vanishes.

*Equations (3.5) and(3.6) become compatible through accidental degeneracy. It is easily shown, however, that there is no such degeneracy in magnetohydrodynamics, and equations (3.5) and (3.6) are incompatible.

$$\Delta_{\pm} v_y = \pm \varepsilon \gamma^{-1} \operatorname{sign}(H_x H_{1y}) \cdot \int_{r_2}^{r_1} (q-1)^{1/2} (rq-1)^{-1/2} c r^{-1} dr. \quad (3.13)$$

The index 2 pertains to the region behind the wave.

As was shown by Lax,[3] the Riemann invariants are conserved in shock waves of low intensity, accurately to quantities of second order inclusive.

4. FAST WAVE WHEN $U_1 \ll c_1$

Equations (3.12) and (3.13) become much simpler when the magnetic pressure $H_1^2/8\pi$ ahead of the simple wave is much less than the hydrostatic pressure p_1 (this is equivalent to satisfying the inequality $U_1 \ll c_1$). In this case

$$r_1 \gg 1, \quad q_{1+} = 1 + U_{1y}^2 / c_1^2 + O(U_1^4 / c_1^4).$$

Since the quantity q_{1+} diminishes and becomes greater than unity in a self-similar wave, the quantity $\xi \equiv q_+ - 1$ is positive and much smaller than unity. It follows from (3.12) that

$$\Delta_+ v_x = -2(c_1 - c_2)/(\gamma - 1). \quad (4.1)$$

In order to transform (3.13), we rewrite (1.2) as

$$dr/d\xi = \theta(r-1)\xi^{-1} + 2\theta, \qquad \xi \ll 1,$$

hence

$$r = 1 + r_1 \xi^{\theta} \xi_1^{-\theta} - \gamma(\gamma-1)^{-1}\xi. \quad (4.2)$$

We denote the smallest value of ξ reached in a complete fast wave by ξ_m. It follows from (4.2) that

$$\xi_m = \xi_1 [\xi_1 / (\gamma - 1) r_1]^{1/(\theta - 1)} \ll \xi_1. \quad (4.3)$$

The corresponding value of r is determined from the expression

$$r_m = 1 - \xi_m \approx 1.$$

If $\xi_2 \gg \xi_m$, (3.13) becomes much simpler

$$\Delta_+ v_y = \varepsilon c_1 \sqrt{\xi_1} \gamma^{-1} r_1^{-1/2\gamma} \operatorname{sign}(H_x H_{1y}) \times \int_{r_2}^{\infty} r^{-(\gamma+1)/2\gamma} (r-1)^{-(\gamma-1)/\gamma} dr. \quad (4.4)$$

The expression (4.4) can be used for any value of ξ_2, since the main contribution to the integral (3.13) is made by values $\xi > \xi_m$. In fact, when $\xi \sim \xi_m$, the value of r is close to unity, and therefore $c_2 \approx c_2 r_1^{-(\gamma-1)/2\gamma}$. The contribution to the integral (3.13) from the values $\xi \sim \xi_m$ is given by the formula

$$\Delta' v_y = \frac{c_1}{r_1^{(\gamma-1)/2\gamma}} \int_{\xi_2}^{\xi_c} \frac{[\theta r_1 \xi^{\theta-1} \xi_1^{-\theta} - \gamma(\gamma-1)^{-1}]\, d\xi}{[r_1 \xi^{\theta-1} \xi_1^{-\theta} - (\gamma-1)^{-1}]^{1/2}},$$

where ξ_c satisfies the inequalities $\xi_m \ll \xi_c \ll \xi_1 r_1^{-1/\theta}$. The main contribution to the integral $\Delta' v_y$ is made by the quantities* $\xi \sim \xi_2$. Therefore

$$\Delta' v_y \sim c_1 \xi_1^{\gamma/2(\gamma-1)} r_1^{-1/2\gamma(\gamma-1)},$$

which is much less than

$$\Delta_+ v_y \sim c_1 \xi_1^{1/2} r_1^{-1/2\gamma}.$$

If $r_2 \to 1$, it follows from (4.4) that

$$\Delta_+ v_y = \varepsilon U_{1y} (U_{1x}/c_1)^{1/\gamma} h(\gamma) \operatorname{sign}(H_x H_{1y}), \quad (4.5)$$

$$h(\gamma) = \frac{\Gamma[(\gamma-1)/2\gamma]\,\Gamma(1/\gamma)}{\gamma \Gamma[(\gamma+1)/2\gamma]}, \qquad h(5/3) = 3.52\ldots$$

5. SLOW WAVE WHEN $U_1 \ll c_1$

In front of a slow wave, where $r_1 \gg 1$, the quantity q_{1-} is defined by the expression

$$q_{1-} = r_1^{-1} + O(U_1^4 / c_1^4) \ll 1. \quad (5.1)$$

As a result of the decrease in pressure r in a self-similar wave, the inequality $r \gg 1$ is valid only in front of the wave, and relation (5.1) soon ceases to hold. However, since q decreases, the inequality $q \ll 1$ holds over the entire slow wave.

From (1.2) it follows that when $q \ll 1$

$$r = (1+\theta) q_1^{-1} - \theta q^{-1}. \quad (5.2)$$

From (5.2) we find that at the point where cavitation sets in ($r_2 = 0$), the value of q is determined by the relation

$$q_2 = \gamma q_1 / 2.$$

Using (5.2) we find the discontinuities in the velocity components in a slow wave

$$\Delta_- v_x = -\frac{\varepsilon U_{1x}}{\sqrt{2\gamma}} \int_{\sigma_2}^{1} \frac{\sigma^{-(\gamma+1)/2\gamma} d\sigma}{[1 - \sigma(\theta+1)^{-1}]^{1/2}}; \quad (5.3)$$

$$\Delta_- v_y = -\frac{\varepsilon c_1}{\gamma} \operatorname{sign}(H_x H_{1y}) \cdot \int_{\sigma_2}^{1} \sqrt{\frac{1 - \sigma(\theta+1)^{-1}}{1-\sigma}}\, \sigma^{-(\gamma+1)/2\gamma} d\sigma, \quad (5.4)$$

$\sigma = r/r_1$. If cavitation sets in on a slow wave ($\sigma_2 = 0$), then the expressions for $\Delta_- v_x$ and $\Delta_- v_y$ are

$$\Delta_- v_x = -\varepsilon U_{1x} f(\gamma), \qquad \Delta_- v_y = -\varepsilon c_1 g(\gamma) \operatorname{sign}(H_x H_{1y}), \quad (5.5)$$

where

$$f(\gamma) = \frac{1}{\sqrt{2\gamma}} \int_0^1 \frac{\sigma^{-(\gamma+1)/2\gamma}}{[1 - \sigma(\theta+1)^{-1}]^{1/2}} d\sigma, \quad (5.6)$$

$$g(\gamma) = \frac{1}{\gamma} \int_0^1 \sqrt{\frac{1 - \sigma(\theta+1)^{-1}}{1-\sigma}}\, \sigma^{-(\gamma+1)/2\gamma} d\sigma. \quad (5.7)$$

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*This can be readily verified by comparing the contribution to $\Delta' v_y$ made by the quantities $\xi/\xi_m \to 1$ and $\xi/\xi_m \gg 1$.

To calculate the quantities $f(\gamma)$ and $g(\gamma)$ it is necessary to expand $[1-\sigma(\theta+1)^{-1}]^{1/2}$ in powers of σ. Integrating term by term, we get

$$f(\gamma)=\sqrt{2\gamma}\left[\frac{1}{\gamma-1}+\frac{1}{2}\frac{1}{\theta+1}\frac{1}{3\gamma-1}+\frac{1\cdot 3}{2^3}\frac{1}{(\theta+1)^2}\frac{1}{5\gamma-1}+\ldots\right],$$

$$g(\gamma)=\frac{\sqrt{\pi}\Gamma[(\gamma-1)/2\gamma]}{\gamma\Gamma[(2\gamma-1)/2\gamma]}\left[1-\frac{1}{2(\theta+1)}\frac{\gamma-1}{2\gamma-1}-\frac{1}{8(\theta+1)^2}\frac{\gamma-1}{2\gamma-1}\frac{3\gamma-1}{4\gamma-1}-\ldots\right].$$

When $\gamma = 5/3$ we have $f(5/3) = 2.78\ldots$ and $g(5/3) = 3.67\ldots$ The transverse magnetic field H_y is given by the relation

$$H_y=\sqrt{8\pi\rho_1(1-\sigma)/\gamma}\, c_1 \operatorname{sign} H_y. \tag{5.8}$$

The formulas obtained make it possible to solve the problems of the magnetohydrodynamic piston and of the escape of a magnetohydrodynamic medium into vacuum.

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