A NEW RESONANCE EFFECT IN METALS AT HIGH FREQUENCIES

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The current density **j** and the electric field intensity **E** in metals is investigated at frequencies exceeding those for which cyclotron resonance was previously studied.[1] It is shown that, in the case of resonance, **j** and **E** on the central cross section of the Fermi surface (when the latter is not an ellipsoid) vary with the depth y in a very peculiar way. At not too great a depth ($y \ll d^2/\delta_0$, where d is the Larmor orbit diameter and δ_0 is the skin depth, $d \gg \delta_0$) the field strength and the current have sharp maxima not only at the surface of the metal but also for $y \approx d, 2d, 3d, \ldots$, (Fig. 2). For $\gtrsim d^2/\delta_0$, the current and the field oscillate at distances of the order of d with a damping depth of the order of d^2/δ_0. In this connection, some new effects are predicted, such as discontinuities in the resonance impedance (with resonance conditions maintained) and discontinuous disappearance of resonance at the harmonics in plates of thickness $D > d$: 1) when the frequency of the field increases and 2) when a constant magnetic field is rotated in the plane of the film. Other new effects are selective transparency of films at resonance, and an electronic "echo" similar to the spin "echo." A study of the impedance of plates permits one to plot the Fermi surface directly. It is also shown that cyclotron resonance rather than diamagnetic resonance takes place in a number of semiconductors and poor metals.

1. PHYSICAL REASON FOR SPIKES IN FIELD AND CURRENT

It is well known that a variable electromagnetic field in a metal is damped the more rapidly the higher the frequency of the field. However, we shall show that there exists a very important special case in which, in a metal at a depth much greater (by two orders of magnitude) than the usual skin depth, there exist sharp maxima of the value of the field. The field in these spikes is of the same order of magnitude as on the surface of the metal.

Inasmuch as "spikes" of the field much deeper in the metal than the skin depth have not been encountered earlier, so far as we know, we shall first study the physical peculiarities of the given case.

Let us first consider the motion of electrons in a metal along one of the orbits (orbit 1, Fig. 1) passing through the skin layer close to the surface of the metal, where the electric field is not small (the constant magnetic field **H** is perpendicular to the plane of the drawing). In a layer of the order of δ, the electrons along a path of length $\sqrt{r\delta}$ receive a directed velocity and produce a current J and a current density **j**.

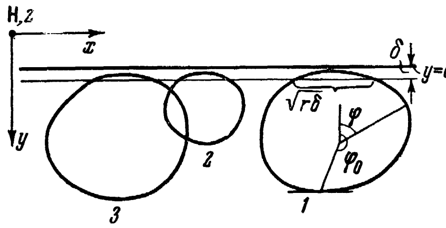

FIG. 1

Downward motion along the orbit changes, in the first place, the electron velocity parallel to the plane of the metal (only then does current flow), causing current to change in proportion to cos φ; in the second place, the electrons will "disperse" throughout the depth, being contained (for φ w $\sqrt{\delta/r}$) not in a layer δ but in a layer $\sqrt{r\delta}$ sin φ.

Thus the current density brought about by electrons of a given orbit is shown to be of the order J cot $\varphi/\sqrt{\delta r}$, i.e., decreasing rapidly with depth; for $\varphi \sim 1$, it will be $\sqrt{r/\delta}$ less than in the layer δ. A value of **j** of the same order as on the surface ($\mathbf{j} \sim J/\delta$) is evidently achieved when $\varphi \sim \sqrt{\delta/r}$, i.e., at a distance of the order of δ. At a depth $y > r$, the current density changes sign, remaining small in absolute value in comparison with J/δ until the angle φ come close enough to φ_0 so that

$|\varphi_0 - \varphi| \lesssim \sqrt{\delta/r}$. At a depth d, the current density would differ in this case from the current density on the surface only in its sign.

Such a current density would create an electric field which would cause directed motion (along the surface) of other electrons moving in the body of the metal, and the appearance of a spike of current and field at y = 2d, etc. As a result, it would be necessary to determine this self-consistent system of currents and fields. However, from earlier considerations given above, it is clear that the graph plotted in Fig. 2 is completely natural.

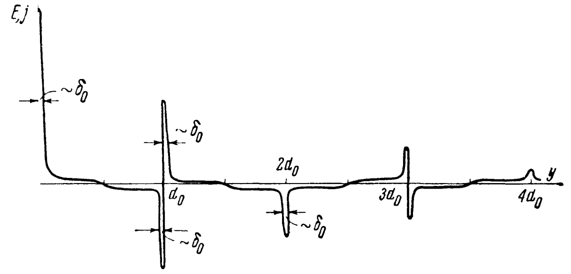

FIG. 2

So far we have discussed one of the orbits penetrating through the skin layer, an orbit with a given radius r. It is clear that all orbits of a given radius penetrating the skin layer (the scatter of the coordinates of their centers is obviously of the order δ) do not change the character of the picture just described.

Another picture arises in the presence of orbits with different radii (for example, orbits 2 or 3 in Fig. 1), corresponding to different cross sections of the Fermi surface [we recall, for example, that for free electrons $r = p_\perp c/eH = c\sqrt{2m\epsilon - p_z^2}/eH$, $0 \le p_z \le \sqrt{2m\epsilon}$, where $\mathbf{p}$ is the quasi-momentum and r changes from 0 to $c\sqrt{2m\epsilon}/eH$]. The scatter in the radii evidently leads to the result that only a small part of the electrons (of order δ/r) is "collected" at an arbitrary depth in a layer of order δ, and the field "pulled along" into the depth will naturally fall off extremely rapidly in subsequent "links" when $y > d$.

In the case in which the cyclotron frequency Ω of rotation around an orbit does not depend on the cross section (i.e., when the Fermi surface is an ellipsoid), this scatter in the radii cannot be eliminated. But if the cyclotron resonance depends on the cross section (i.e., on p_z, where z is the direction of the constant magnetic field), then it is possible to make use of the cyclotron resonance to eliminate a significant amount of the scatter in the radii; in this resonance (see reference 1), only electrons close to the extremal cyclotron frequencies, with scatter in p_z of the order of Δp_z, take part:

$$|\omega - \Omega(p_z)| \sim 1/\tau, \quad \Delta p_z \sim p_0/\sqrt{\omega\tau}$$

(p_0 is of the order of the limiting Fermi momentum, τ is the time of free flight of the electrons, and ω is the frequency of the rf field); for the existence of resonance it is necessary in any case that $\omega\tau \gg 1$.

The scatter in the radii possible for a given $\omega\tau$ is quite different for the central cross section where, as is known from symmetry considerations, Ω and d have extrema simultaneously, and for other cross sections, corresponding to an extremum of Ω, where there is no reason to expect an extremum of d.

In the first case, when $d'(0) = 0$, $\Delta d = (\frac{1}{2})d''(0)(\Delta p_z)^2 \sim d/\omega\tau$, for the existence of the structure of Fig. 2 it is necessary (and, as will be seen below, sufficient) that $\Delta d \sim \delta_0$, i.e.,

$$\omega\tau \gtrsim r/\delta_0 \qquad (r \sim v/\omega, \quad \delta_0 \sim c/\omega_0) \tag{1.1}$$

(v is the velocity of electron, c is the velocity of light, ω_0 is the plasma frequency, $\omega_0 \sim \epsilon_0/\hbar$, ϵ_0 is the limiting Fermi energy). This is observed at frequencies much higher than those studied in reference 1.

The condition (1.1) can be rewritten in the form

$$\omega \gtrsim v/\sqrt{l\delta_0}, \qquad H = m^*c\omega/e \quad (l \sim v\tau), \tag{1.2}$$

which for $l \sim 10^{-1}$ cm, $\delta_0 \sim 10^{-5}$ cm, v $\sim 10^8$ cm/sec corresponds to a wavelength of the order of 1 cm and $H \sim 10^4$ oe, i.e., values feasible in experiments on cyclotron resonance.

In the remaining cases (extremal Ω corresponding to non-central cross sections) we have $\Delta d = d'\Delta p_z \sim d/\sqrt{\omega\tau}$, and the condition for the existence of the structure shown in Fig. 2 is

$$\omega\tau \gtrsim (r/\delta_0)^2.$$

It is easy to prove that this conclusion is valid also for elliptical reference points (where $\mathbf{v} \parallel \mathbf{H}$), in which the role of p_z is played by the angle φ, measured along the arc $\epsilon = \epsilon_0$, $v_y = 0$ (see reference 1). However, in this case the condition $\Delta d \sim \delta_0$ corresponds to the simultaneous disappearance of the strong cyclotron resonance, since it calls for $\Delta d \gg \delta_0$ near the reference points.

We note that the stronger inequality $\omega\tau \gg (r/\delta_0)^2$ cannot be achieved in general. In fact, for cyclotron resonance (when even a small deflection of the electron by collision with a phonon takes the electron out of the skin layer and out of the "resonance" central cross section and is therefore very important), the number of electron-phonon collisions is

$$\nu_{\text{eff}} = 1/\tau_{\text{eff}} \sim \frac{k\Theta}{\hbar}\left(\frac{\hbar\omega + kT}{k\Theta}\right)^3 \gtrsim \frac{k\Theta}{\hbar}\left(\frac{\hbar\omega}{k\Theta}\right)^3$$

(Θ is the Debye temperature). Therefore, since $\tau \lesssim \tau_{\text{eff}}$, we have

$$\omega\tau(\delta_0/r)^2 \lesssim \left(\frac{k\Theta}{\hbar} \Big/ \omega_0 \frac{v}{c}\right)^2 \sim \left(\frac{k\Theta}{\varepsilon_0}\frac{c}{v}\right)^2 \sim 1,$$

because in practice $\epsilon_0 v/c \sim k\Theta$ for all metals.

Thus, "spikes" of current and field are possible for central cross sections if

$$\omega\tau \sim (r/\delta_0)^2, \quad \hbar\omega \gtrsim kT, \quad l \sim l_{\text{eff}} \gtrsim (v\hbar/k\Theta)(\Theta/T)^3.$$

For $l \sim 10^{-1}$ cm, this corresponds to $T \gtrsim 10°$ K, $\lambda \lesssim 1$ mm, $H \sim 10^5$ oe, which is (in the case of λ and H) at the limit of today's experimental capabilities.

If conditions (1.3) are fulfilled, an important effect is exerted on the resonance by the electric field perpendicular to the surface of the metal, and also by Fermi-liquid effects which, however, do not complicate the picture of Fig. 2. This is connected with the circumstance that if condition (1.3) is satisfied the effective conductivity is

$$\sigma_{\text{eff}} \sim \sigma(\delta/r)\sqrt{\omega\tau} \sim \sigma$$

(δ/r takes into account the "ineffectiveness" of electrons outside the skin layer, and $\sqrt{\omega\tau}$ is the number of rotations at resonance averaged over the "essential" orbits).

A subsequent paper will be devoted to consideration of the case in which the condition (1.3) is satisfied, which makes it possible to obtain some information on the correlation function in a Fermi liquid, but which is much more complicated. In the present paper, for simplicity of presentation, only resonance of the central cross section is considered, with

$$(r/\delta_0)^2 \gg \omega\tau \gtrsim r/\delta_0.$$

Naturally, the main interest attaches to the case in which the stronger inequality on the right also holds.

We note in passing the following circumstance, which does not apply directly to the theme of this paper but which is worthy of mention. For a square law of dispersion, a sharp increase (by a factor of $\omega\tau$) in the conductivity at resonance can lead at sufficiently large $\omega\tau$ to the result that the anomalous skin effect will correspond to the conductivity $\sigma\omega\tau$ in a poor metal or even in an alloyed semiconductor, and the diamagnetic resonance will be replaced by cyclotron resonance. For this case it is necessary that

$$\delta^2 \sim c^2 m/2\pi n e^2 \omega\tau \sim \delta_0^2/\omega\tau \ll r^2 \sim v^2/\omega^2,$$

that is,

$$\delta_0^2 \ll vl/\omega.$$

For bismuth, for example, at $\omega \sim 10^{11}$ sec^{-1} and $l \sim 10^{-1}$ cm, this inequality is very well satisfied.

It is possible that the divergence of experimental results with ordinary theory is explained by precisely this circumstance (see, for example, the work of Lax[2]), inasmuch as one must treat the experimental data according to the theory of cyclotron resonance (Lax also pointed out the fact that the data correspond to the theory of cyclotron resonance).

2. DETERMINATION OF THE FIELD INTENSITY IN THE METAL

It is well known that in the one-dimensional case the Maxwell equations (with the obvious neglect of the displacement current) reduce to the form

$$E''_\alpha = (4\pi i\omega/c^2)\, j_\alpha, \quad \alpha = x, z; \qquad j_y = 0, \tag{2.1}$$

where the relation between **j** and **E** must be found from the kinetic equation for the electron distribution function in the metal.

It can be shown (for example, in a fashion similar to what was done in reference 1) that when $\omega\tau \ll (r/\delta_0)^2$ the equations separate: the terms containing E_y, which does almost no work, are small in the expression for j_α (and we can formally set $E_y = 0$ in j_α), while $j_y = 0$ is an equation that determines only E_y, which does not present much interest to us. [In a paper devoted to the case $\omega\tau \sim (r/\delta_0)^2$, a method will be given for separating the equations in the general case.]

A general formula for j_α will be found in reference 1; however, taking it into account that a relaxation time can be introduced for the anomalous skin effect,[1] j_α can be determined much more simply (see the work of Chambers[3]). Evidently,

$$\mathbf{j}e^{i\omega t_1} = \frac{2e}{h^3}\int \mathbf{v}n\,dp_x dp_y dp_z = \frac{2e}{h^3}\frac{eH}{c}\int \mathbf{v}n\,d\varepsilon\, dp_z dt, \tag{2.2}$$

where n is the distribution function, t is the time for rotation of the electron about the orbit (which is determined by the equation $d\mathbf{p}/dt = \dot{\mathbf{p}} = \frac{e}{c}\mathbf{v} \times \mathbf{H}$, $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$), and t_1 is the "ordinary" time. The quantities ϵ and p_z determine the position of the orbit (p_z) on the Fermi surface $\epsilon(\mathbf{p}) = \epsilon$. The quantity t can be interpreted in this sense, that it determines the location of the center of the orbit, in this case taking the integration over t from 0 to $T = 2\pi/\Omega = 2\pi m^* c/eH$ (m^* is the effective mass, Ω is the cyclotron frequency) to mean integration over the centers of all orbits passing through the given point y.

In fact, let t be the time of transit of the electron from the top of the trajectory to the given point; then the statement that the electron has arrived at the point after a time t means that the coordinate of the center of the orbit is $y - r(t)$ (see Fig. 3; the determination of the center of the orbit is clear from the drawing), with

$$r(t) = \int^{t} v_y dt = \frac{cp_x(t)}{eH} + \text{const.}$$

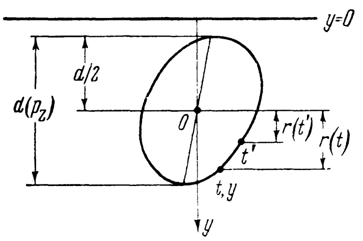

FIG. 3

Such an approach makes immediately obvious the consideration of the boundary condition in the region close to resonance ($|\omega - q\Omega| \ll \omega$, $q = 1, 2, \ldots$), where the only electrons of importance are those which do not collide with the surface. It is clear that an electron does not collide with the surface if the top of the orbit lies inside the metal, i.e., if

$$y - r(t) - r_0 > 0, \quad r_0 = d/2.$$

Therefore, in order to take into account only electrons that do not collide with the surface, it is necessary to add the factor $s[y - r(t) - r_0]$ under the integral sign in (2.2), with $s(w) = 1$ for $w > 0$ and $s(w) = 0$ for $w < 0$.

Now let us determine n. Inasmuch as the number of electrons remains unchanged, the distribution function changes only as the result of change in energy of the electrons, so that

$$n(y, t_1) = n_0(\varepsilon - \Delta\varepsilon) = n_0(\varepsilon) - \frac{\partial n_0}{\partial \varepsilon}\Delta\varepsilon(y, t_1), \tag{2.3}$$

where $\Delta\epsilon(y, t_1)$ is the energy acquired by an electron incident at the point y at the time t_1 ($\dot{\epsilon} = e\mathbf{v}\cdot\mathbf{E}$). Determining $\Delta\epsilon$ and substituting it in (2.3), and then substituting (2.3) in (2.2), we obtain

$$j_\alpha(y) e^{i\omega t_1} = \frac{2e^2}{h^3}\frac{eH}{c} e^{i\omega t_1}\int v_\alpha(t) e^{-i\omega t - t/\tau} s(y - r(t) - r_0)\frac{\partial n_0}{\partial \varepsilon} d\varepsilon dp_z dt \times \int_{-\infty}^{t} \mathbf{v}(t')\mathbf{E}(y - r(t) + r(t')) e^{i\omega t' + t'/\tau} dt', \tag{2.4}$$

which gives the relation between $\mathbf{j}$ and $\mathbf{E}$.

Now, taking into account the periodicity of the velocities and $r(t)$ in t with period T, we transform $\int_{-\infty}^{t}$ into $\int_{t-T}^{t}$ and keep only the principal term in $1/\omega\tau \ll 1$. Noting that $\partial n_0/\partial\epsilon \approx -\delta(\epsilon - \epsilon_0)$, we obtain the following expression for $\mathbf{j}$ near resonance ($|\omega - q\Omega| \ll \omega$).

$$j_\alpha(y) = \frac{2e^3H}{h^3c}\int [1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau)]^{-1} dp_z \times \int_0^T v_\alpha(t) e^{-i\omega t} s(y - r(t) - r_0) dt \times \int_0^T v_\beta(t') E_\beta(y - r(t) + r(t')) e^{i\omega t'} dt',$$

$$v_\beta E_\beta \equiv v_x E_x + v_z E_z, \tag{2.5}$$

where $[1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau)]^{-1}$ evidently gives a large number of rotations for an orbit with a given p_z and $\epsilon = \epsilon_0$: the error in an incomplete rotation is insignificant — it makes a non-resonant contribution to j_α.

Equations (2.1) and (2.5) can be continued as even functions in the region $y < 0$ if we set $E_\alpha(-y) = E_\alpha(y)$ and substitute $s(y - r(t) - r_0)$ for $s(|y - r(t)| - r_0)$, which is equal to it when $y > 0$; it is easy to see [if we take into account the central symmetry of the Fermi surface: $\epsilon(\mathbf{p}) = \epsilon(-\mathbf{p})$] that in this case $j_\alpha(y) = j_\alpha(-y)$.

The basic mathematical difference from the calculations of reference 1 is that in the latter case the minimum parameter is the "anomaly" parameter δ_0/r and in the taking of asymptotes one assumes $\omega\tau \ll r/\delta_0$; in the present case the minimum parameter is $1/\omega\tau$, and $\omega\tau \gg r/\delta_0$. However, in exactly the same fashion as was done earlier,[1] it can be shown that account of the boundary condition (which leads, as we have seen, to the substitution of unity for $s[y - r(t) - r_0]$) leads only to multiplication of the quantities of interest to us by a constant of the order of unity.

Physically this is quite clear: the important role, in any case, could be played even in the absence of a boundary only by trajectories on which the electron covers in a skin layer a path on the order of $\sqrt{r\delta}$, i.e., close to the maximum possible value (trajectories 1, 2, and 3, but not 4, in Fig. 4). Account of the boundary eliminates the trajectories of type 1, 3, i.e., it merely decreases somewhat the effective conductivity. This decrease is the more insignificant in that the quantities of interest to us (for example, the impedance) are determined by the cube root of the effective conductivity. Therefore, for simplicity of presentation only, we shall disregard collisions of an electron with the surface, and investigate Eqs. (2.1) and (2.5) with unity in place of $s[y - r(t) - r_0]$.

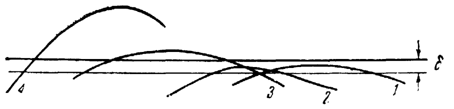

FIG. 4

Such a system of equations is quickly solved by the method of Fourier components, for which it suffices to multiply both sides of the equation by e^{iky} and integrate over y from $-\infty$ to $+\infty$. We have

$$-k^2\mathscr{E}_\alpha(k) - 2E'_\alpha(0) = iK_{\alpha\beta}(k)\mathscr{E}_\beta(k),$$

$$K_{\alpha\beta}(k) = \frac{8\pi\omega e^3 H}{c^3 h^3}\int\left[1 - \exp(-2\pi i\omega/\Omega - 2\pi/\Omega\tau)\right]^{-1} dp_z$$

$$\times\int_0^T \exp\{-i\omega t + ikr(t)\} v_\alpha(t)\,dt \int_0^T \exp\{i\omega t' - ikr(t')\} v_\beta(t')\,dt',$$

$$\mathscr{E}_\alpha(k) = \int_{-\infty}^{\infty} e^{iky} E_\alpha(y)\,dy,$$

$$E_\alpha(y) = \frac{1}{\pi}\int_0^\infty \mathscr{E}_\alpha(k)\cos ky\,dk, \tag{2.6}$$

whence it is easy to find $\mathscr{E}_\alpha(k)$ and $E_\alpha(y)$.

Although this solves the problem in principle, we carry out, for convenience of investigation of the function $E_\alpha(y)$, a further simplification of Eqs. (2.6). To determine the impedance and the "spikes" of current and field (the "what-not shelves" in Fig. 2), as is clear from Sec. 1, only electrons close to the "upper" and "lower" points of the central cross section are important, where $p_z = 0$, $v_y = 0$, so that $r(t)$ has a minimum or a maximum. This statement (as well as many others which will be encountered later, statements made without proof) can be rigorously verified.

At the points mentioned,

$$v_\beta(\varepsilon, p_z, p_x) \approx v_\beta(\varepsilon_0, 0, p_x^{max})\,\mathrm{sign}\,v_x \equiv v_\beta^0 \nu, \tag{2.7}$$

and $K_{\alpha\beta}$ [and with it Eq. (2.6)] is automatically diagonalized by the choice of the new orthogonal axes α and β, where α is the direction of the velocity $\mathbf{v}$ at the point $\epsilon = \epsilon_0$, $p_z = 0$, $p_x = p_x^{max}$, so that $v_\alpha^0 = v_0$. The fact that in this case $K_{\alpha\beta} = K_{\beta\alpha} = K_{\beta\beta} = 0$ apparently means only that $K_{\beta\beta}$, $K_{\alpha\beta}$, $K_{\beta\alpha}$ do not have a resonant character. Therefore, only E_α has "spikes" while E_β has none.

We note that we disregard cross terms in the determination of both E_α and E_β. [In the determination of E_β one must write down the non-resonant $K_{\beta\beta}$ correctly according to the formula (2.6).] This is connected with the fact that, as can be proved by direct calculation,

$$K_{\alpha\alpha} \sim (\omega\tau)^{-1/2}, \quad K_{\beta\beta} \sim K_{\alpha\beta} \sim K_{\beta\alpha} \sim 1,$$

$$E_\alpha \sim (\omega\tau)^{-1/6}, \quad E_\beta \sim 1.$$

Naturally, the diagonalized equation

$$-k^2\mathscr{E}_\alpha - 2E'_\alpha(0) = iK_{\alpha\alpha}\mathscr{E}_\alpha \tag{2.8}$$

is not suitable for the determination of E_α between the spikes, where all points of the orbit are important and not only its vertices. However, inasmuch as almost everywhere $v_{\alpha,\beta} \sim v_{\alpha,\beta}^0$ and the cross terms do not change the behavior of the functions, Eq. (2.8) is useful for a qualitative analysis of the behavior of E_α for all y, as can be established directly by simple (though cumbersome) calculations.

From (2.8) and (2.6), by setting

$$k = x/d_0, \quad d_0 = d|_{p_z} = 0; \quad y = d_0\zeta \tag{2.9}$$

and omitting the index α everywhere in what follows, we get

$$E(\zeta) = -\frac{2d_0}{\pi}E'(0)\int_0^\infty \frac{\cos\zeta x\,dx}{x^2 + id_0^2 K(x/d_0)}. \tag{2.10}$$

3. INVESTIGATION OF THE STRUCTURE OF THE FIELD IN A METAL

For simplicity, let us consider the case of "exact" resonance: $\omega = q\Omega_0$. It is easy to prove that for $\omega\tau \gg r/\delta_0$ one can set $p_z = 0$ in (2.6) in the inner integrals, so that we get from (2.10)

$$E(\zeta) = -\frac{2d_0}{\pi}E'(0)J(\zeta), \quad J(\zeta) = \int_0^\infty \frac{\cos\zeta x\,dx}{x^2 + i\exp(-i\pi\sigma/4)M^3A^2(x)},$$

$$A(x) = A(-x)$$

$$= \sqrt{|\dot{v}_y^0|/2\pi d_0}\left|\int_0^T \nu(t)\exp[-i\omega t + ixr(t, 0)/d_0]\,dt\right|. \tag{3.1}$$

Here,

$$r = r(t, p_z), \quad M^3 = \frac{12\sqrt{2}\,qv_0^2\sqrt{\omega\tau_0}}{\Omega_0|\dot{v}_y^0|\sqrt{|m_0^{*\prime\prime}m_0^*|}}\frac{1}{\delta_0^2} \sim 10\,q^3\sqrt{\omega\tau_0}\left(\frac{r_1}{\delta_0}\right)^2,$$

$$1/\delta_0^2 = 2\pi e^2 8\pi(p_x^{max})^3(3c^2 m_0^* h^3)^{-1}, \quad r_1 \sim v/\omega,$$

$$d_0/2 = r_{00} = p_x^{max}/m_0^*\Omega_0, \quad \sigma = \mathrm{sign}(\Omega_0/\Omega'')_0. \tag{3.2}$$

Here the dot indicates the derivative with respect to t and the prime the derivative with respect to p_z; the index 0 denotes the point $\epsilon = \epsilon_0$, $p_z = 0$, $p_x = p_x^{max}$. [The fact that $A(-x) = A(x)$ can be established by making use of the substitution $t \to T_0/2 + t = \pi/\omega + t$.]

Taking it into account that

$$J = \int_0^\infty \ldots\cos\zeta x\,dx = \int_{-\infty}^\infty \ldots\exp(i\zeta x)\,dx, \tag{3.3}$$

where $A^2(x)$ is an analytic function, and applying the theorem of residues, we obtain for ζ not too close to an integer ($\zeta = a + \zeta'$; a is an integer, $\zeta' \gg M^{-1}$),

$$J \approx 2\pi M^{-3/2} e^{-i\pi(2-\sigma)/8} \sum_{n=1}^{\infty} \frac{\cos \zeta x_n \exp(-\zeta\beta_n)}{x_n |A'(x_n)|}; \qquad (3.4)$$

$$\beta_n = M^{-3/2} e^{-i\pi(2-\sigma)/8} |x_n / A'(x_n)|; \quad A(x_n) = 0, \quad x_n > 0. \qquad (3.4a)$$

We note that for $x \gg 1$

$$A(x) \sim \sqrt{2/x} \cos(x/2 - \pi/4), \qquad x_n \sim (2n + 1/2)\pi,$$
$$\beta_n \sim M^{-3/2} n^{3/2}. \qquad (3.5a)$$

If $\zeta' \sim 1$, the series (3.4) converges rapidly: only $n \sim 1$ are important. The function $J(\zeta)$ oscillates at distances of the order of unity $(y \sim r)$ and falls off extremely slowly at distances of the order of $M^{3/2}$ [$y \sim r^2(\omega\tau)^{1/4}/\delta_0$; for $\omega \sim 10^{11}\,\text{sec}^{-1}$, $1 \sim 10^{-1}$ cm, $n \sim 10^{22}\,\text{cm}^{-3}$, this corresponds to a depth of the order of 1 mm]; for $\zeta \gg M^{3/2}$ it is enough to keep only the first term in the series (3.4).

For $\zeta' \ll 1$, the terms $n \gg 1$ become important in the series (3.4), which makes it possible to make direct use of the asymptotic expression (3.5a) for $A(x)$ in the integral $J(\zeta)$. In this case, $J(\zeta)$ is conveniently calculated in the following fashion, with accuracy to M^{-3}. Transforming $J(\zeta)$:

$$J(\zeta) = \int_0^{\infty} \frac{x \cos(a + \zeta')\, x\, dx}{x^3 + iM^3 \exp(-i\pi\sigma/4)(1 + \sin x)} = \sum_{n=0}^{\infty} \int_{2\pi n}^{2\pi(n+1)} dx \ldots$$

$$\approx \sum_{n=0}^{\infty} \int_0^{2\pi} \frac{2\pi n \cos(ax + 2\pi n\zeta')\, dx}{(2\pi n)^3 + i \exp(-i\pi\sigma/4) M^3 (1 + \sin x)}$$

and replacing the sum by an integration, we obtain

$$J(\zeta) = \frac{1}{2\pi M} \int_0^{\infty} y\,dy \int_0^{2\pi} \frac{\cos(ax + \zeta' M y)\, dx}{y^3 + i \exp(-i\pi\sigma/4)(1 + \sin x)}$$

$$= \frac{(-1)^a}{M} \int_0^{\infty} \frac{y \cos(\zeta' M y + \pi a/2)\, dy}{\sqrt{g^2 - 1}\,(g + \sqrt{g^2 - 1})^a},$$

$$g = 1 + y^3 \exp[-i\pi(2 - \sigma)/4],$$

$$\sqrt{g^2 - 1} \equiv g\sqrt{1 - 1/g^2}, \qquad \sqrt{1} = +1. \qquad (3.5)$$

Equations (3.5) are also suitable for the calculation of the derivatives of $J^{(k)}(\zeta)$ which are used below in Sec. 6 [in the series for derivatives, obviously, $n \gg 1$ are important for any ζ, but substitution of summation for integration, i.e., keeping only the zero term in the Poisson formula which can be readily justified for $\zeta' \ll 1$ and $k \geq 0$, and the transition to a formula analogous to (3.5), are no longer valid when $\zeta' \sim 1$.]

4. CONCLUSIONS FROM ANALYSIS OF THE BEHAVIOR OF THE FIELD

Let us formulate without proof the results of the investigation of Eqs. (2.10), (3.4), (3.5). These results substantiate the conclusions of Sec. 1 (see Fig. 2).

1. Near $y \approx d_0, 2d_0, \ldots$, the absolute value of the field has a sharp maximum and is $\sqrt{M}$ times greater than the average field between the maxima.

2. The distance from a maximum at which a value of the order of the maximum M^{-1} is achieved corresponds to a distance y from $\delta_0(r/\delta_0)^{1/6}$ to δ_0. This is easily understood if we note that for $\zeta' M \gg 1$ we obtain $J(\zeta') \sim |\zeta' M|^{-1/2}$.

3. With increase in the number of the maximum, its height falls off (for $a \gg 1$) as $a^{-1/3}$, and the "width" grows. The sharp maxima gradually disappear; at large distances the field oscillates over distances $y \sim r$ and slowly decays at a depth $rM^{3/2} \sim (r^2/\delta_0)(\omega\tau)^{1/4}$.

4. For $y_b = 2bd_0$ $(b = 0, 1, 2, \ldots)$ there are separate extrema of the value of the field, the signs of which alternate as $(-1)^b$.

5. For $y \approx y_b = (2b + 1)d_0$ $(b = 0, 1, 2, \ldots)$ there are two extrema which differ only in sign; the field close to these points is antisymmetric; $E(y_b + y') = -E(y_b - y')$. The signs of the first of the two neighboring extrema alternate as $(-1)^{b+1}$; the first extremum at $y \approx d_0$ has the sign opposite the sign of the field for $y = +0$. The current density changes in a fashion similar to the field intensity, with

$$j(\zeta') \sim [|\zeta' M| + 1]^{-3/2}, \qquad j(a) \sim a^{-1} \qquad (a \gg 1).$$

6. For $\Omega_0\tau \ll r/\delta_0$, the relative growth in the field close to $y = d_0$ is of the order $(\Omega_0\tau)^{1/2}$; the a-th spike is of order

$$E(0)\, a^{-1/3} (\delta_0/r)^{a/3} (\Omega_0\tau)^{5a/12}.$$

Thus, the condition $\Omega_0\tau \geq r/\delta_0$ is extremely important, for unless it is fulfilled the spikes decay rapidly. These conclusions can be obtained from (2.6) where naturally it is not possible to set $p_z = 0$ in the inner integrals, and the term periodic for $kr \gg \omega\tau$ has the order $\sqrt{\omega\tau/kr}$.] Here, inasmuch as the effective τ depends strongly on the degree of parallelism of the magnetic field to the surface of the metal, strict parallelism is essential if spikes are to be observed.

5. MATHEMATICAL REASON FOR THE PRESENCE OF SPIKES OF THE FIELD. CHARACTERISTIC FREQUENCIES OF PLASMA OSCILLATIONS

At first glance, the fact that the field rapidly decays in the interior appears to be an obvious consequence of Eq. (2.1). Actually, inasmuch as

$$j_\alpha(y) \sim (\sigma_{eff}/r) \int K_{\alpha\beta}(y, y') E_\beta(y')\, dy'$$

(σ_{eff} is the effective conductivity) and

$$\frac{1}{N} E''_\alpha \sim \frac{i}{r^2} \int \frac{1}{r} K_{\alpha\beta}(y, y') E_\beta(y')\, dy', \quad N \gg 1, \qquad (5.1)$$

while $K \sim 1$ and falls off at distances of the order r, then

$$E'' \sim iN\bar{E}/r^3, \quad \bar{E} \sim \int E(y')\, dy',$$

and E must fall off at distances of the order of $r/\sqrt[3]{N} \ll r$.

However, this conclusion is valid only when for the given $K_{\alpha\beta}(y, y')$ the integral

$$\int K_{\alpha\beta}(y, y') E_\beta(y')\, dy'$$

is of the order of $\bar{E}$, i.e., if (5.1) does not have a zero eigenvalue.

In the considered case, for an equation corresponding to (2.8), we have

$$(\omega\tau)^{-1/2} E''_\alpha = (\omega\tau)^{-1/2} i\hat{K}_{\alpha\alpha} E_\alpha = i\hat{K}^{(1)}_{\alpha\alpha} E_\alpha.$$

For $\omega = q\Omega_0$, where q is an integer, $\tau = \infty$ is an infinitely multiply degenerate eigenvalue with eigenfunctions $A_n \cos(x_n y/d_0)$, where x_n is a root of (3.4a). We are extremely close to this eigenvalue and it is not difficult to see that (3.4) is a superposition of eigenfunctions, and that $E(\zeta)$, averaged over the interval $M^{-1} \ll \Delta\zeta \ll 1$, is close to the eigenfunction. The "spikes" in this case are the natural singularities of the solution of the equation close to its eigenvalue.

It is interesting that as $\tau \to \infty$ the natural frequencies ω_q of the plasma oscillations in a strong magnetic field ($\Omega_0\tau \gg 1$) are shown in this fashion to be discrete multiples of Ω_0: $\omega_q = q\Omega_0$ (this means several orders of magnitude less than the usual natural frequencies $\omega_0 \sim c/\delta_0$) and the corresponding discrete values of the wave vector

$$k_n = x_n/d_0 = \omega_q x_n m^*/2qp_x^{max}.$$

6. DETERMINATION OF THE SURFACE IMPEDANCE

We shall investigate how the phenomenon of "spikes" affects the surface impedance. We immediately note that, inasmuch as the impedance has a minimum at cyclotron resonance, a significant resonance will be observed, as noted earlier,[1] only for a definite polarization of the electromagnetic wave, when **E** is directed along $\mathbf{v}_0$. Therefore, as has been shown,[4] it is more convenient to measure the derivative of the impedance with respect to the magnetic field, which is determined essentially by $Z_{\alpha\alpha}$, where

$$Z_{\alpha\alpha} = (-4\pi i\omega/c^2)(E_\alpha(0)/E'_\alpha(0)) \equiv Z.$$

Making use of (3.1), and carrying out transformations similar to those applied in Sec. 3, we find

$$Z = \frac{4i\omega d_0}{\pi c^2 M} (q\sqrt{\varkappa})^{1/3} \int_0^\infty z\,dz$$
$$\times \int_0^{2\pi} \left[z^3 + \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\{1 + \sin(x + s_2 N_1 z w^2)\}\, dw}{w^2 + s\sigma \exp(-is \tan^{-1}|\varkappa_1|^{-1})} \right]^{-1} dx; \qquad (6.1)$$

where

$$\varkappa = \sqrt{1 + \varkappa_1^2}, \qquad \varkappa_1 = (\omega - q\Omega_0)\tau_0,$$
$$s = \operatorname{sign}(\omega - q\Omega_0), \quad s_2 = \operatorname{sign} r''_0,$$
$$N_1 = |r''_0 m^*_0 / r_0 m^{*''}_0| M\varkappa^{5/6}(\omega\tau_0)^{-1} \sim (r/\delta_0)^{2/3} \varkappa^{5/6} (\omega\tau_0)^{-5/6}. \qquad (6.2)$$

The integration in (6.1) is easily carried out; the equations obtained which are too complicated to write down, make it possible to plot the entire resonance curve. Thus, for $s\sigma = 1$, $N_1 \ll 1$, we have

$$Z(N_1) \approx Z(0)\Big\{1 - 2^{2/15}\pi^{-1/2}\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{7}{6}\right)$$
$$\times N_1^{1/10} e^{i\varphi} \int_0^\infty \frac{dy}{y^2}\left[1 - \frac{1}{\sqrt{1 + y^5}}\right]\Big\},$$
$$\varphi = \frac{\pi}{60}\left(1 - \sigma + \frac{\sigma}{\pi}\tan^{-1}|\varkappa_1|^{-1}\right),$$

and comparing $Z(0)$ with $Z(\infty)$, we find that the appearance of "spikes" of the field increases the impedance (that is, it acts "against" the resonance) by a factor of $\Gamma(1/6)[\sqrt[3]{2}\sqrt{\pi}\,\Gamma(2/3)]^{-1} \approx 1.84$, compared with the "ordinary" case examined in reference 1, without changing all the dependences. This corresponds to a decrease of $Z(\infty)$ with changing $\omega\tau_0$ to approximately 1/40.

We shall make clear how the presence of "spikes" in the case of films of thickness $a = D/d_0 \gg 1$ affects the impedance and its derivatives. According to the definition of impedance in (3.1),

$$Z = E(0) \Big/ \int_0^D j(y)\, dy \approx Z_\infty\{1 + J'(a)/J'(0)\},$$
$$J'(0) = -\frac{\pi}{2} \qquad (6.3)$$

where Z_∞ is the impedance of the half-space and the integral $J(a)$ is given by Eq. (3.1). Thus for the determination of $Z^{(k)}(a)$ it is sufficient to know $J^{(k+1)}(a)$. (Strictly speaking, one should also take into account the presence of the surface $y = D$, writing down the boundary condition corresponding to it, extending E and its equation in even fashion in the region $0 > y > -D$ and beyond periodically through all space with period 2D, and solving the equation by expansion in a Fourier series. However, for $a \gg 1$, in the zeroth approximation, use can be made of the solution for $E(\zeta)$ in the case of a half-space.)

Without carrying out the detailed derivations here, we only note that the jump in $J^{(k+1)}$ is of the order $M^k a^{-(2k+3)/3}$, where even $a-k$ gives a single jump, and odd $a-k$ gives two jumps very close together, equal in magnitude and opposite in sign. Physically, this is evident from (6.3) and Fig. 2: if a is varied, the current has alternately a single jump and two jumps, and these compensate for each other.

7. NEW RESONANCE EFFECTS

The basic predictable effect is the gradually damped and broadened "spikes" of field and current density in the metal at a depth that is a multiple of the orbit corresponding to the central cross section of the Fermi surface, in a direction normal to the surface of the metal, and the slow damping of the field in the body of the metal [at distances of the order $r^2(\omega\tau)^{1/4}/\delta_0$]. The peculiarities of the structure of the field were analyzed in Sec. 4; they are shown in Fig. 2.

Such a structure of the field leads to a number of new resonance effects.

1. It is simplest to observe experimentally the change in the magnetic field of the derivatives of the impedance $Z = R + iX$ (dR/dH, dX/dH, $d\ln X/dH$), since the impedance itself has, as has been pointed out, a resonant character only for a definite polarization of the electromagnetic field. However, one could observe with certainty only a non-monotonic change of the impedance with the appearance of "spikes" (which is connected with the fact that the approach of the magnetic field to resonance leads to resonance reduction of the impedance and then to increase of it as a result of the appearance of "spikes") which does not always take place. It is more convenient to measure the quantity

$$\Delta = (H - H_{res})\, d\ln X/dH$$

(H_{res} is the resonance value of H). Away from resonance it is constant without "spikes"; the presence of spikes leads to an additional "bump" since for $N_1 \gg 1$ and for $N_1 \ll 1$ (see Sec. 6) we will have $\Delta \approx -1/6$. Naturally such a criterion for the appearance of "spikes" is not conclusive.

2. It appears to us that the effects on layers of thickness $D \sim 10^{-3} - 10^{-1}$ cm are the simplest of all observed and also clearly account for the "what-not shelves" in the field structure (Fig. 2).

Upon increase in the frequency ω, one ought to observe jumps of the resonance values of the impedance and its derivatives (the latter, as was pointed out above, are much more convenient), which correspond to the fundamental ($\omega = \Omega_0$) when $D = ad_0 \equiv a2cp_x^{max}/eH$ (a is an integer), and also an abrupt increase in the number of observed harmonics by one (from a to $a+1$), so that the number of observed harmonics also makes it possible to determine a and $d_0 = D/a$. The reason for this is clear from Fig. 5 (we recall that the change in the number of "spikes" of the field in the depth of the metal changes the impedance).

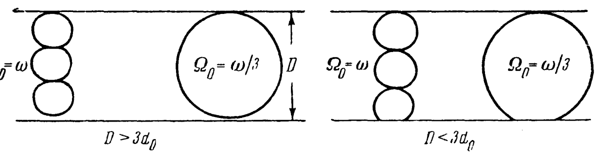

FIG. 5

A similar effect should also take place for fixed ω (which greatly simplifies the experiment) upon rotation of a constant magnetic field in the plane of the plate.

For angles φ at which the impedance changes abruptly and the last harmonic disappears, one can also determine the diameter of the Fermi surface $2p_x^{max} = eHd_0/c$ in the corresponding directions. Naturally, in order to construct the Fermi surface directly, experiments are necessary on layers of different thickness and with different orientations of the surface relative to the crystallographic axes.

It is easy to estimate the magnitude and width in φ of the jumps in impedance and its derivatives due to rotation of the magnetic field. Since the jump corresponds to a change of ad_0 by a magnitude of the order of $a^{2/3}\delta_0$, then

$$\Delta\varphi \sim a^{2/3}\delta_0 / ad_0'(\varphi) \sim \delta_0 / a^{1/3}d_0,$$

that is, of the order of ten of seconds.

It is evidently most convenient to measure dZ/dH as a function of the angle of rotation φ and to study the variation of the tangent to the curve

$$f(\varphi) = H\,\frac{d(Z/Z_\infty - 1)}{dH} \approx a\,\frac{d(Z/Z_\infty - 1)}{da}.$$

Inasmuch as

$$\frac{df}{d\varphi} = \frac{\partial f}{\partial\varphi} + \frac{\partial f}{\partial a}\frac{da}{d\varphi},\quad \frac{da}{d\varphi} \sim a,$$

it is easy, knowing J''' (see Sec. 6), to verify that the jumps of $df/d\varphi$ are of the order $M^2a^{-1/3}$. Moreover, inasmuch as $M^2 \gtrsim (r/\delta_0)^{5/3} \sim 10^3$, one can hope to observe jumps in plates of thickness up to $M^{3/2}d_0 \sim 100d_0$, that is, with $D \sim 1$ mm (because of the extremely slow damping of the "spikes," the thickness D is limited only by the inequality $D < M^{3/2} d_0$).

3. The third effect in which the "spikes" enter is the selective transmission of films at resonance, when the thickness of the film is $D = ad_0 \sim H$ (to observe this effect, it is obviously necessary to change the frequency ω and the magnetic field simultaneously or to rotate the magnetic field in the plane of the film), and to observe the field, which "penetrates" to a depth $r^2(\omega\tau)^{1/4}/\delta_0$. However, observation of these effects is very difficult because of the almost ideal reflectivity of the metal: reflection from only the two surfaces of the film leads to a reduction in the intensity of the transmitted wave by a factor $\sim (10\delta_0/\lambda)^2$, where λ is the wavelength of the incident wave. For a "penetrating" field which is not connected with the "spikes," the additional attenuation is of the order of $r/\delta_0 \sim v_0\lambda(2\pi c\delta_0)^{-1}$.

4. The next effect is a "spatial" electron "echo," similar in a certain sense to the well-known spin "echo." If a pulse of duration $\Delta t \ll 2\pi/\Omega_0$ is incident on a metal placed in a constant magnetic field H such that $\Omega_0\tau_0 \gg 1$, then within the time $2\pi/\Omega_0$, $4\pi/\Omega_0$, . . . (Ω_0 is the frequency corresponding to the central cross section), when the largest number of accelerating electrons reaches the surface of the metal, "echoes" will be observed — these are spikes of the field. (For a square dispersion law all of the accelerated electrons come together and the "echo" will be maximum.)

5. For $\Omega_0\tau \gg 1$, the fluctuations in the metal will have an unusual character. This problem will be considered in a separate paper.

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