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THE SCATTERING OF SPIN WAVES AND PHONONS BY IMPURITIES IN FERROMAGNETIC DIELECTRICS

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The effect of impurities on the heat conductivity of a ferromagnetic dielectric at low temperatures is examined. The conductivity of a ferromagnetic dielectric containing impurities is calculated.

THERMAL energy can be carried in ferromagnetic dielectrics by spin waves and by phonons. Akhiezer and Shishkin[1] have calculated the thermal conductivity determined by the interaction of spin waves and phonons with one another, and also by the interaction of spin waves with phonons. According to these authors, the conductivity varies exponentially with temperature at low temperatures. This relation is connected with the fact that it is only possible to obtain a finite value for the conductivity of a crystal if umklapp processes are considered, as was shown by Peierls.[2] In these, the quasi-momentum and consequently also the energy of at least one of the interacting particles must be sufficiently great.

If a crystal contains impurities, the scattering of long-wave spin waves and phonons by the impurities must be taken into account in calculating the thermal conductivity. Since there are considerably more long-wave phonons and spin waves than short-wave ones at low temperatures, the low-temperature conductivity will be determined by scattering of spin waves and phonons by impurities, and not by umklapp processes, the probability of which falls off exponentially. We shall not consider the excitation of optical-mode phonons, since their contribution to the thermal conductivity at sufficiently low temperatures is also exponentially small.

The aim of the present work is to study the influence of impurities on the thermal conductivity of ferromagnetic dielectrics. We can write the Hamiltonian for a ferromagnetic dielectric in the form

$$H = H^{(s)} + H^{(l)}, \tag{1}$$

where the first term is the exchange energy and the second is the energy of elastic lattice vibrations.

The exchange energy of a ferromagnetic dielectric containing impurities is

$$\begin{aligned} H^{(s)} = &-\frac{1}{2}\sum_{l,n} J_1(R_{ln})\,\mathbf{s}_l\mathbf{s}_n \\ &-\sum_{l,n} J_{12}(R_{ln})\,\mathbf{s}_l\boldsymbol{\sigma}_n - \frac{1}{2}\sum_{l,n} J_2(R_{ln})\,\boldsymbol{\sigma}_l\boldsymbol{\sigma}_n, \end{aligned} \tag{2}$$

where $\mathbf{s}_l$ is the spin of an atom of the basic (ferromagnetic) substance, $\boldsymbol{\sigma}_l$ is the spin of a paramagnetic impurity atom, $J(R_{ln})$ are the exchange integrals and $R_{ln} = |\mathbf{R}_l - \mathbf{R}_n|$ is the distance between the l-th and n-th lattice sites. The first term in the Hamiltonian $H^{(s)}$ is the exchange energy between atoms of the basic material, and the summation is taken over those lattice sites which are occupied by atoms of the basic substance. The second term is the exchange energy between basic atoms and paramagnetic impurity atoms, with the summation over l taken, in this case, over those lattice sites occupied by basic atoms, and over n taken over those sites occupied by impurity atoms. The third term is the energy of the exchange interaction between paramagnetic impurity atoms, with the sum taken over sites occupied by these impurity atoms.

The exchange energy $H^{(s)}$ is not changed formally if there are diamagnetic impurities. We should bear in mind that the summation in (2) is already not taken over all lattice sites.

The energy of elastic vibrations of the lattice can be expressed, as far as terms quadratic in the atomic displacements, $\mathbf{u}_n$, from the equilibrium position, in the form

$$\begin{aligned} H^{(l)} = &\frac{1}{2}\sum_{n} m_n\dot{\mathbf{u}}_n^2 + \frac{1}{2}\sum_{l,n} A_{ik}^{(1)}(R_{ln})\,u_l^i u_n^k \\ &+\sum_{l,n} A_{ik}^{(12)}(R_{ln})\,u_n^i u_l^k + \frac{1}{2}\sum_{l,n} A_{ik}^{(2)}(R_{ln})\,u_l^i u_n^k, \end{aligned} \tag{3}$$

where m_n is the mass of the atom at the n-th site and A_{ik} are quantities characterizing the elastic interaction between atoms. The first term in the

Hamiltonian (3) represents the kinetic energy of the atoms, the second the elastic energy of atoms of the basic material, the third the elastic energy related to the interaction of the basic and impurity atoms and, finally, the fourth term represents the elastic energy of the impurity atoms. The summation in the first term of the Hamiltonian $H^{(l)}$ is taken over all lattice sites, in the second it is taken over those sites where there are atoms of the basic material, in the third term the summation over l is taken over those sites where there are atoms of the basic material, while over n it is taken over sites occupied by paramagnetic or diamagnetic impurities and in the fourth term the summation is taken over those sites where there are impurity atoms.

It is convenient to express the Hamiltonian H as the sum of a Hamiltonian H_0 describing free spin waves, phonons and the exchange energy of paramagnetic impurities and a Hamiltonian H_{int} characterizing the scattering of spin waves and phonons by impurities:

$$H = H_0 + H_{int},$$

$$H_0 = -\frac{1}{2}\sum_{l,n} J_1(R_{ln})\,\mathbf{s}_l\mathbf{s}_n + \frac{1}{2}m\sum_{ln}\dot{\mathbf{u}}_l^2 + \frac{1}{2}\sum_{l,n} A_{ik}^{(1)}(R_{ln})\,u_l^i u_n^k - \sum_{n,l} J_{12}(R_{ln})\,\mathbf{s}_l^0\sigma_n, \tag{4}$$

$$H_{int} = \sum_l\sum_n J_1(R_{ln})\,\mathbf{s}_l\mathbf{s}_n - \sum_l\sum_n J_{12}(R_{ln})(\sigma_n, \mathbf{s}_l - \mathbf{s}_l^0) + \sum_l\sum_n [A_{ik}^{(12)}(R_{ln}) - A_{ik}^{(1)}(R_{ln})]\,u_l^i u_n^k + \frac{1}{2}\sum_n \Delta m_n\dot{\mathbf{u}}_n^2. \tag{5}$$

Here m is the mass of a basic atom and $\Delta m_n = m - m_n$. In expression (5) for H_{int}, the summation over n is taken over the impurity sites. Besides impurities, we must also consider the existence of a rarer isotope of the basic substance. Corresponding to this we must also take the summation in the last term of the Hamiltonian H_{int} over sites occupied by atoms of the rarer isotope. We should point out that in writing the Hamiltonian H_{int} we have omitted terms proportional to the square of the impurity concentration.

Transforming from the operators $\mathbf{s}_l$ and $\mathbf{u}_l$ to creation operators $\overset{+}{a}_{\mathbf{k}}$ and annihilation operators $a_{\mathbf{k}}$ of the spin waves[3] and creation operators $\overset{+}{b}_{\mathbf{k}j}$ and annihilation operators $b_{\mathbf{k}j}$ of the phonons[4], we obtain

$$H_0 = \Sigma\varepsilon_{\mathbf{k}}\overset{+}{a}_{\mathbf{k}}a_{\mathbf{k}} + \Sigma\hbar\omega_{\mathbf{k}j}\overset{+}{b}_{\mathbf{k}j}b_{\mathbf{k}j}, \tag{6}$$

$$H_{int} = \Sigma\Phi_{\mathbf{k}_1\mathbf{k}_2}\overset{+}{a}_{\mathbf{k}_1}a_{\mathbf{k}_2} + \Sigma\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3j}\overset{+}{a}_{\mathbf{k}_1}a_{\mathbf{k}_2}b_{\mathbf{k}_3j} + \Sigma\chi_{\mathbf{k}_1j_1\mathbf{k}_2j_2}\overset{+}{b}_{\mathbf{k}_1j_1}b_{\mathbf{k}_2j_2} + \text{compl. conj.} \tag{7}$$

$\epsilon_{\mathbf{k}} = 2J_1 s(ak)^2$ is the energy of a spin wave with wave vector $\mathbf{k}$, a is the lattice constant, $\hbar\omega_{\mathbf{k}j} = \hbar c_j k$ is the energy of a phonon with wave vector $\mathbf{k}$ and polarization j and the summation is taken over all the vectors $\mathbf{k}$ and polarization j.

The quantities Φ, Ψ and χ are determined from the equations

$$\Phi_{\mathbf{k}_1\mathbf{k}_2} = \frac{1}{2}J_{12}\sigma f_1(\mathbf{k}_1, \mathbf{k}_2) + J_1 s a^2(\mathbf{k}_1, \mathbf{k}_2)[f_1(\mathbf{k}_1, \mathbf{k}_2) + f_2(\mathbf{k}_1, \mathbf{k}_2)], \tag{8}$$

$$\Psi_{\mathbf{k}_1\mathbf{k}_2, \mathbf{k}_3j} = 2i\sigma J_{12}(\hbar/2mN)^{1/2}\omega_{\mathbf{k}_3j}^{-1/2}\mathbf{e}_{\mathbf{k}_3j}\mathbf{k}_3 f_1(\mathbf{k}_2 + \mathbf{k}_3, \mathbf{k}_1), \tag{9}$$

$$\chi_{\mathbf{k}_1j_1, \mathbf{k}_2j_2} = -\frac{1}{4}\hbar\sqrt{\omega_{k_1j_1}\omega_{k_2j_2}}\{[m^{-1}\Delta m_1\mathbf{e}_{\mathbf{k}_1j_1}\mathbf{e}_{\mathbf{k}_2j_2} + \lambda_1(\mathbf{n}_1, \mathbf{n}_2)]f_1(\mathbf{k}_1, \mathbf{k}_2) + [m^{-1}\Delta m_2\mathbf{e}_{\mathbf{k}_1j_1}\mathbf{e}_{\mathbf{k}_2j_2} + \lambda_2(\mathbf{n}_1, \mathbf{n}_2)]f_2(\mathbf{k}_1, \mathbf{k}_2) + m^{-1}\Delta m_3 f_3(\mathbf{k}_1, \mathbf{k}_2)\}, \tag{10}$$

where J_1 and J_{12} are the exchange integrals for neighboring atoms; $\mathbf{e}_{\mathbf{k}j}$ is the unit polarization vector; $\lambda_1(\mathbf{n}_1, \mathbf{n}_2)$ and $\lambda_2(\mathbf{n}_1, \mathbf{n}_2)$ are dimensionless quantities of the order of unity, dependent on the directions of the wave vectors and polarizations of the phonons scattered by the impurities; Δm_1, Δm_2 and Δm_3 are respectively the differences between the masses of an atom of the basic material and of a paramagnetic impurity atom, a diamagnetic impurity atom and an atom of a rarer isotope. Finally, the function $f(\mathbf{k}_1, \mathbf{k}_2)$ describes the phase difference produced in the scattering of spin waves and phonons by impurities, and f_1, f_2 and f_3 refer, respectively, to scattering by paramagnetic and diamagnetic impurities and by atoms of a rarer isotope. The function $f(\mathbf{k}_1, \mathbf{k}_2)$ is determined by the formula

$$f(\mathbf{k}_1, \mathbf{k}_2) = \frac{1}{N}\sum_n \exp\{-i(\mathbf{k}_1 - \mathbf{k}_2, \mathbf{R}_n)\},$$

where the summation over n is taken over those lattice sites occupied by the corresponding impurity.

If we know the interaction Hamiltonian (6) we can calculate the thermal conductivity of a ferromagnetic dielectric determined by scattering of spin waves and phonons by impurities. For this purpose we write the kinetic equations, giving the change in the number of spin waves $n_{\mathbf{k}}$ and of phonons $N_{\mathbf{k}j}$ in the presence of a small temperature gradient:

$$\dot{n}_{\mathbf{k}}^{\text{CT}} \equiv L_{\mathbf{k}}^{ss}\{n\} + L_{\mathbf{k}}^{sl}\{n, N\} = n_{\mathbf{k}}^0(n_{\mathbf{k}}^0 + 1)T^{-2}\varepsilon_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}\nabla T, \tag{11}$$

$$\dot{N}_{\mathbf{k}j}^{\text{CT}} \equiv L_{\mathbf{k}j}^{ll}\{N\} + L_{\mathbf{k}j}^{ls}\{n, N\} = N_{\mathbf{k}j}^0(N_{\mathbf{k}j}^0 + 1)T^{-2}\hbar\omega_{\mathbf{k}j}\mathbf{c}_{\mathbf{k}j}\nabla T,$$

where $\mathbf{V}_{\mathbf{k}} = \hbar^{-1}\partial\epsilon_{\mathbf{k}}/\partial\mathbf{k}$ is the velocity of a spin wave; $\mathbf{c}_{\mathbf{k}j}$ the velocity of a phonon of polarization j and wave vector $\mathbf{k}$; $n_{\mathbf{k}}^0$ and $N_{\mathbf{k}j}^0$ are the equilibrium Bose functions. The collision operator L is equal to:

$$L^{ss}_{\mathbf{k}_1}\{n\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2} |\Phi_{\mathbf{k}_1\mathbf{k}_2}|^2 [(n_{\mathbf{k}_1}+1)n_{\mathbf{k}_2} - n_{\mathbf{k}_1}(n_{\mathbf{k}_2}+1)]\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}), \tag{12}$$

$$\begin{aligned} L^{sl}_{\mathbf{k}_1}\{n, N\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3 j} \{ & |\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 j}|^2 [(n_{\mathbf{k}_1}+1)n_{\mathbf{k}_2}N_{\mathbf{k}_3 j} \\ & - n_{\mathbf{k}_1}(n_{\mathbf{k}_2}+1)(N_{\mathbf{k}_3 j}+1)]\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}-\hbar\omega_{\mathbf{k}_3 j}) \\ & + |\Psi_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3 j}|^2 [(n_{\mathbf{k}_1}+1)(N_{\mathbf{k}_3 j}+1)n_{\mathbf{k}_2} \\ & - n_{\mathbf{k}_1}N_{\mathbf{k}_3 j}(n_{\mathbf{k}_2}+1)]\,\delta(\varepsilon_{\mathbf{k}_1}+\hbar\omega_{\mathbf{k}_3 j}-\varepsilon_{\mathbf{k}_2})\}, \end{aligned} \tag{13}$$

$$L^{ll}_{\mathbf{k}j}\{N\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2 j_2} |\chi_{\mathbf{k}_1 j_1 \mathbf{k}_2 j_2}|^2 [(N_{\mathbf{k}_1 j_1}+1)N_{\mathbf{k}_2 j_2} - N_{\mathbf{k}_1 j_1}(N_{\mathbf{k}_2 j_2}+1)]\,\delta(\hbar\omega_{\mathbf{k}_1 j_1} - h\omega_{\mathbf{k}_2 j_2}), \tag{14}$$

$$L^{ls}_{\mathbf{k}j}\{N, n\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3} |\Psi_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1 j}|^2 [(N_{\mathbf{k}_1 j_1}+1)(n_{\mathbf{k}_2}+1)n_{\mathbf{k}_3} - N_{\mathbf{k}_1 j_1}n_{\mathbf{k}_2}(n_{\mathbf{k}_3}+1)]\,\delta(\hbar\omega_{\mathbf{k}_1 j_1}+\varepsilon_{\mathbf{k}_2}-\varepsilon_{\mathbf{k}_3}). \tag{15}$$

We shall look for solutions of (11) of the form

$$\begin{aligned} n_{\mathbf{k}} &= n^0_{\mathbf{k}} + n^0_{\mathbf{k}}(n^0_{\mathbf{k}}+1)T^{-2}\varepsilon_{\mathbf{k}}g(\varepsilon_{\mathbf{k}})(\mathbf{k}\nabla T), \\ N_{\mathbf{k}j} &= N^0_{\mathbf{k}j} + N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)T^{-2}\hbar\omega_{\mathbf{k}j}f_j(\hbar\omega_{\mathbf{k}j})(\mathbf{k}\nabla T). \end{aligned} \tag{16}$$

Substituting (16) for the distribution functions of spin waves and phonons in the kinetic equations (11), and linearizing the collision operator for a small temperature gradient, we can obtain the following equations for the distribution functions g and f if we assume that $\epsilon_{\mathbf{k}}$ and $\omega_{\mathbf{k}}$ depend only on the magnitude of the wave vector **k**:

$$\begin{aligned} g(\varepsilon_{\mathbf{k}})[\delta L^{ss}_{\mathbf{k}}/\delta n_{\mathbf{k}} + \delta L^{sl}_{\mathbf{k}}/\delta n_{\mathbf{k}}] &= 1, \\ f_j(\hbar\omega_{\mathbf{k}j})[\delta L^{ll}_{\mathbf{k}j}/\delta N_{\mathbf{k}j} + \delta L^{ls}_{\mathbf{k}j}/\delta N_{\mathbf{k}j}] &= 1, \end{aligned} \tag{17}$$

where $\delta L_{\mathbf{k}}/\delta n_{\mathbf{k}}$ and $\delta L_{\mathbf{k}j}/\delta N_{\mathbf{k}j}$ are the variational derivatives of the collision operators with respect to the distribution functions, taken at $n_{\mathbf{k}} = n^0_{\mathbf{k}}$ and $N_{\mathbf{k}j} = N^0_{\mathbf{k}j}$.

From (17) we obtain

$$\begin{aligned} g^{-1}(\varepsilon_{\mathbf{k}_1}) = & -\frac{2\pi}{\hbar}\sum_{\mathbf{k}_2}|\Phi_{\mathbf{k}_1\mathbf{k}_2}|^2\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}) \\ & -\frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3 j}\{|\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 j}|^2(n_{\mathbf{k}_2}+N_{\mathbf{k}_3}+1)\,\delta(\varepsilon_{\mathbf{k}_1}-\varepsilon_{\mathbf{k}_2}-\hbar\omega_{\mathbf{k}_3 j}) \\ & - |\Psi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 j}|^2(n_{\mathbf{k}_2}-N_{\mathbf{k}_3 j})\,\delta(\varepsilon_{\mathbf{k}_1}+\hbar\omega_{\mathbf{k}_3 j}-\varepsilon_{\mathbf{k}_2})\}, \end{aligned} \tag{18}$$

$$\begin{aligned} f^{-1}_{j_1}(\omega_{\mathbf{k}_1 j_1}) = & -\frac{2\pi}{\hbar}\sum_{\mathbf{k}_2 j_2}|\chi_{\mathbf{k}_1 j_1,\,\mathbf{k}_2 j_2}|^2\,\delta(\hbar\omega_{\mathbf{k}_1 j_1}-\hbar\omega_{\mathbf{k}_2 j_2}) \\ & + \frac{2\pi}{\hbar}\sum_{\mathbf{k}_2\mathbf{k}_3}|\Psi_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1 j}|^2(n_{\mathbf{k}_3}-n_{\mathbf{k}_2})\,\delta(\hbar\omega_{\mathbf{k}_1 j_1}+\varepsilon_{\mathbf{k}_2}-\varepsilon_{\mathbf{k}_3}). \end{aligned} \tag{19}$$

We shall now determine the heat currents $\mathbf{S}_s$ and $\mathbf{S}_l$, carried by spin waves and phonons

$$\mathbf{S}_s = \sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}n_{\mathbf{k}}, \quad \mathbf{S}_l = \sum_{\mathbf{k}j}\hbar\omega_{\mathbf{k}j}\mathbf{c}_{\mathbf{k}j}N_{\mathbf{k}j}.$$

Using the distribution functions (16) and neglecting the dependence of the velocity of sound $\mathbf{c}_{\mathbf{k}j}$ on the direction of the wave vector **k**, we obtain[6]

$$\mathbf{S}_s = \frac{2}{3}\frac{V}{(2\pi)^3}\frac{T}{\hbar}\nabla T\int\left(\frac{\varepsilon}{T}\right)^3 n^0_{\mathbf{k}}(n^0_{\mathbf{k}}+1)g(\varepsilon_{\mathbf{k}})\,d\mathbf{k},$$

$$\mathbf{S}_l = \frac{1}{3}\frac{V}{(2\pi)^3}\frac{T}{\hbar}\nabla T\sum_j\left\{\int_{T/\hbar S_j}^{\infty}\left(\frac{\hbar\omega_{\mathbf{k}j}}{T}\right)^3 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)f_j(\hbar\omega_{\mathbf{k}j})\,d\mathbf{k} + \int_0^{T/S_j\hbar} f_j(T)\left(\frac{\hbar\omega_j}{T}\right)^2 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)\,d\mathbf{k}\right\}$$

From this the thermal conductivity

$$\varkappa = \varkappa_s + \varkappa_l,$$

where κ_s and κ_l are the thermal conductivities of spin waves and phonons:

$$\varkappa_s = -\frac{2}{3(2\pi)^3}\frac{T}{\hbar}\int\left(\frac{\varepsilon_{\mathbf{k}}}{T}\right)^3 n^0_{\mathbf{k}}(n^0_k+1)g(\varepsilon_{\mathbf{k}})\,d\mathbf{k}, \tag{20}$$

$$\varkappa_l = -\frac{1}{3(2\pi)^3}\frac{T}{\hbar}\sum_j\left\{\int_{T/\hbar S_j}^{\infty}\left(\frac{\hbar\omega_{\mathbf{k}j}}{T}\right)^3 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)f_j(\hbar\omega_{\mathbf{k}j})\,d\mathbf{k} + \int_0^{T/S_j\hbar} f_j(T)\left(\frac{\hbar\omega_j}{T}\right)^2 N^0_{\mathbf{k}j}(N^0_{\mathbf{k}j}+1)\,d\mathbf{k}\right\}. \tag{21}$$

The functions $g(\epsilon_{\mathbf{k}})$ and $f_j(\hbar\omega_{\mathbf{k}j})$ which appear in (20) and (21) depend, in general, on the actual location of the impurity in the solid. If the impurity concentration is small enough so that $d\bar{k} \gg 1$ (d is the mean distance between the impurities and $\bar{k}$ is the mean value of the wave vector of a spin wave and phonon at a given temperature) we can neglect interference in the scattering of spin waves and phonons by separate impurities. This means that we can replace the term

$$|f(\mathbf{k}_1, \mathbf{k}_2)|^2 = \frac{1}{N^2}\Big|\sum_n \exp\{i(\mathbf{k}_1-\mathbf{k}_2, \mathbf{R}_n)\}\Big|^2$$

by ξ/N, where ξ are the mean concentrations of the respective impurities. In this limiting case expressions (20) and (21) for κ_s and κ_l take the form

$$\varkappa_s = \frac{4J_1 s}{3\pi\hbar a}\left(\frac{T}{2J_1 s}\right)^2\int_{\mu M_0/T}^{\infty}\frac{e^x x^3\,dx}{(e^x-1)^2\bigl[[\xi_p(J_{12}\sigma/2J_1 s)^2+\frac{2}{3}(T/2J_1 s)^2 x^2\xi_d\bigr]}, \tag{22}$$

$$\varkappa_l = \frac{2\pi}{3}\sum_j\left\{\frac{\Theta_j^2}{\hbar aT}\int_1^{\infty} f_j(y)N^0_y(N^0_y+1)\,dy + f_j(1)\int_0^1 y^2 N^0_y(N^0_y+1)\,dy\right\}, \tag{23}$$

where μ is the Bohr magneton, $M_0 = \mu/a^3$ is the magnetic moment per unit volume and

$$\begin{aligned} f_j^{-1}(y) = & \left[\left(\frac{\Delta m_1}{m}\right)^2\overline{(\mathbf{e}_1\mathbf{e}_2)^2}+\overline{\lambda_1^2}\right]\xi_p + \left[\left(\frac{\Delta m_2}{m_0}\right)^2\overline{(\mathbf{e}_1\mathbf{e}_2)^2}+\overline{\lambda_2^2}\right]\xi_d \\ & + \xi_i\left(\frac{\Delta m_3}{m}\right)^2\overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \xi_p\frac{2J_{12}\sigma}{mc_j^2}\left(\frac{\Theta_j}{2J_1 s}\right)^3\frac{2J_{12}\sigma}{T}\frac{(e^y-1)}{y^3} \\ & \times\int_0^{\infty}\frac{e^x\sqrt{x(x+y)}}{(e^x-1)(e^{x+y}-1)}\,dx. \end{aligned}$$

ξ_p, ξ_d and ξ_i are the concentrations of paramagnetic and diamagnetic impurities and of atoms of a rarer isotope; the bar over the functions λ_1^2, λ_2^2, and $(\mathbf{e}_1\mathbf{e}_2)^2$ indicates integration over the angles, and $\Theta_j = \hbar c_j/a$.

In certain limiting cases, Eqs. (22) and (23) simplify considerably. We derive first expressions for the phonon conductivity.

If $\Theta_D \ll J_1$,

$$\varkappa_l \approx \Theta_D^2/\hbar a\xi T,$$

$$\xi = [(\Delta m_1/m)^2 \overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \overline{\lambda_1^2}]\,\xi_p + [(\Delta m_2/m)^2 \overline{(\mathbf{e}_1\mathbf{e}_2)^2} + \overline{\lambda_2^2}]\,\xi_d + (\Delta m_3/m)^2 \overline{(\mathbf{e}_1\mathbf{e}_2)^2}\,\xi_i. \tag{24}$$

If $\Theta_D \gg J_1$

$$\begin{aligned} \varkappa_l &\approx \Theta_D^2/\hbar a\xi T, & \xi_p &\ll (J_1/J_{12})^2 (\xi_d + \xi_i)\, J_1T/\Theta_D^2; \\ \varkappa_l &\approx (J_1/J_{12})^2 J_1/\hbar a\xi_p, & \xi_p &\gg (J_1/J_{12})^2 (\xi_d + \xi_i)\, J_1T/\Theta_D^2. \end{aligned} \tag{25}$$

The expression for the conductivity given by equation (24) and by the first formula of (25) agrees with the phonon conductivity found earlier by Pomeranchuk[5] and by Klemens[6] from a consideration of the thermal conductivity of dielectrics.

The conductivity of the spin waves, κ_s is given by:

$$\begin{aligned} \varkappa_s &\approx (J_1/a\hbar\xi_p)\,(T/J_1)^2, & \xi_p &\gg (T/J_1)^2\,\xi_d, \\ \varkappa_s &\approx (J_1/a\hbar\xi_d)\ln(T/\mu M_0), & \xi_p &\ll (T/J_1)^2\,\xi_d. \end{aligned} \tag{26}$$

By using (24) — (26) it is easy to obtain expressions for the thermal conductivity of ferromagnetic dielectrics in various limiting cases.

We confine ourselves to the case when the impurity is either paramagnetic or diamagnetic. If the impurity is paramagnetic, then

$$\begin{aligned} \varkappa &\approx \Theta_D^2/\hbar a\xi_p T, & \Theta_D &\ll J_1; \\ \varkappa &\approx (J_1/J_{12})^2 J_1/a\hbar\xi_p, & \Theta_D &\gg J_1. \end{aligned} \tag{27}$$

If the impurity is diamagnetic, then

$$\begin{aligned} \varkappa &= (J_1/a\hbar\xi_d)\ln(T/\mu M_0), & \Theta_D \gg T &\gg \Theta_D^2/J_1; \\ \varkappa &= \Theta_D^2/a\hbar\xi_d T, & T &\ll \Theta_D^2/J_1. \end{aligned} \tag{28}$$

For paramagnetic impurities, therefore, the conductivity is determined by the scattering of phonons by impurities. For diamagnetic impurities, if $\Theta_D \gg T \gg \Theta_D^2/J_1$ the conductivity of a ferromagnetic dielectric is determined by scattering of spin waves by impurities, and for $T \ll \Theta_D^2/J_1$ by scattering of phonons by impurities.

According to the work of Akhiezer and one of the present authors,[7] the conductivity determined by umklapp processes is

$$\begin{aligned} \varkappa &\sim (mc^2/9\hbar)\,(C_l + 2C_s)^2 a^5 \exp(\pi\Theta_D/T), & \Theta_D &\ll J_1; \\ \varkappa &\sim (T/a\hbar)\exp(\pi^2 J_1/T), & \Theta_D &\gg J_1. \end{aligned} \tag{29}$$

Here $C_s \sim a^{-3}(T/J_1)^{3/2}$ and $C_l \sim a^{-3}(T/\Theta_D)^3$ are the heat capacities of the spin waves and phonons respectively. By comparing expressions (27), (28), and (29) it can be seen that collisions of spin waves and phonons with impurities are the dominant factors determining the conductivity if the impurity concentration is great enough to satisfy the inequality

$$\begin{aligned} \xi_p &\gg \frac{9\Theta_D^2}{mc^2a^6\,(C_l + 2C_s)^2} \exp\left(-\frac{\pi\Theta_D}{T}\right) \text{ при } \Theta_D \ll J_1; \\ \xi_p &\gg \left(\frac{J_1}{J_{12}}\right)^2 \frac{J_1}{T} \exp\left(-\frac{\pi^2 J_1}{2T}\right) \text{ при } \Theta_D \gg J_1 \end{aligned} \tag{30}$$

for the case of paramagnetic impurities, and the inequality

$$\begin{aligned} \xi_d &\gg \frac{J_1}{mc^2}\left(\frac{\Theta_D}{T}\right)^6 \ln\frac{T}{\mu M_0} \exp\left(-\frac{\pi\Theta_D}{T}\right) \text{ при } \Theta_D \gg T \gg \frac{\Theta_D^2}{J_1}; \\ \xi_d &\gg \left(\frac{\Theta_D}{T}\right)^2 \exp\left(-\frac{\pi^2 J_1}{2T}\right), \\ &\frac{\Theta_D^2}{mc^2 T}\left(\frac{J_1}{T}\right)^3 \exp\left(\frac{-\pi\Theta_D}{T}\right) \text{ при } T \ll J_1,\ \frac{\Theta_D^2}{J_1} \end{aligned} \tag{31}$$

for diamagnetic impurities.

We have not considered the scattering of spin waves and phonons on the boundaries of the specimen in calculating the conductivity. This is permissible if the mean free paths, l_l and l_s, of the phonons and spin waves are considerably smaller than the specimen dimensions L. The mean free paths can be derived from the conductivities κ_s and κ_l by using the relation which is well known in the kinetic theory of gases

$$\varkappa \approx vlC, \tag{32}$$

where C is the heat capacity, v the average velocity, and l the mean free path.

The mean velocity of spin waves is $v_s\,(T/J_1)^{1/2} \times J_1 a/\hbar$, and the mean phonon velocity is the velocity of sound, c. Using expressions (24) — (26) for κ_l and κ_s, it is easy to see that the inequalities $l_s \ll L$ and $l_l \ll L$ hold if

$$\begin{aligned} \xi_p + (T/J_1)^2\,\xi_d &\gg a/L, \\ (J_{12}/\Theta_D)^2\,(T/J_1)^3\xi_p + (T/\Theta_D)^4\,(\xi_p + \xi_d + \xi_i) &\gg a/L. \end{aligned} \tag{33}$$

If the inequalities (33) are not satisfied the conductivity can be derived from the equation

$$\varkappa \sim (C_s v_s + C_l c)\,L. \tag{34}$$

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