THE SCATTERING MATRIX WITH THE PAULI INTERACTION TAKEN INTO ACCOUNT

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Submitted to JETP editor November 2, 1959; resubmitted May 5, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 293-303 (August, 1960)

A ten-dimensional formulation of electromagnetic field theory, making use of first-order equations, is employed. Within the framework of this formalism, a general expression is derived for the quantum-electrodynamic scattering matrix taking into account the so-called Pauli interaction without introducing scalar photons. The free field longitudinal photon states are used in order to simplify the calculations of the transition probabilities for unpolarized light.

In an earlier paper by the present authors[1] a ten-dimensional form of the quantum theory of the electromagnetic field in vacuum was developed on the basis of the "generalized" first-order wave equation[2]

$$(\gamma_\rho \nabla_\rho + \gamma_0)\psi(x) = 0 \quad .$$

The wave function ψ consists of the components of the potential ψ_μ and of the field $\psi_{\mu\nu} = -\psi_{\nu\mu}$; γ_ρ are the ten-dimensional matrices of the vector meson theory satisfying the Duffin-Kemmer algebra, while γ_0 is the projection matrix:

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{1}$$

The advantage of this approach is that scalar photons are automatically eliminated from the theory, and one does not have to impose on the wave function any restrictions associated with the Lorentz condition. States of right- and left-handed circular polarization appear naturally in the theory as the fundamental states of the photon. The second-quantized wave function of the electromagnetic field ψ and the conjugate function $\bar{\psi}$ have the form

$$\psi(x) = \frac{\sqrt{\hbar c}}{2\pi}\int \frac{d^3k}{\sqrt{k_0}} \sum_{q=0,\pm 1} [c_q(\mathbf{k})\psi^{(q)}(k)e^{ikx} + c_q^+(\mathbf{k})\psi^{(q)}(-k)e^{-ikx}],$$

$$\bar{\psi}(x) = \frac{\sqrt{\hbar c}}{2\pi}\int \frac{d^3k}{\sqrt{k_0}} \sum_{q=0,\pm 1} [c_q^+(\mathbf{k})\bar{\psi}^{(q)}(k)e^{-ikx} + c_q(\mathbf{k})\bar{\psi}^{(q)}(-k)e^{ikx}]. \tag{2}$$

Here the operators $c_q(\mathbf{k})$ and $c_q^+(\mathbf{k})$ satisfy the commutation relations

$$[c_q(\mathbf{k}), c_{q'}^+(\mathbf{k}')]_- = \delta_{qq'}\delta(\mathbf{k}-\mathbf{k}'),$$
$$[c_q(\mathbf{k}), c_{q'}(\mathbf{k}')]_- = [c_q^+(\mathbf{k}), c_{q'}^+(\mathbf{k}')]_- = 0, \quad q, q' = 0, \pm 1. \tag{3}$$

The quantities $\psi^{(q)}(\pm k)$ are defined by the following formulas (cf. reference 1)

$$\psi^{(\pm 1)}(\pm k) = \begin{pmatrix} 0 \\ \mathbf{e} \\ \pm i[\mathbf{k}\times\mathbf{e}] \\ \mp ik_4\mathbf{e} \end{pmatrix}, \quad \psi^{(\mp 1)}(\pm k) = \begin{pmatrix} 0 \\ \mathbf{e}^* \\ \pm i[\mathbf{k}\times\mathbf{e}^*] \\ \mp ik_4\mathbf{e}^* \end{pmatrix},$$

$$\psi^{(0)}(\pm k) = \begin{pmatrix} k_4 \\ \mathbf{k} \\ 0 \\ 0 \end{pmatrix}, \tag{4}$$

$$\bar{\psi} = \psi^*\eta,$$

where η is the matrix of the bilinear invariant form[1]:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{6}$$

All the components of the vectors $\psi(k)$, with the exception of the first, are three-dimensional vectors in the notation of (4); correspondingly in the matrices γ_0 and η all the elements with the exception of the first row and the first column are three-dimensional square matrices; $k = (\mathbf{k}, k_4)$ is the four-momentum of the photon, $\mathbf{e}$ and $\mathbf{e}^*$ are three-dimensional complex "circular" vectors having the following properties

$$[\mathbf{k}\times\mathbf{e}] = i|\mathbf{k}|\mathbf{e}, \quad [\mathbf{k}\times\mathbf{e}^*] = -i|\mathbf{k}|\mathbf{e}^*, \quad |\mathbf{e}| = 1.$$

We shall need the commutation relations for the transverse part of $\psi(x)$. By utilizing the methods employed previously[3] and by taking into account the fact that the truncated minimal polynomial of the operator $B(k) = ik_\rho\gamma_\rho + \gamma_0$ for the transverse part of the electromagnetic field has the form $S^2(k)\,[B(k) - 1]^2$, we obtain

$$[\psi'(x')\cdot\bar{\psi}'(x'')]_- = \frac{2\hbar c}{(2\pi)^2 i}\int S^2(\mathbf{k})(B(k)-1)^2\frac{\sin k_0x_0}{k_0}e^{i\mathbf{kx}}d^3k,$$
$$x = x' - x''. \tag{7}$$

Here $S(\mathbf{k}) = ik_a|\mathbf{k}|^{-1}\delta_{abc}\gamma_b\gamma_c$ is the projection operator giving the component of the spin along the direction of motion;[1] δ_{abc} is the Levi-Civita symbol; a, b, c = 1, 2, 3; the prime indicates that only the transverse part of the function $\psi(x)$ is taken.

In order to obtain functions that are solutions of the Dirac equation, we shall use the general method described in reference 3. The Dirac equation for the plane wave $\varphi(x) = \varphi(p)\exp(ipx)$ takes the form

$$(i\hat{p} + \varkappa)\varphi(x) = 0, \tag{8}$$

where $\hat{p} = p_\mu\beta_\mu$; β_μ are the Dirac matrices. The projection operator giving the component of spin along the direction of motion of the particle which commutes with $\hat{p}$ has the form

$$\sigma = i\delta_{abc}p_a\beta_b\beta_c/4|\mathbf{p}|. \tag{9}$$

The minimal equations for the matrices $i\hat{p}$ and σ can be written in the following form:

$$(i\hat{p}+\varkappa)(i\hat{p}-\varkappa) = 0, \quad (\sigma + {}^1/_2)(\sigma - {}^1/_2) = 0. \tag{10}$$

Therefore, in accordance with reference 3, the projection operators corresponding to the four linearly independent solutions of equation (8) have the following form

$$\tau_\pm^{(1)}(p) = ({}^1/_2 + \sigma)(\varkappa \mp i\hat{p})/2\varkappa = \pm\varphi^{(1)}(\pm p)\cdot\bar{\varphi}^{(1)}(\pm p), \tag{11}$$

$$\tau_\pm^{(2)}(p) = ({}^1/_2 - \sigma)(\varkappa \mp i\hat{p})/2\varkappa = \pm\varphi^{(2)}(\pm p)\cdot\bar{\varphi}^{(2)}(\pm p), \tag{12}$$

where $\bar{\varphi} = \varphi^*\beta_4$. Expressions of the form $\varphi(p)\cdot\bar{\varphi}(p)$ denote the dyads $(\varphi\cdot\bar{\varphi})_{ik} = \varphi_i\bar{\varphi}_k$. The vectors $\varphi^{(1)}(p)$, $\varphi^{(3)}(p)$ are amplitudes of the plane wave $\varphi(x) = \varphi(p)\exp(ipx)$ which describes a free electron of four-momentum p, and correspond to the two possible components of spin along the direction of motion. Similarly, the vectors $\varphi^{(1)}(-p)$, $\varphi^{(2)}(-p)$ are the amplitudes of the plane wave $\varphi(x) = \varphi(-p)\exp(-ipx)$ which according to Feynman[4] can be used to describe a free positron of the same four-momentum and polarization. The following normalization holds[3] for the vectors $\varphi^{(i)}(\pm p)$

$$\varphi^{(i)}(\pm p)\bar{\varphi}^{(k)}(\pm p) = \pm\delta_{ik}, \tag{13}$$

as can be easily verified directly by multiplying the operators $\tau(p)$ when $i \neq k$, and by evaluating their traces when i = k.

The general solution of the Dirac equation can be represented as usual in the form of the three-dimensional Fourier integral (cf., for example, reference 5)

$$\varphi(x) = (2\pi)^{-3/2}\int\sqrt{\frac{\varkappa}{p_0}}\sum_{r=1,2}[a_r(\mathbf{p})\varphi^{(r)}(p)e^{ipx} + b_r^+(\mathbf{p})\varphi^{(r)}(-p)e^{-ipx}]d^3p,$$
$$\bar{\varphi}(x) = (2\pi)^{-3/2}\int\sqrt{\frac{\varkappa}{p_0}}\sum_{r=1,2}[a_r^+(\mathbf{p})\bar{\varphi}^{(r)}(p)e^{-ipx} + b_r(\mathbf{p})\bar{\varphi}^{(r)}(-p)e^{ipx}]d^3p, \tag{14}$$

where

$$[a_r(\mathbf{p}), a_{r'}^+(\mathbf{p}')]_+ = \delta_{rr'}\delta(\mathbf{p}-\mathbf{p}'),$$
$$[b_r(\mathbf{p}), b_{r'}^+(\mathbf{p}')]_+ = \delta_{rr'}\delta(\mathbf{p}-\mathbf{p}'), \tag{15}$$

and all the other anticommutators vanish.

The general relation obtained in reference 3 immediately yields, with the aid of (11) and (12), the well-known commutation relations for $\varphi(x)$ and $\bar{\varphi}(x)$:

$$[\varphi(x')\cdot\bar{\varphi}(x'')]_+ = \frac{i}{(2\pi)^3}(\beta_\mu\nabla_\mu - \varkappa)\int\frac{d^3p}{p_0}e^{i\mathbf{px}}\sin p_0x_0,$$
$$x' - x'' = x. \tag{16}$$

With the aid of the same formulas (11) and (12) we can very simply obtain the pairing of the operators $\varphi(x)$ and $\bar{\varphi}(x)$. By utilizing the dyad notation we can write this relation in the form

$$\underbrace{\varphi(x')\cdot\bar{\varphi}(x'')} = S^F(x'-x'') = T[\varphi(x')\cdot\bar{\varphi}(x'')] - N[\varphi(x')\cdot\bar{\varphi}(x'')], \tag{17}$$

where T and N are the symbols denoting the mixed and the normal products respectively, S^F is the matrix operator of the pairing. On substituting (14) into (17) and taking (11) and (12) into account we easily obtain the well-known expression

$$S^F(x) = \frac{1}{2(2\pi)^3}(\varkappa - \beta_\mu\nabla_\mu)\int e^{i\mathbf{px}}e^{-ip_0|x_0|}\frac{1}{p_0}d^3p. \tag{18}$$

The ten-dimensional representation for the electromagnetic field functions enables us to generalize the Lagrangian in a natural manner to the case of the interaction of the Pauli type. As is well known, the usual Lagrangian for the interaction between the electromagnetic and the Dirac fields has the form

$$L_{\text{int}}^{(1)} = -ie\bar{\varphi}(x)\psi_\mu(x)\beta_\mu\varphi(x), \tag{19}$$

while the interaction Lagrangian of the Pauli type has the form

$$L_{\text{int}}^{(2)} = \frac{1}{2}iel\bar{\varphi}(x)\psi_{\mu\nu}(x)\beta_\mu\beta_\nu\varphi(x), \tag{20}$$

where l is a real constant having the dimension of length.[6]

We introduce the ten-dimensional matrix vector

$$\Gamma = (\beta_4, \beta_1, \beta_2, \beta_3, \beta_2\beta_3, \beta_3\beta_1, \beta_1\beta_2, \beta_1\beta_4, \beta_2\beta_4, \beta_3\beta_4). \quad (21)$$

Then, by combining (19) and (20) we can write*

$$L_{\text{int}} = L_{\text{int}}^{(1)} + L_{\text{int}}^{(2)} = -ie\bar{\varphi}(x)[\Gamma(1-\gamma_0-l\gamma_0)\psi(x)]\varphi(x). \quad (22)$$

By starting with the fact that the Lagrangian of the free electromagnetic field has in the ten-dimensional formulation the form[1,2] $L_{ph} = \bar{\psi}(x)(\gamma_\rho\nabla_\rho + \gamma_0)\psi(x)/8\pi$ we can construct the Lagrangian for the interacting electromagnetic and electron-positron fields:†

$$L(x) = (8\pi)^{-1}\bar{\psi}(x)(\gamma_\rho\nabla_\rho + \gamma_0)\psi(x) + \hbar c\bar{\varphi}(x)(\beta_\mu\nabla_\mu + \varkappa)\varphi(x)$$
$$-\tfrac{1}{2}ie\bar{\varphi}(x)\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\varphi(x)$$
$$-\tfrac{1}{2}ie\bar{\varphi}(x)\bar{\psi}(x)(1-\gamma_0+l\gamma_0)\Gamma\varphi(x). \quad (23)$$

In the classical case, we obtain from (23) by variation the following system of equations for the coupled fields:

$$(\gamma_\mu\nabla_\mu + \gamma_0)\psi(x) = 4\pi ie\bar{\varphi}(x)(1-\gamma_0+l\gamma_0)\Gamma\varphi(x), \quad (24)$$

$$(-\tilde{\gamma}_\mu\nabla_\mu + \gamma_0)\bar{\psi}(x) = 4\pi ie\bar{\varphi}(x)\Gamma(1-\gamma_0-l\gamma_0)\varphi(x), \quad (25)$$

$$(\beta_\mu\nabla_\mu + \varkappa)\varphi(x) = (ie/\hbar c)\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\varphi(x), \quad (26)$$
$$(\tilde{\beta}_\mu\nabla_\mu - \varkappa)\bar{\varphi}(x) = -(ie/\hbar c)\Gamma(1-\gamma_0-l\gamma_0)\psi(x)\bar{\varphi}(x).$$

Since[1]

$$\bar{\psi}(x) = (1-2\gamma_0)\psi(x), \quad \gamma_0(1-\gamma_0) = 0,$$
$$\tilde{\gamma}_\mu = \gamma_\mu, \quad \gamma_\mu(1-2\gamma_0) = -(1-2\gamma_0)\gamma_\mu,$$

(24) and (25) coincide. Equation (24) is equivalent to the second-order equations for the potentials with the only difference that in the right-hand side the Pauli current appears as well as the Dirac current. In particular, it follows from (24) that

$$\Box\psi_4(x) - \nabla_4(\nabla_\mu\psi_\mu(x)) = -4\pi ie\{\bar{\varphi}(x)\beta_4\varphi(x)$$
$$-l\nabla_k[\bar{\varphi}(x)\beta_4\beta_k\varphi(x)]\}. \quad (27)$$

Equation (24) can be solved by utilizing the Lorentz or the Coulomb gauge for the potentials. Usually the Lorentz gauge is used for the potentials, and this causes, after second quantization, a subsidiary condition to be imposed on the state vector. The longitudinal and the scalar fields describing the Coulomb interaction are then eliminated with the aid of this condition (cf., for example, references 7 and 8). In the final result the Coulomb field remains unquantized. The use of the Coulomb gauge enables us to avoid this procedure, since the physical distinction between the transverse and the Coulomb fields is already apparent in the equations of motions themselves. Thus, the possibility appears of taking care of the Coulomb interaction in the classical theory without subjecting the field components describing this interaction to second quantization.

Such an approach has been employed by Lipmanov[9] to describe the ordinary Coulomb interaction on the basis of a relativistic generalization of the Coulomb gauge. This generalization is associated with an arbitrary choice of the space-time surface, on which all the characteristics of the system (energy, momentum etc.) are given in the Heisenberg representation. However, such a generalization of the gauge contributes nothing new to the construction of the scattering matrix. Therefore, we shall limit ourselves to taking the gauge in the form

$$\nabla_k\psi_k(x) = 0, \quad (28)$$

as a result of which the Coulomb interaction is described only by the component $\psi_4(x)$.

As will be shown later, the choice of the gauge (28) allows us to take into account the "generalized" Coulomb interaction* already in the classical theory. This leads to the result that in the interaction representation the operators of the electromagnetic field obey a first order ten-dimensional wave equation whose solutions have been given earlier.[1]

On taking (28) into account, Eq. (27) goes over into an equation for the purely Coulomb part of the potential

$$\Delta\psi_4(x) = -4\pi ie\{\bar{\varphi}(x)\beta_4\varphi(x) - l\nabla_k[\bar{\varphi}(x)\beta_4\beta_k\varphi(x)]\}. \quad (29)$$

We break up the potential $\psi_4(x)$ into two parts $\psi_4'(x)$ and $\psi_4''(x)$ which satisfy the following equations

$$\Delta\psi_4'(x) = 4\pi iel\nabla_k[\bar{\varphi}(x)\beta_4\beta_k\varphi(x)], \quad (30)$$

$$\Delta\psi_4''(x) = -4\pi ie\bar{\varphi}(x)\beta_4\varphi(x). \quad (31)$$

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*We note that $1-\gamma_0$ picks out the first four components of ψ, while γ_0 picks out the last six components. Therefore, all the terms in (22) have the same dimensions.

†We note that because of the properties[1] of the function $\psi(x)$ the following equation holds

$$\Gamma(1-\gamma_0-l\gamma_0)\psi(x) = \bar{\psi}(x)(1-\gamma_0+l\gamma_0)\Gamma;$$

so that the interaction terms in (23) are equal to $L_{\text{int}}(x)$ in (22). In carrying out the variation $\psi(x)$ and $\bar{\psi}(x)$ in (23) are regarded as independent.

*The "generalized" Coulomb interaction, just as the ordinary one, includes the Coulomb part of the interaction due to the Pauli terms.

We obtain the solution of these equations with the aid of the invariant function $\mathscr{D}(x)$ (cf., for example, reference 9). It has the form

$$\psi_4'(x) = -4\pi iel\int \frac{\partial \mathscr{D}(x-x')}{\partial x_0}\Big|_{x_0=x_0'} \nabla_k [\bar{\varphi}(x')\beta_4\beta_k\varphi(x')]\, d^3x', \qquad (32)$$

$$\psi_4''(x) = 4\pi ie\int \frac{\partial \mathscr{D}(x-x')}{\partial x_0}\Big|_{x_0=x_0'} \bar{\varphi}(x')\beta_4\varphi(x')\, d^3x', \qquad (33)$$

where $\mathscr{D}(x - x')$ is determined in our case by the relation

$$\mathscr{D}(x-x') = \frac{1}{(2\pi)^3}\int k_0^{-3}\sin k_0(x_0 - x_0')\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\, d^3k. \qquad (34)$$

We define the transverse part of the wave function $\psi(x)$ in the following manner:

$$\psi' = (0, \psi_1, \psi_2, \psi_3, \psi_{23}', \psi_{31}', \psi_{12}', \psi_{14}', \psi_{24}', \psi_{34}'). \qquad (35)$$

Here $\psi_{\mu\nu}' = \psi_{\mu\nu} - \psi_{\mu\nu}''$, where by definition $\psi_{\mu\nu}''$ are expressed in terms of the Coulomb part of the potential ψ_4'': $\psi_{k4}'' = -\psi_{k4}'' = \nabla_k\psi_4''$, $\psi_{kn}'' = 0$. Then from (24) we obtain the following equation for $\psi'(x)$:

$$(\gamma_\mu\nabla_\mu + \gamma_0)\psi'(x) = 4\pi ie\bar{\varphi}(x)(1 - \gamma_0 + l\gamma_0)\Gamma'\varphi(x) - P(x). \qquad (36)$$

Here Γ' differs from Γ by the fact that zero appears in place of β_4; $P(x)$ is a ten-dimensional vector of the following form

$$P(x) = \begin{pmatrix} 0 \\ \nabla\nabla_4\psi_4''(x) \\ 0 \\ -\nabla\psi_4'(x) \end{pmatrix}. \qquad (37)$$

We have thus obtained the equation of motion for the transverse part of the electromagnetic field, with a right-hand side completely determined, in accordance with (32) and (33), by the current due to the charged Dirac particles with anomalous magnetic moment.

On going over to the quantum theory we define the canonical commutation relations for the functions $\varphi(x)$ and $\psi'(x)$ in the following manner [cf. (7) and (16)]:

$$[\beta_4\varphi(\mathbf{x}, x_0)\cdot\bar{\varphi}(\mathbf{x}', x_0)]_+ = \delta(\mathbf{x} - \mathbf{x}'), \qquad (38)$$

$$[\gamma_4\psi'(\mathbf{x}, x_0)\cdot\bar{\psi}(\mathbf{x}', x_0)] = 4\pi\hbar c\hat{S}^2\gamma_4^2(1 - \gamma_k\nabla_k)\,\delta(\mathbf{x} - \mathbf{x}'), \qquad (39)$$

where

$$\hat{S} = (\nabla_k/\sqrt{\nabla^2})\,\delta_{klm}\gamma_l\gamma_m,$$

while the result of applying the operator $1/\sqrt{\nabla^2}$ to the function $\exp(i\mathbf{k}\cdot\mathbf{x})$ is defined by the formula

$$(1/\sqrt{\nabla^2})\, e^{i\mathbf{k}\mathbf{x}} = e^{i\mathbf{k}\mathbf{x}}/ik$$

(cf., for example, reference 7).

Since equations (36) contain subsidiary conditions, i.e., row-equations that do not contain the time derivative, the transition to the interaction representation cannot be carried out in the usual manner (c.f., for example, reference 10).

In order to get around this difficulty we use the following procedure. We alter the Lagrangian for the interacting fields in such a way that on introducing the interaction the subsidiary conditions remain unaltered. At the same time we require that the altered Lagrangian $L_a(x)$ should lead to the ten-dimensional equation for the electromagnetic field from which the former second-order equations for the potentials can be obtained. On taking this into account we can write

$$L_a(x) = (8\pi)^{-1}\bar{\psi}(x)(\gamma_\nu\nabla_\nu + \gamma_0)\psi(x) + \hbar c\bar{\varphi}(x)(\beta_\mu\nabla_\mu + \varkappa)\varphi(x)$$
$$+ \tfrac{1}{2}iel\bar{\psi}(x)\, Rx + \tfrac{1}{2}ielR(x)\psi(x) - \tfrac{1}{2}ie\bar{\varphi}(x)\bar{\psi}(x)(1 - \gamma_0$$
$$+ l\gamma_0)\Gamma_a\varphi(x) - \tfrac{1}{2}ie\bar{\varphi}(x)\Gamma_a(1 - \gamma_0 - l\gamma_0)\bar{\psi}(x)\varphi(x), \qquad (40)$$

$$\Gamma_a = (\beta_4, \beta_1, \beta_2, \beta_3, 0, 0, 0, \beta_1\beta_4, \beta_2\beta_4, \beta_3\beta_4),$$

$$R(x) = \begin{pmatrix} 0 \\ \nabla_k\bar{\varphi}(x)\beta\beta_k\varphi(x) \\ 0 \\ 0 \end{pmatrix}. \qquad (41)$$

We note that from (40) we obtain the former equations (26) for the electron-positron field. By utilizing the previously described method of picking out the transverse part of the electromagnetic field we obtain from $La(x)$

$$(\gamma_\mu\nabla_\mu + \gamma_0)\psi'(x) = 4\pi ie\bar{\varphi}(x)\Gamma_a'(1 - \gamma_0 + l\gamma_0)\varphi(x)$$
$$- P(x) - 4\pi ielR(x), \qquad (42)$$

where Γ_a' differs from Γ_a by the fact that zero appears in place of β_4. After this the transition to the interaction representation for equations (26) and (42) with the aid of the commutation relations (38) and (39) presents no difficulty. The unitary transformation $U(x_0)$ which accomplishes this transition satisfies the equation

$$i\hbar c\, dU(x_0)/dx_0 = \int H(x)\, d^3x\cdot U(x_0), \qquad (43)$$

where $H(x)$ has the following form*

$$H(x) = -ie\bar{\varphi}(x)\bar{\psi}'(x)(1 - \gamma_0 + l\gamma_0)\Gamma_a'\varphi(x) + iel\bar{\psi}(x)R(x)$$
$$- \tfrac{1}{2}ie\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x)$$
$$+ \tfrac{1}{2}iel\nabla_k\psi_4(x)\cdot\bar{\varphi}(x)\beta_k\beta_4\varphi(x). \qquad (44)$$

Since all the operators in (44) satisfy the free field equations:

$$(\gamma_\mu\nabla_\mu + \gamma_0)\psi'(x) = 0, \qquad (\beta_\mu\nabla_\mu + \varkappa)\varphi(x) = 0,$$
$$(-\gamma_\mu\nabla_\mu + \gamma_0)\bar{\psi}'(x) = 0, \qquad (\tilde{\beta}_\mu\nabla_\mu - \varkappa)\bar{\varphi}(x) = 0, \qquad (45)$$

*All the terms in H(x) are normal products of operators taken in the interaction representation.

we can write H(x) in the following form

$$H(x) = -ie\bar{\varphi}(x)\hat{\psi}'(x)\varphi(x) - \tfrac{1}{2}ie\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x) + \tfrac{1}{2}iel\nabla_k\psi_4(x)\cdot\bar{\varphi}(x)\beta_k\beta_4\varphi(x) + iel\nabla_n(\psi_k(x)\cdot\bar{\varphi}(x)\beta_k\beta_n\varphi(x)),$$

where the following notation has been introduced

$$\hat{\psi}'(x) = \Gamma(1-\gamma_0 - l\gamma_0)\psi'(x). \qquad (46)$$

When H(x) is integrated over all space the term $iel\nabla_n[\psi_k(x)\cdot\bar{\varphi}(x)\beta_k\beta_n\varphi(x)]$ disappears, and in constructing the scattering matrix we can utilize for H(x) the following expression

$$H(x) = H_{\text{ph}}(x) + H_{\text{C}}(x), \qquad (47)$$

where

$$H_{\text{ph}}(x) = -ie\bar{\varphi}(x)\hat{\psi}'(x)\varphi(x), \qquad (48)$$

$$H_{\text{C}}(x) = -\tfrac{1}{2}ie\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x) + \tfrac{1}{2}iel\nabla_k\psi_4(x)\cdot\bar{\varphi}(x)\beta_k\beta_4\varphi(x). \qquad (49)$$

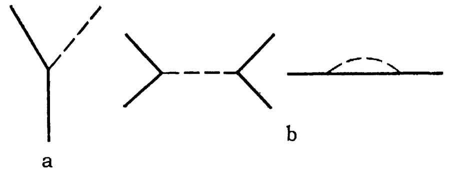
By utilizing (43) we find the following expression for the S matrix:

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{(\hbar c)^n n!}\int dx_1\ldots\int dx_n T\{H(x_1)H(x_2)\ldots H(x_n)\}. \qquad (50)$$

Thus, with the aid of the foregoing procedure the S matrix is finally expressed in terms of quantities referring to the Lagrangian (23).

The scattering matrix in the form (50) is not convenient for practical use since H(x) consists of two terms of the first and the second degree in terms of the Sommerfeld fine structure constant. However, it can be written in the form of a series in powers of $e/\sqrt{\hbar c}$ by a method similar to that proposed by Lipmanov.[9] It turns out to be possible formally to introduce symmetrical pairings of photon operators, with the state function, as before, not containing scalar photon states. Just as in the case of the usual scattering matrix, it is convenient in our case to use a graphical representation of the matrix elements. The part of the interaction operator $H_{\text{ph}}(x)$ is represented by the simplest graph (cf., the figure, graph a) with the only difference that in the momentum representation the end of the photon line corresponds, as will be seen from subsequent discussion, to the matrix $\gamma_\mu + \tfrac{1}{2}il(\gamma_\mu\hat{k} - \hat{k}\gamma_\mu)$ instead of to γ_μ. It can be easily shown that the graph corresponding to the Coulomb part of the interaction operator $H_{\text{C}}(x)$ must contain four electron lines, while the photon line must connect two vertices (graph b in the figure).

Therefore, it is convenient to deal with the Coulomb parts of the matrix element by considering the sum of these parts and the corresponding



parts arising from the "pairing" of the photon operators $\psi'(x)$. The summation of the pairings of the photon operators with the corresponding Coulomb parts is similar to that carried out in reference 9. We shall carry this out for the sum of the second order terms of the S matrix:

$$S^{(2)}_{\text{pair}} = \frac{(-i)^2}{2!(\hbar c)^2}\int dx_1 dx_2 T\{[-ie\bar{\varphi}(x_1)\underbrace{\hat{\psi}'(x_1)\varphi(x_1)][-ie\bar{\varphi}(x_2)\hat{\psi}'}(x_2)\varphi(x_2)]\} - \frac{i}{\hbar c}\int dx_0\left\{-\frac{ie}{2}\int\psi_4(x)\cdot\bar{\varphi}(x)\beta_4\varphi(x)d^3x + \frac{iel}{2}\int\nabla_m\psi_4(x)\cdot\bar{\varphi}(x)\beta_m\beta_4\varphi(x)d^3x\right\}. \qquad (51)$$

Since for the solution of practical problems we are interested in pairings of operators in convolution with the matrix vectors Γ, it appears advantageous to take this fact into account from the outset. Let us evaluate an expression of the following form

$$F'(x-x') = \underbrace{\hat{\psi}'(x)\ldots\hat{\psi}'}(x') = T[\hat{\psi}'(x)\ldots\hat{\psi}'(x')] - N[\hat{\psi}'(x)\ldots\hat{\psi}'(x')], \qquad (52)$$

where the dots denote intermediate terms appearing in the scattering matrix, while the symbols T and N refer only to the operators appearing in the expressions $\psi'(x)$. By utilizing the method employed in reference 3, and on substituting (2) into (52), we obtain

$$F'(x-x') = \frac{2\hbar c}{(2\pi)^2}\int\sum_{q=\pm1}(\hat{\psi}^{(q)}(k)\ldots\hat{\psi}^{(q)}(-k))e^{-ik_0|x_0-x_0'|}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{d^3k}{k_0}. \qquad (53)$$

Let us transform the quantities $\hat{\psi}^{(\pm1)}(\pm k)$. For example, on taking (4) into account we obtain

$$\hat{\psi}^{(+1)}(k) = \hat{e} + il[e_1\beta_1(k_2\beta_2 + k_3\beta_3 + k_4\beta_4) + e_2\beta_2(k_1\beta_1 + k_3\beta_3 + k_4\beta_4) + e_3\beta_3(k_1\beta_1 + k_2\beta_2 + k_4\beta_4)], \qquad (54)$$

where the indices 1 – 4 denote the components of the vectors **e** and **k**; $\hat{e} = e_a\beta_a$. On adding to (54) a sum of terms of the form

$$il(e_1\beta_1\cdot k_1\beta_1 + e_2\beta_2\cdot k_2\beta_2 + e_3\beta_3\cdot k_3\beta_3) = il\mathbf{ek} = 0,$$

we obtain the expression

$$\hat{\psi}^{(+1)}(k) = \hat{e} + \tfrac{1}{2}il(\hat{e}\hat{k} - \hat{k}\hat{e}),$$

where $\hat{k} = k_\mu \beta_\mu$. In a similar manner we obtain for the remaining expressions $\hat{\psi}^{(\pm 1)}(\pm k)$ the relations

$$\hat{\psi}^{(\pm 1)}(\pm k) = \hat{e} \pm \tfrac{1}{2} il(\hat{e}\hat{k} - \hat{k}\hat{e}),$$
$$\hat{\psi}^{(\mp 1)}(\pm k) = \hat{e}^* \pm \tfrac{1}{2} il(\hat{e}^*\hat{k} - \hat{k}\hat{e}^*). \quad (55)$$

Similar relations also hold for $\hat{\psi}^{(0)}(\pm k)$, where the vectors $\psi^{(0)}(\pm k)$ describe the free field longitudinal photon states:

$$\hat{\psi}^{(0)}(\pm k) = \hat{k} = \hat{k} \pm \tfrac{1}{2} il(\hat{k}\hat{k} - \hat{k}\hat{k}). \quad (56)$$

On substituting (55) into (53) and on taking into account the fact[1] that $\mathbf{e}\cdot\mathbf{e}^* + \mathbf{e}^*\cdot\mathbf{e} = 1 - \mathbf{k}\cdot\mathbf{k}/k^2$, we can write

$$F'(x-x') = 4\pi\hbar c\left\{\left[\beta_a + \frac{l}{2}(\beta\hat{\nabla} - \hat{\nabla}\beta_a)\right]\ldots \right.$$
$$\left.\ldots\left[\beta_b - \frac{l}{2}(\beta_b\hat{\nabla} - \hat{\nabla}\beta_b)\right]\right\}\left(1 - \frac{\nabla\cdot\nabla}{\nabla^2}\right)_{ab} D^F(x-x'). \quad (57)$$

Here $\hat{\nabla}$ denotes $\beta_\mu \nabla_\mu$, while $D^F(x-x')$ is the well-known singular Feynman function.

We write expression (57) as a sum of two terms:

$$F'(x-x') = 4\pi\hbar c\{[\beta_\mu + \tfrac{1}{2} l(\beta_\mu\hat{\nabla} - \hat{\nabla}\beta_\mu)]\ldots$$
$$\times[\beta_\mu - \tfrac{1}{2} l(\beta_\mu\hat{\nabla} - \hat{\nabla}\beta_\mu)]\} D^F(x-x')$$
$$- 4\pi\hbar c\{[\beta_4 + \tfrac{1}{2} l(\beta_4\hat{\nabla} - \hat{\nabla}\beta_4)]\ldots[\beta_4 - \tfrac{1}{2} l(\beta_4\hat{\nabla} - \hat{\nabla}\beta_4)]$$
$$+ [\beta_a + \tfrac{1}{2} l(\beta_a\hat{\nabla} - \hat{\nabla}\beta_a)]\ldots$$
$$\times[\beta_b + \tfrac{1}{2} l(\beta_b\hat{\nabla} - \hat{\nabla}\beta_b)]\nabla_a\nabla_b/\nabla^2\} D^F(x-x'). \quad (58)$$

After some transformations, the first term of the sum can be rewritten in the form

$$F(x-x') = \frac{4\pi\hbar c}{(2\pi)^4 i}\int \frac{d^4k}{k^2 - i\varepsilon}\left[\beta_\mu + \frac{il}{2}(\beta_\mu\hat{k} - \hat{k}\beta_\mu)\right]\ldots$$
$$\times\left[\beta_\mu - \frac{il}{2}(\beta_\mu\hat{k} - \hat{k}\beta_\mu)\right] e^{ik(x-x')}. \quad (59)$$

By utilizing the equation of continuity for the current density we can show that the substitution of the second term of (58) into (51) leads to an expression that cancels the second term of (51).

Thus, if we formally define the pairing of the photon operators in convolution with the matrix vectors Γ by equation (59), then the expression for $S^{(2)}_{\text{pair}}$ may be written in the form

$$S^{(2)}_{\text{pair}} = \frac{1}{2!}\left(-\frac{e}{\hbar c}\right)^2 \int dx\,dx' T\{\bar{\varphi}(x)\hat{\psi}'(x)\varphi(x)\cdot\bar{\varphi}(x')\hat{\psi}'(x')\varphi(x'), \quad (60)$$

where in contrast to formula (51) we have adopted a different notation for the pairing since it is defined in a different manner. From a comparison of (51) and (60) it follows that

$$\frac{e^2}{\hbar c}\int dx\,dx' T[\bar{\varphi}(x)\underbrace{\hat{\psi}'(x)\varphi(x)\cdot\bar{\varphi}(x')}\hat{\psi}'(x')\varphi(x')]$$
$$= \frac{e^2}{\hbar c}\int dx\,dx' T[\bar{\varphi}(x)\hat{\psi}'(x)\varphi(x)\cdot\bar{\varphi}(x')\hat{\psi}'(x')\varphi(x')]$$
$$+ 2i\int H_\kappa(x)\,dx. \quad (61)$$

It may be easily shown that (61) holds also in the case when the left-hand side appears as an integral factor in the higher order terms of the S matrix.

This enables us to draw a more general conclusion, viz., that the complete scattering matrix (50) is equivalent to the following expansion:*

$$S \sim \sum_{n=0}^{\infty}\left(-\frac{e}{\hbar c}\right)^n \frac{1}{n!}\int dx_1\ldots\int dx_n T[\bar{\varphi}(x_1)\hat{\psi}'(x_1)$$
$$\times\varphi(x_1)\ldots\bar{\varphi}(x_n)\hat{\psi}'(x_n)\varphi(x_n)]. \quad (62)$$

The equivalence should be interpreted in the sense that the matrix elements of S are obtained from the matrix elements of the right-hand side of (62) if all the pairings of the photon operators combined with Γ are replaced in them by expressions (59).

For the solution of problems involving the electromagnetic interaction of two different Dirac particles the interaction Lagrangian is constructed from two terms of type (22). In this case the scattering matrix (62) assumes the following form

$$S \sim \sum_{n=0}^{\infty}\left(-\frac{e}{\hbar c}\right)^n \frac{1}{n!}\int dx_1\ldots\int dx_n T\{[\bar{\varphi}_1(x_1)\hat{\psi}_1'(x_1)\varphi_1(x_1)$$
$$+\bar{\varphi}_2(x_1)\hat{\psi}_2'(x_1)\varphi_2(x_1)]\ldots$$
$$\ldots[\bar{\varphi}_1(x_n)\hat{\psi}_1'(x_n)\varphi_1(x_n) + \bar{\varphi}_2(x_n)\hat{\psi}_2'(x_n)\varphi_2(x_n)]\}, \quad (63)$$

where the indices 1 and 2 carried by φ and $\bar{\varphi}$ indicate different types of Dirac fields, while the indices 1 and 2 carried by $\hat{\psi}'$ indicate different Pauli constants l_1 and l_2.

The application of the matrix (62) to the calculations of the Compton scattering by a proton, and correspondingly, the application of (63) to the calculation of the Møller scattering of two different Dirac particles leads to the same results as Feynman's method.

In the scattering matrix (62) only the transverse photon operators appear. In reference 1 in addition to the transverse photon operators the free field longitudinal photon operators were also formally introduced. The longitudinal photon states need not have been taken into account, since the energy of these states is equal to zero. However, in the case when effects involving interaction with

*The proof of this is completely analogous to the one given in reference 9, and is therefore not given here.

unpolarized light are being discussed all the calculations can be considerably simplified with their aid. It may be easily shown that just as in the case of the usual scattering matrix (cf., for example, reference 4) the $\psi^{(0)}$ states do not give rise to any effects in real processes. In the momentum representation the matrix element which corresponds to a transition from the state in which the electron has momentum p_0 to the state in which it has momentum $p_N + k$, is of the form

$$M_q = \bar{\varphi}(p_N + k)\Big\{\hat{\psi}_N \prod_{j=1}^{N-1} A_j\hat{\psi}_j A_0 \hat{\psi}^{(q)}(k) + \sum_{s=1}^{N-1} \hat{\psi}_N \prod_{j=s+1}^{N-1} A_j\hat{\psi}_j A_s \hat{\psi}^{(q)}(k) \prod_{j=1}^{s} B_j\hat{\psi}_j + \hat{\psi}^{(q)}(k)\prod_{j=1}^{N} B_j\hat{\psi}_j\Big\}\varphi(p_0). \quad (64)$$

Here the quantities $\psi_j = \psi^{(q_j)}(k_j)$ are the Fourier components of the electromagnetic field functions which represent the corresponding perturbing fields

$$A_j = [\varkappa - i(\hat{p}_j + \hat{k})]/[\varkappa^2 + (p_j + k)^2],$$

$$B_j = (\varkappa - i\hat{p}_j)/(\varkappa^2 + p_j^2), \quad p_j = p_{j-1} + k_j,$$

and in the products Π the factors are written from left to right in the order of decreasing j. In the matrix element the Fourier component of the perturbing field $\psi^{(q)}(k)$ is specifically picked out. The terms of the sum in (64) correspond to graphs which differ from one another by the position of the photon line representing $\psi^{(q)}(k)$.

Let $\psi^{(q)}(k)$ correspond to the emission or absorption of a longitudinal free field photon, i.e., $q = 0$. Proceeding in a manner analogous to that employed by Feynman,[4] it can then be easily shown that

$$M_0 = 0. \quad (65)$$

This enables us to simplify considerably the calculation of the transition probabilities for unpolarized light. In this case the quantity $|M_{+1}|^2 + |M_{-1}|$ is to be evaluated. We introduce the notation

$$\mathfrak{M}_\mu = \bar{\varphi}(p_N + k)\Big\{\hat{\psi}_N \prod_{j=1}^{N-1} A_j\hat{\psi}_j A_0 \beta_\mu (1 + il\hat{k}) + \sum_{s=1}^{N-1} \hat{\psi}_N \prod_{j=s+1}^{N-1} A_j\hat{\psi}_j A_s \beta_\mu (1 + il\hat{k}) \prod_{j=1}^{N} B_j\hat{\psi}_j + \beta_\mu(1 + il\hat{k})\prod_{j=1}^{s} B_j\hat{\psi}_j\Big\}\varphi(p_0). \quad (66)$$

By utilizing the dyadic notation and by taking (55) into account, we obtain

$$|M_{+1}|^2 + |M_{-1}|^2 = \mathfrak{M}_a \mathfrak{M}_b^* (\mathbf{e}\cdot\mathbf{e}^* + \mathbf{e}^*\cdot\mathbf{e})_{ab}. \quad (67)$$

On adding to (67) a term of the form

$$|\mathbf{k}|^{-2}[\mathfrak{M}_a\mathfrak{M}_b^*(\mathbf{k}\cdot\mathbf{k})_{ab} - \mathfrak{M}_4\mathfrak{M}_4^* k_4^2],$$

which in consequence of (65) and (56) is equal to zero, we obtain

$$|M_{+1}|^2 + |M_{-1}|^2 = \mathfrak{M}_a\mathfrak{M}_b^*(\mathbf{e}\cdot\mathbf{e}^* + \mathbf{e}^*\cdot\mathbf{e} + \mathbf{k}\cdot\mathbf{k}/k^2)_{ab} + \mathfrak{M}_4\mathfrak{M}_4^* = \mathfrak{M}_\mu\mathfrak{M}_\mu^*. \quad (68)$$

Thus, the method of simplifying the calculation of the transition probabilities for unpolarized light proposed by Feynman[4] is generalized in a natural manner also to the Pauli interaction. In subsequent papers the method just presented will be applied to the calculation of different effects due to interactions.

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Translated by G. Volkoff
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Errata

Volume	No.	Author	page	col.	line	Reads	Should read
10	5	Bogachev et al.	872	1	21	± 0.3 cm	± 0.7 cm
11	6	Gol'danskii et al.	1229	r	Eq. (13)	$\frac{1}{4\pi^2}\ \frac{h}{Mc}$	$\frac{1}{4\pi^2}\ \frac{h}{Mc}$
			1331	r	4	$\ldots + \frac{1}{4} + \frac{\gamma_a}{2}$	$\ldots + \frac{1}{4} \cos \quad + \frac{\lambda}{2}$
12	2	Moroz and Fedorov	210	1	Eq. (7)	$\ldots \frac{\sin k_0 x_0}{k_0} e^{i\mathbf{kx}} d^3k,$	$\ldots \frac{ik_0 \delta(k^2)}{\lvert k_0 \rvert} e^{ikx} d^4k.$
			212	1	Eq. (39)	$\ldots = 4\pi\hbar c \ldots$	$\ldots = -4\pi\hbar c \ldots$
			212-3	r-1	Eqs. (44) and (39)	$\ldots + \frac{1}{2} iel \nabla_k \psi_4 (x) \ldots$	$\ldots + \frac{1}{2} iel \nabla_k \psi_4'' (x) \ldots$
			213	r	Eq. (51), line 2	$\ldots \frac{iel}{2} \int \nabla_m \psi_4 (x) \ldots$	$\ldots \frac{iel}{2} \int \nabla_m \psi_4'' (x) \ldots$
			213	r	Eq. (53)	$\ldots e^{-ik_0 \lvert x_0 - x_0' \rvert} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \frac{d^3k}{k_0}.$	$\ldots e^{ik(x-x')} \frac{d^4k}{2\pi i (k^2 - i\varepsilon)}$
12	3	Nikishov	530	1	Eq. (10)	—	$\mu^{(2)} = \frac{1}{2\beta_2\zeta} \ln\left[\frac{y_1 - 1}{y_1 + 1} \cdot \frac{-y_2 - 1}{-y_2 + 1}\right]$
			533	r	Fig. 4	The dashed curve of Fig. 4 has been incorrectly calculated (corrections to μ^+ scattering on electrons). Its value ranges from -6 to -8.	
12	1	Anisovich	72, 75		Eqs. (4a), (4b), (11)	$\sigma(\pi^+ + p \to n + \pi^+ + \pi^+)$	$2\sigma(\pi^+ + p \to n + \pi^+ + \pi^+)$
	5	,,	948		Eq. (6)	$\sigma(\pi^- + p \to n + \pi^0 + \pi^0)$	$2\sigma(\pi^- + p \to n + \pi^0 + \pi^0).$