ON THE EQUILBRIUM SHAPE OF ATOMIC NUCLEI

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The equilibrium shape of atomic nuclei has been found for the case when the deformations are small and the external nucleons do not interact among themselves.

In the initial phases of the development of the collective model of A. Bohr and Mottelson[1] it was assumed that the equilibrium shape of deformed atomic nuclei has an axis of symmetry. With this assumption it was possible to explain the properties of the experimentally observed rotational excitations.[2] A number of calculations[3,4] also confirmed this hypothesis. However, as new experimental data accumulated, there appeared in the rotational spectra of some nuclei certain anomalies, which could not be accounted for by introducing simple corrections to the rotational states corresponding to an axially symmetric equilibrium shape. These anomalies become understandable if their appearance is related to deviations of the equilibrium shape from axial symmetry.[5,6] In this connection it becomes necessary to reconsider the theoretical evidence for the existence of a symmetry axis in deformed nuclei which has up to now appeared in the literature.

The problem of the equilibrium shape of a nucleus with a single external nucleon has been investigated by A. Bohr.[3] The operator for the energy of the interaction of the external nucleon with the deformations of the core was chosen in the form

$$H_{int} = (k\beta / j(j+1))\left[\cos\gamma\,(3j_\zeta^2 - j^2) + \sqrt{3}\sin\gamma\,(j_\xi^2 - j_\eta^2)\right], \quad (1)$$

where k is some constant of the order of magnitude of the kinetic energy of the outer nucleon; j is the total angular momentum of the outer nucleon; j_ξ, j_η, and j_ζ are its projections on the axes fixed in the core; β and γ are collective coordinates which characterize the deviation of the shape of the nuclear core from spherical symmetry. Expression (1) is valid for the case in which the spin-orbit interaction of the external nucleon is large in comparison with H_{int}, so that the angular momentum j is an approximate integral of the motion.

Averaging the operator H_{int} over the coordinates of the external nucleon, we can find the interaction energy ϵ as a function of the coordinates of the core β and γ and then determine the values of β and γ for which $\epsilon(\beta, \gamma)$ summed with the potential energy of the free oscillations of the core, $V(\beta) = \frac{1}{2}C\beta^2$, has a minimum. These values of β and γ determine the equilibrium shape of the core completely.

To simplify the problem further, A. Bohr assumed that the equilibrium shape of the core is axially symmetric. Then not only j, but also j_ζ (the symmetry axis coincides with the ζ axis) becomes an approximate integral of the motion, and H_{int} is to be averaged over the state of the external nucleon with a definite angular momentum j and definite projection j_ζ on the ζ axis of the core.

Let $j_\zeta = \Omega$ and $\Phi(j, \Omega)$ be the wave function of the outer nucleon; then

$$\varepsilon_{\Omega 0}(\beta, \gamma) = (\Phi(j, \Omega) | H_{int} | \Phi(j, \Omega)) \equiv (j, \Omega | H_{int} | j, \Omega)$$
$$= (k\beta / j(j+1))\cos\gamma\,(3\Omega^2 - j(j+1)). \quad (2)$$

It is easily seen that the interaction energy $\epsilon_{\Omega 0}(\beta, \gamma)$ has a minimum for $\gamma = 0$, if $3\Omega^2 - j(j+1) < 0$, and for $\gamma = \pi$, if $3\Omega^2 - j(j+1) > 0$.

This result, therefore, would seem to justify the initial assumption about the axial symmetry of the equilibrium shape. This, however, is not actually the case: the energy of the interaction of the external nucleon with the deformations of the core has been computed only in first order of perturbation theory, and we must know the contribution from higher-order terms.

In second-order perturbation theory the correction to the interaction energy due to the nondiagonal (with respect to j) terms of the operator H_{int} is different from zero. This correction $[\epsilon_{\Omega 1}(\beta, \gamma)]$ is always negative, since we are interested in the equilibrium shape of the ground state of the nucleus: moreover, it is proportional to $\beta\sin^2\gamma$, so that we can write it in the form

$$\varepsilon_{\Omega 1}(\beta, \gamma) = -b\beta \sin^2 \gamma,$$

where $b > 0$.

Similarly we write

$$\varepsilon_{\Omega 0}(\beta, \gamma) = a\beta \cos \gamma,$$

where we set $a > 0$ for definiteness. Then

$$\varepsilon(\beta, \gamma) \approx \varepsilon_{\Omega 0}(\beta, \gamma) + \varepsilon_{\Omega 1}(\beta, \gamma) = a\beta \cos \gamma - b\beta \sin \gamma. \quad (3)$$

At the point $\gamma = \pi$ the interaction energy will have a minimum even if $\epsilon_{\Omega 1}(\beta, \gamma)$ is included, so long as $\frac{1}{2}a - b > 0$. If the opposite inequality holds, we obtain a maximum instead of a minimum. The quantity $\frac{1}{2}a - b$ depends on j and Ω. Therefore, a special investigation is required for each nucleon state in order to find the value of $\frac{1}{2}a - b$.

Bohr's proof of the existence of an axis of symmetry for a nucleus with a single external nucleon can therefore not be considered completely correct. There is still less justification for taking this proof over for the case of nuclei with a large number of nucleons.

Birbrair, Peker, and Sliv[4] showed under very general assumptions that $\partial\epsilon/\partial\gamma = 0$ for $\gamma = 0, \pi$.

However, from this it is also impossible to draw any conclusions about the axial symmetry of the equilibrium shape of the nucleus, since we must still find the sign of the second derivative.

For a more exact solution of the problem of the equilibrium shape of a nucleus with a single external nucleon, we do not make Bohr's assumption about the axial symmetry of the equilibrium shape, but keep all other restrictions. We seek the wave function of the external nucleon in the form of a superposition of states with different values of j and require that it be an eigenfunction of the operator H_{int}. As a result, we obtain the secular equation

$$|\varepsilon\delta_{\Omega\Omega'} - (j, \Omega | H_{int} | j, \Omega')| = 0, \quad (4)$$

the roots of which are the average values of the interaction energy in the $\frac{1}{2}(2j+1)$ different states of motion of the outer nucleon.

The diagonal matrix elements of the operator H_{int} were introduced above [Eq. (2)]. Among the nondiagonal elements, the following are different from zero:

$$(j, \Omega | H_{int} | j, \Omega + 2) = (j, \Omega + 2 | H_{int} | j, \Omega)$$
$$= (k\beta / 2j(j+1)) \sin\gamma\, [3(j-\Omega)(j-\Omega-1)$$
$$\times (j+\Omega+1)(j+\Omega+2)]^{1/2}. \quad (5)$$

The secular equations for different values of j are easily found:

$$x = 0 \quad (j = 1/2), \quad (6a)$$
$$x^2 - 9 = 0 \quad (j = 3/2), \quad (6b)$$
$$x^3 - 84x - 160\cos 3\gamma = 0 \quad (j = 5/2), \quad (6c)$$
$$x^4 - 378x^2 - 1728\, x\cos 3\gamma + 8505 = 0 \quad (j = 7/2), \quad (6d)$$
$$x^5 - 1188x^3 - 9504\cos 3\gamma x^2 + 171072x$$
$$+ 1119744\cos 3\gamma = 0 \quad (j = 9/2), \quad (6e)$$
$$x^6 - 3003x^4 - 36608\, x^3\cos 3\gamma$$
$$+ 1550835x^2 + 22214400\, x\cos 3\gamma$$
$$- 63149625 + 39424000\cos^2 3\gamma = 0 \quad (j = 11/2). \quad (6f)$$

The ϵ and x belonging to a given j are connected by the relation $\epsilon = k\beta x/j(j+1)$.

The solutions of equations (6c) to (6e) are shown in Figs. 1 to 3 as functions of γ. It is easily seen that a single external nucleon in the states under consideration leads to an axially-symmetric deformation. The exact solution, therefore, gives the same result as the approximate treatment of A. Bohr.

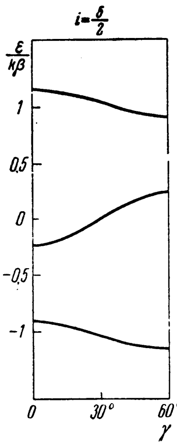

FIG. 1

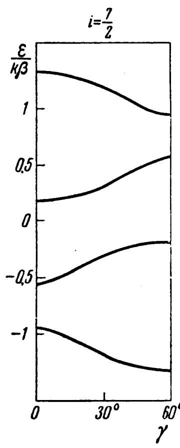

FIG. 2

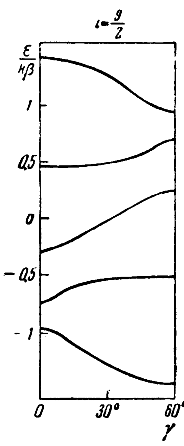

FIG. 3

Let us now turn our attention to many-particle configurations. If only one of the shells is filled, the equilibrium shape of the nucleus will remain axially symmetric as before. The only exception to this is the configuration with three nucleons in the $j = 5/2$ shell, which has an energy minimum at $\gamma = \pi/6$. It is noteworthy, however, that if the shell is less than half-filled the interaction energy has a minimum at $\gamma = \pi/3$ (oblate ellipsoid of revolution) and a maximum at $\gamma = 0$ (prolate ellipsoid of revolution). Once the shell is more than half-filled, the maximum and minimum change places. Special consideration must therefore be given to those nuclei in which the external nucleons fill up less than half of one shell and more than half of another.

By direct calculation, using the curves of Figs.

1 to 3, we convince ourselves that the configurations

$$({}^5/_2)^1({}^7/_2)^{-1},\quad ({}^5/_2)^1({}^7/_2)^{-3},\quad ({}^5/_2)^2({}^7/_2)^{-1},$$

$$({}^5/_2)^2({}^7/_2)^{-2},\quad ({}^5/_2)^2({}^7/_2)^{-3},\quad ({}^5/_2)^1({}^9/_2)^{-1}$$

etc. correspond to an equilibrium shape without axial symmetry. As an example we show in Fig. 4 the interaction energy ϵ_1 corresponding to four nucleons in the $j = {}^5/_2$ shell, the energy ϵ_2 corresponding to two nucleons in the $j = {}^7/_2$ shell, and the total energy $\epsilon = \epsilon_1 + \epsilon_2$, which has a minimum at the point $\gamma \approx {}^2/_3\pi$.

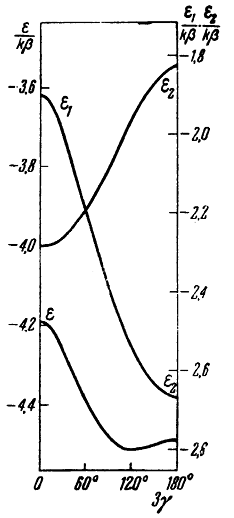

FIG. 4

We have shown, therefore, that the many-particle configurations lead in many cases to a non-axially symmetric equilibrium shape. To realize these configurations it is necessary that the neutrons and protons fill different shells, as is indeed the case in heavy and intermediate nuclei.

In the case of strongly deformed nuclei H_{int} cannot be regarded as a small quantity, so that our calculations cannot be directly applied. However, the qualitative result illustrated by Figs. 1 to 3 should be preserved in the case of strong deformations: the energy of one part of the states of the external nucleon is a minimum for an axially symmetric oblate shape of the core and a maximum for a prolate shape, while the energy of another part of the states of the external nucleon has the opposite behavior — it has a maximum for an axially symmetric oblate shape and a minimum for a prolate shape. Even for strongly deformed nuclei there exist, therefore, configurations in which the competition between the external nucleons leads to a non-axially symmetric shape. This is confirmed by the calculations of Gursky,[7] Geĭlikman,[8] and Zaikin,[9] who considered the equilibrium deformation of the core using explicit forms of the single particle potential but neglecting the spin-orbit interaction.

In many-particle configurations the motions of the external nucleons may turn out to be strongly correlated, which leads to an additional interaction of the nucleons with the deformations of the core. The equilibrium shape for one of the possible types of correlations was investigated earlier.[10]

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[1] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **27**, Nr. 16 (1953).

[2] Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. **28**, 432 (1956).

[3] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **26**, Nr. 14 (1953).

[4] Birbrair, Peker, and Sliv, JETP **36**, 803 (1959), Soviet Phys. JETP **9**, 566 (1959).

[5] A. S. Davydov and G. F. Filippov, JETP **35**, 440 and 703 (1958), Soviet Phys. JETP **8**, 303 (1959).

[6] D. van Patter, Nucl. Phys. **14**, 42 (1959/60).

[7] M. Gursky, Phys. Rev. **98**, 1205 (1955).

[8] B. T. Geĭlikman, JETP **35**, 989 (1958), Soviet Phys. JETP **8**, 690 (1959).

[9] D. A. Zaikin, JETP **35**, 529 (1958), Soviet Phys. JETP 8, 365 (1959).

[10] A. S. Davydov and G. F. Filippov, JETP **36**, 1497 (1959), Soviet Phys. JETP **9**, 1061 (1959).

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